

# Investigation of Nonlinear Fluid Flow Equation in a Porous Media and Evaluation of Convection Heat Transfer Coefficient, By Taking the Forchheimer Term into Account

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*Abstract:* - The developed fluid flow in a channel filled with porous media is known as one of the classical issues in the field of fluid mechanics. Darcy's, Brinkman's and Brinkman-Forchheimer's laws are well-known models for describing this kind of fluid. Darcy equation, as the most useful equations, is based on the description of fluid friction and porous matrix. In Brinkman equation, the term of viscosity similar to that of Laplacian in the Navier Stokes equation is added to the Darcy equation, and finally, Forchheimer term is able to account for second-order drag term due to the impact of solid in the fluid. Adding the Forchheimer term to the Darcy-Brinkman equation causes the nonlinearity of the equation. In this paper, in addition to the analytical response of this equation, the convective heat transfer coefficient is estimated. The effect of all parameters on the the Nusselt number is estimated. The results show, as the Forchheimer coefficient increases, Nusselt number declines; this downward trend is sharp in smaller Darcy numbers; hence Nusselt number tends to its asymptotic values. While, as Darcy number increase, the downtrend is getting close to a linear one.

*Key-Words:* Porous media, Brinkman-Forchheimer, Developed fluid flow, Instability of fluid flow

## 1. Introduction

Many fundamental heat transfer analyses have focused on convective flows in porous media in recent decades. Their vast range of applications in industrial applications, as well as in many natural circumstances such as high-performance building insulation, chemical catalytic reactors, packed spherical beds, grain storage, geothermal energy systems, oil recovery processes and so forth have piqued attention [1]. The Darcy equation is the most widely used equation to express fluid flow in porous media [2]. Considering the limitations of this method, especially in order to modify and include effects between fluid friction and porous matrix, Forchheimer improved the Darcy equation. The added term in Forchheimer equation cause non-linearity of Brinkman equation [3].

Munaf et al [4] showed that the effect of the inertia term on fluid flow in the porous medium would be significant under the circumstances. In some cases, such as recovering oil wells that are driven by high-pressure steam, when the pressure gradient is too high or the dense fluid flow, the inertia term will be effective.

Subramaniam and Rajagopal [5] modeled and studied high-pressure fluid flow with high pressure gradient, taking into account the pressure-dependent viscosity. Kannan and Rajagopal [6] studied high-pressure fluid flow on a slopping plate with high pressure gradient resulted from gravity, and their result indicated the development of a boundary layer in which viscosity is focused. In both studies, the flow is steady and because of the particular assumed form, the inertia term is neglected.

Vafai and Kim presented an analytical solution for laminar flow in a porous channel [7]. In this model, it was assumed that the boundary layer is not developed up to the center of the channel. Comparison of this solution to numerical solution reveals an appropriate agreement of results, if Darcy number is less than one. The reason for this is the increased thickness of the momentum boundary layer as the result of an increase in Darcy number [8].

Nield et al presented another analytical solution for the same problem without using an approximation of boundary layer. Contrary to the presented solution by Vafai and Kim, this solution is useable for flow with a Darcy number of more than one; and for the flows

with a Darcy number less than one does not have a desirable accuracy, since the error in calculating numerical integration, which is the base of this solution, increases [9].

From another perspective, Fluid flow in a porous media is another form of a two-phase flow in which the relative motion of a fluid phase in a solid matrix is investigated. Although it is generally assumed that the solids matrix is rigid and static, and therefore the flow is considered as a single-phase flow. Various analyses have been presented by various researchers on the above-mentioned models of fluid flow in porous media. [10-13].

Due to the non-linearity of the Brinkman-Forschimmer equation, this model has been studied frequently by numerical methods. In this paper, using the Liao homotopy [14] and the Kourosch method [15], an analytical solution suitable for different flow models in the channel and on the porous media is presented. The most important point of the proposed solution is the ease of use of the velocity profile, especially for the analysis of the convective heat transfer problem and the exergy analysis of flow.

