FEM based work equivalent nodal load derivations for plates resting on Winkler foundations

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Abstract: Transmission of rational, vertical or horizontal forces to plates in case of elastic foundations is a frequently recurring in many engineering structures. In general it is often difficult to find suitable nodal loading models for plates on elastic foundation problems. In this study, it is proposed to extend an application of finite elements (FEM) for derivation nodal loads to provide solution of plates resting on Winkler foundation. The derivations of the governing differential equations and exact shape functions extended to the nodal load modeling for solving plate bending problems.

Key-Words: Nodal loads, Shape functions, Finite element, Plates, Elastic foundation

1 Introduction

In many cases assessment of vertical or horizontal forces to transmit foundations is a frequent problem of design in structural engineering. In order to include behaviour of foundation properly it is necessary to represent mathematical formulation of external loads acting on plates with some simple assumptions. Winkler model is one of the well known useful simplified model assumes the foundation behaves elastically, and vertical displacement and pressure underneath foundation are linearly proportional to the external loads with assumptions of supporting medium is isotropic, homogeneous and linearly elastic for small deflections. There are many two parameter foundation models based on Winkler model with more sophisticated proper mathematical formulations [1-3]. The procedure incorporating the finite strip method together with spring systems proposed for treating plates on elastic supports is a convenience in solution of plate problems as a numerical method have attracted much attention [4-7]. The problem can be simplified as to use a grilage of beam elements to define continuous plates. The governing equation for transverse displacement of plates subjected to lateral loads is given in Eqn 1.

$$D\left(\frac{\partial^4 w(x,y)}{\partial x^4} + 2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial y^4}\right) = q(x,y) \qquad (1)$$

Rrepresenting the soil response underneath plates by Winkler parameter the governing equation for transverse displacement of plates subjected to lateral loads. The equation can be rearranged for plates resting for Winkler foundation by using two-dimensional Laplacian operator in Eqn 2 as follows:

$$D\nabla^2\nabla^2 w + kw = q(x,y) \qquad (2)$$

where k is the Winkler parameter with the unit of force per unit area/per unit length (force/length^3) and D is the flexural rigidity of the plate element.

This equation is applicable to all types of rectangular plates including Winkler foundation problems for bending. In many cases classical methods that provide mathematically exact solutions of plate problems are available for a limited number of load and boundary conditions [8]. There are a few load and boundary conditions that permit Equation 2 to be solved exactly. Therefore approximate and numerical methods have a great importance to solve the governing differential equations of plates resting on one-parameter elastic foundation for transverse displacement. Some numerical and approximate methods, such as finite element, finite difference, boundary element and framework methods have been developed to overcome such problems [9-12].

2 Formulations of the problem

The objective of the present study is to present a numerical solution for plates on Winkler foundations. In this form, plates are idealized as a grillage of beams of a given geometry satisfying given boundary conditions. The Shape functions and stiffness matrices of a beam element on Winkler foundation can be derived to obtain nodal loads forl plate bending problems. A representation of the foundation with closely linear translational springs underlying a beam element can be considered instead of plate elements.

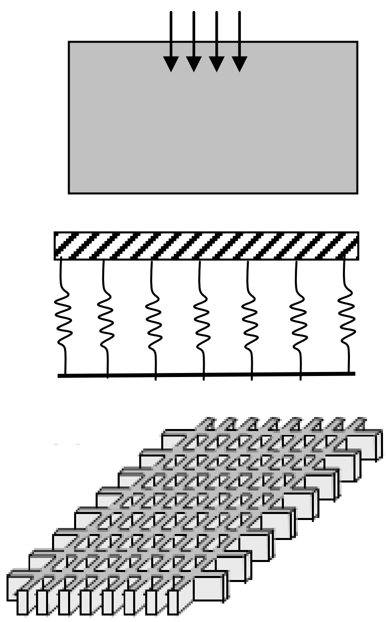


Fig 1. The idealized discrete system as parallel sets of one-dimensional elements replaced by the continuous surface.

For particular plate problems, closed form solutions have been obtained for Equation 2. The properties of beam elements that resembles that of plates resting on Winkler foundations will be a very useful tool to solve such complicate problems. However, the equation of the elastic curve derived for a beam element from the equilibrium equations of an infinitesimal segment of the structural member is given in Eqn 3.

$$EI\frac{d^4w(x)}{dx^4} + kw(x) = q(x) \tag{3}$$

For different types of loading and boundary conditions it is possible to extend the exact solution of Equation 3 for a beam element supported on a two-parameter elastic foundation to plates on generalized foundations when the plate is represented by a discrete