### 2. Governing Equations

The incompressible flow of the Newtonian viscose fluid in the porous media and the steady state conditions are expressed by the equation of continuity and the Darcy-Brinkman Forchheimer equation.

$$\nabla \cdot V = 0$$

$$(V \cdot \nabla)V = -1/\rho \nabla P + \nu \nabla^2 V + \frac{\nu}{K} \cdot V + \frac{C_F V |V|}{\sqrt{K}}$$

Where  $V$  is the velocity vector,  $\rho$  is the density of the fluid,  $\nu$  is the fluid viscosity, and  $P$  denotes the pressure of the fluid. Also, the third term on the right hand side of the equation is the Darcy term, in which  $K$  denotes Permeability of porous media; and the fourth term represents Forchheimer term, in which  $C_F$  denotes Forchheimer coefficient.

Considering the developed flow inside a channel with flat walls filled with porous material (Figure 1), and as a result, regardless of the variations along the axis ( $\partial/\partial x = 0$ ), the velocity profile would be one-dimensional ( $v = 0$ ).

In this case, according to Eq. (2) we can write:

$$0 = -1/\rho \frac{\partial p}{\partial x} - \nu u_{yy} - \frac{\nu}{K} u - \frac{C_F}{\sqrt{K}} u |u| \tag{3}$$

$$\frac{\partial p}{\partial y} = 0 \tag{4}$$

In other words:

$$\begin{cases} u = u(y) \\ P = P(x) \end{cases} \tag{5}$$

The Eq. (3) and Eq. (4) are a general form for Darcy, Darcy-Forchheimer and Darcy-Lapwood-Brinkman

models, which will result in one of the mentioned models, ignoring different terms.

$$1/\rho \bar{P}_{1\bar{x}} + \nu/k \bar{u} = 0 \tag{6-1}$$

$$1/\rho \bar{P}_{1\bar{x}} + \nu/k \bar{u} + \frac{C_F}{\sqrt{k}} |\bar{u}| \bar{u} = 0 \tag{6-2}$$

$$\nu \bar{u}_{yy} = 1/\rho \bar{P}_{1\bar{x}} + \nu/k \bar{u} \tag{6-3}$$

By introducing dimensionless parameters as  $\Delta P = \bar{\Delta P}/\rho \nu^2$ ,  $Re = \rho \nu h/\mu$ ,  $u = \bar{u}/U$ ,  $y = \bar{y}/H$ , we can restate governing equations as following:

$$\begin{aligned} Da Re A + u &= 0 \\ Da Re A + u + C_F Re \sqrt{Da} |u| &= 0 \end{aligned} \tag{7}$$

$$Da Re A + u = Da u_{yy}$$

In general, the Darcy-Brinkman-Forchheimer equation is expressed as below:

$$Da u_{yy} = Da Re A + u + C_F Re \sqrt{Da} |u| \tag{8}$$

Where  $A = \Delta P/L$ ,  $L = \bar{L}/h$ ,  $\Delta P = \bar{\Delta P}/\rho U^2$  and finally  $Da = K/H^2$ .

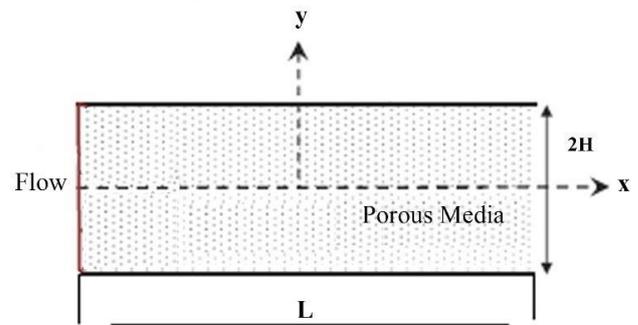


Figure 1. A channel with flat walls filled with porous material

### 3. Asymptotic Solution for Large Darcy Numbers

For  $Da \gg 1$  and by assuming that  $1/\sqrt{Da} = \epsilon$ , we can rewrite Eq. (8) as following:

$$u_{yy} = Re A + \epsilon C_F Re |u| + \epsilon^2 u \tag{9}$$

By considering  $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots$ , the resulting equations in terms of different exponentiation are:

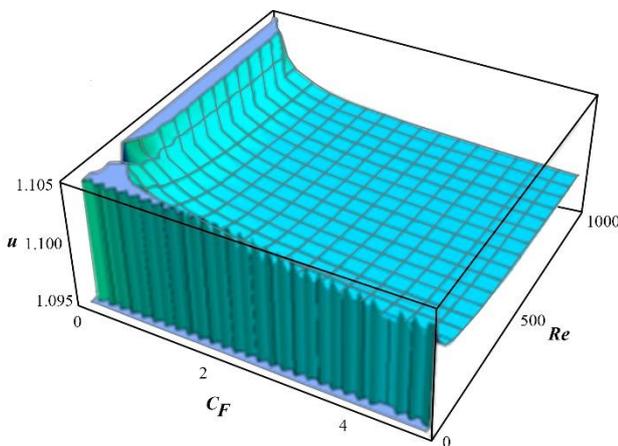
$$\begin{aligned} \epsilon^0: & u_{0yy} = Re A \\ \epsilon^1: & u_{1yy} = \epsilon C_F Re |u_0| \\ \epsilon^2: & u_{2yy} = 2Re C_F u_0 |u_1| + u_0 \\ \epsilon^3: & u_{3yy} = Re C_F (u_1^2 + 2u_0 u_2) + u_1 \\ & \dots \end{aligned} \tag{10}$$

The boundary condition for all of these equations is:  $u_i(1) = u_i(-1) = 0$  is due to the no-slip condition on the wall. The yield set of linear equations can

easily be solved. The following equation expresses analytical solution till (up to) second order of  $\varepsilon$ :

$$u(y) = \frac{1}{2} A Re (y^2 - 1) + \frac{(11A^2 C_F Re^3 - 15A^2 C_F Re^3 y^2 + 5A^2 C_F Re^3 y^4 - A^2 C_F Re^3 y^6)}{120 Da^{0.5}} + \frac{(-31500 A Re - 4919 A^3 C_F^2 Re^5 + 37800 A Re y^2)}{151200 Da} + \frac{6930 A^3 C_F^2 Re^5 y^2 - 6300 A Re y^4 - 2730 A^3 C_F^2 Re^5}{151200 Da} + \dots$$

But the answer can be obtained with higher degrees of accuracy. Figure (2) shows the velocity profile for a Darcy number of 1,000 for different Re and  $C_F$  Number.



**Figure 2.** Velocity profile for a Darcy number of 1,000 with different Reynolds and  $C_F$  number.

### 4. Velocity profile in different models

Since Darcy and Darcy-Forchheimer equations are algebraic equations, velocity profiles can easily be found in them. Although, these two equations do not satisfy boundary conditions.

$$u = -Da Re A \tag{12}$$

$$u(y) = \frac{-1 + \sqrt{1 - 4Re^2 C_F (Da)^{3/2}}}{2C_F Re \sqrt{Da}} \tag{13}$$

Where Eq. (12) and Eq. (13) show velocity profiles in Darcy equation and in Darcy-Forchheimer equation, respectively.

The same solution to the Darcy-Lapword-Brinckman equation, considering the boundary conditions  $u(1) = u(-1) = 0$ , can be shown as:

$$u(y) = A Re Da (-1 + Cosh[\frac{x}{\sqrt{Da}}] Sech[\frac{1}{\sqrt{Da}}]) \tag{14}$$

The Forchheimer term transforms the flow equation into a nonlinear equation.

In general, for different values of the Darcy number Darcy-Brinckman-Forchheimer equation is analyzed

with numerical methods by various estimates. Here is an analytical asymptotic method for obtaining a solution for the equation.

By rewriting equation (8) as the following form, and introducing the  $\mathcal{K}$  parameter, we can obtain this structure:

$$Da u_{yy} = Da Re A + \mathcal{K}u + \mathcal{K}^2 C_F Re \sqrt{Da} |u|u \tag{15}$$

Specifically, when  $\mathcal{K}$  tend to 1 ( $\mathcal{K} \rightarrow 1$ ), the Eq. (15) will be same as Eq. (8). Assuming the solution of Eq. (15) as:

$$u = u_0 + \mathcal{K}u_1 + \mathcal{K}^2 u_2 + \dots \tag{16}$$

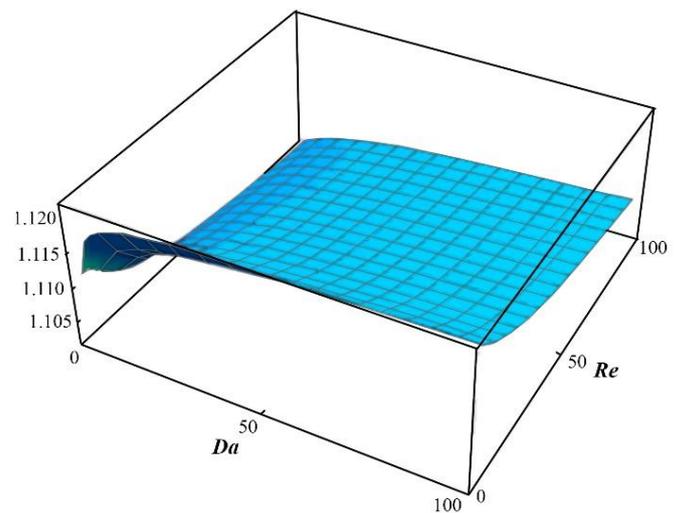
And by calculating the limit of that for  $\mathcal{K} \rightarrow 1$ , an asymptotic solution for Eq. (8) would be derived. By solving the resulting linear equations, the following solution is obtained till the second order of  $\mathcal{K}$ .

$$u(x) = A Re \left(\frac{1-x^2}{2}\right) + Da^{-1} (0.0083 - 0.25x^2 + 0.04167x^4) + AC_F Da^{0.5} Re^2 (0.09167 - 0.125x^2 + 0.04167x^4 - 0.00833x^6) + Da^{-2} (0.08472 - 0.10417x^2 + 0.02083x^4 - 0.00139x^6) + \dots \tag{17}$$

We can calculate the coefficient A by considering the fact that  $\int_0^1 u dy = 1$ .

$$A = \frac{8.75 Re}{C_F Re^3} \left( \left( -0.33 - \frac{0.053}{Da^2} - \frac{0.133}{Da} \right) + \left( \left( 0.33 + \frac{0.053}{Da^2} - \frac{0.133}{Da} \right)^2 + \frac{0.228 C_F Re}{Da^{0.5}} \right)^{0.5} \right) \tag{18}$$

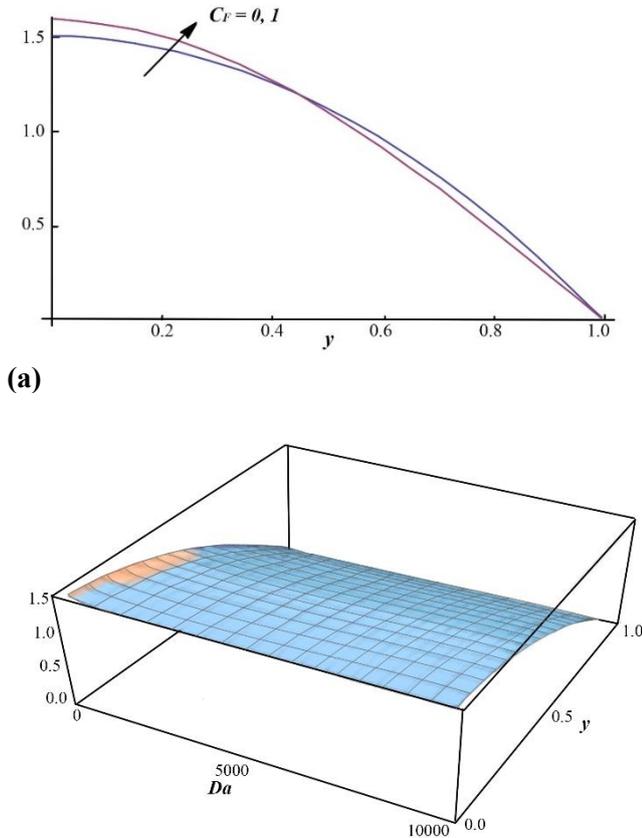
Figure (3) shows the velocity profile of Eq. (17) for  $C_F = 0$  (Brinckman flow) for different Re and Da Number.



**Figure 3.** Velocity profile for  $C_F = 0$  (Brinckman flow) for different Re and Da Number.

In Figure (4.a), the impact of Forchheimer coefficient on variation of velocity,  $Da=100$  and

Re=100, is shown. Also, the effect of Darcy number on velocity profile for  $C_F = 0.5$  and  $Re = 10$  can be seen in Fig (4.b).



**Figure 4.** a) Variation of velocity for different values of in  $C_F = 0$  and  $C_F = 1$  b) Effect of Darcy number on velocity profile for  $C_F = 0.5$  and  $Re = 10$ .

### 5. The solution of energy equation

The energy equation for the developed flow in fully porous channel can be shown as:

$$\rho C_p \bar{u} \frac{\partial \bar{T}}{\partial x} = K \frac{\partial}{\partial y} \left( \frac{\partial \bar{T}}{\partial y} \right) \tag{19}$$

Introducing dimensionless variable  $\theta = (T - T_w) / (T_m - T_w)$ , in which  $T_w$  is wall temperature and  $T_m = \int_0^1 T dy$ . Eq. (19) can be rewritten in this form:

$$\frac{H \rho C_p U}{k} u \frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial y^2} \tag{20}$$

On the other hand

$$(T_m - T_w) \rho C_p U \frac{\partial \theta}{\partial x} = q'' = 2h(T_m - T_w) \tag{21}$$

And by replacing the final equation, energy can be expressed in terms of the dimensionless variable  $\theta$ :

$$2\theta''(y) + u \cdot Nu = 0 \tag{22}$$

And the boundary conditions are

$$\begin{cases} \theta'(0) = 0 \\ \theta(1) = 0 \end{cases} \tag{23}$$

Placement of the specific velocity rate from the solution of various models in Eq. (22) and the solution of the resulting linear differential equation, will result in the temperature rate for each of the models.

For the velocity of the Darcy-Brinkman-Forchheimer model, the result dimensionless temperature equation can be written as:

$$\begin{aligned} \theta(y) = & \frac{1}{Da^{5.5}} (0.0172 ADa^3 \cdot Nu Re + \\ & 0.0424 ADa^4 \cdot Nu Re + 0.1042 ADa^5 \cdot Nu Re - \\ & 0.02128 ADa^3 \cdot Nu Rey^2 - \\ & 0.052ADa^4 \cdot Nu Rey^2 - \\ & 0.125ADa^5 \cdot Nu Rey^2 + \\ & 0.00043ADa^3 \cdot Nu Rey^4 + \\ & 0.01042ADa^4 \cdot Nu Rey^4 + \\ & 0.0208ADa^5 \cdot Nu Rey^4 - \\ & 0.00035ADa^3 \cdot Nu Rey^6 - \\ & 0.0007ADa^4 \cdot Nu Rey^6 + \\ & 0.00001ADa^5 \cdot Nu Rey^6) + \dots \end{aligned} \tag{24}$$

Also considering the definition of  $T_m$ :

$$UT_m H = \int_0^H \bar{u} T dy \tag{25a}$$

$$1 = \frac{T_m - T_w}{T_m - T_w} = \int_0^1 u \left( \frac{T - T_w}{T_m - T_w} \right) dy = \int_0^1 u \theta dy \tag{25b}$$

And finally using the equations obtained for  $\theta$  and  $u$ , we can obtain Nu number for each model by calculating the integral of the problem from their product and equating it with one.

For the general Darcy-Brinkman-Forchheimer model, the general form of Nusselt number can be written as:

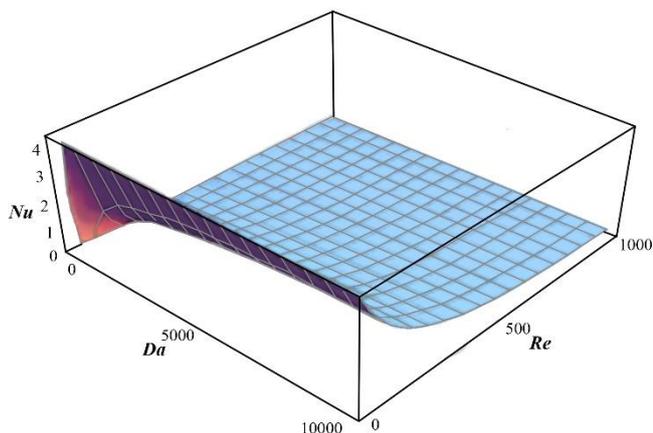
$$Nu = \frac{24Da^{11}}{M + N} \tag{26}$$

Where:

$$M = A^2 Re^2 (0.0175 Da^7 + 0.08623 Da^8 + 0.3190 Da^9 + 0.6476 Da^{11}) + \dots$$

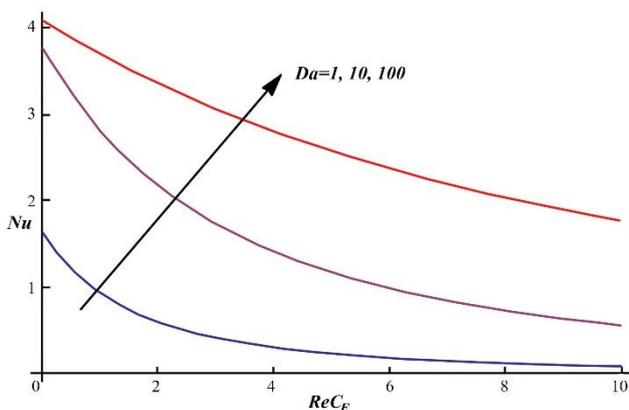
$$N = A^2 Re^2 \left( 0.0372 AC_F Da^{8.5} Re^2 + 0.0918 AC_F Da^{9.5} Re^2 + 0.2264 AC_F Da^{10.5} Re^2 + Da^{10} \cdot (0.5249 + 0.0198 A^2 C_F^2 Re^4) \right) + \dots$$

Figure (5) depicts the variations of Nusselt number in terms of Darcy number for Darcy-Brinkman-Forchheimer flow, when  $C_F$  is 0.5 .

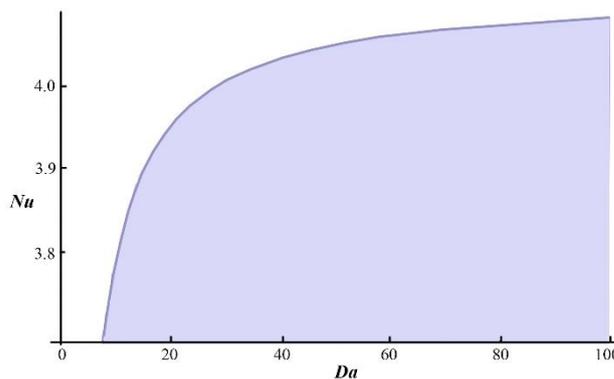


**Figure 5.** Variations of Nusselt number in terms of Darcy number for Darcy-Brinkman-Forchheimer flow in  $C_F = 0.5$

Figure (6) shows the effect of the Forchheimer coefficient on the Nusselt number for certain Darcy values in comparison with each other. By increasing Forchheimer coefficient, Nusselt number decreases; for smaller quantities of Darcy number, this downtrend is sharp and Nusselt number tends to its asymptotic values. While, as the Darcy number increases, the downtrend approaches to linear trend. which can be shown in form of  $24Da^{11}/M$  for Darcy-Lapwood-Brinkman model, neglecting terms that consist of  $C_F$ . The variation of Nusselt number in terms of Darcy number, for Brinkman flow, can be shown as Figure (7).



**Figure 6.** The effect of Forchheimer coefficient on Nusselt number for certain Darcy values



**Figure 7.** The variation of Nusselt number in terms of Darcy number, for Brinkman flow.

As can be seen, when Darcy number tends to extremely large values ( $Da \gg$ ), it is observed that the flow approaches poiseuille flow in channel; Therefore, Nusselt number tends to the conventional value of 4.117.

### 6. Conclusion

In this paper, an analytical solution is proposed for different flow models in a porous media channel. The most important point of the proposed solution is the ease of use of the velocity profile, especially for the analysis of the heat transfer problem and the exergy analysis of flow.

Adding the Forchheimer term to the Darcy-Brinkman equation leads to the nonlinearity of the equation. In this article in addition to presenting an analytical solution for this equation, convective heat transfer coefficient is obtained. The effect of all of the parameters on the Nusselt number is also estimated.

The results indicate that an increase in the Forchheimer coefficient causes a decline in the Nusselt number. In small values of Darcy number, this downtrend is sharp and happens suddenly; also, Nusselt number tends to its asymptotic values. While the increase of Darcy number makes this downtrend become close and closer to a linear trend.

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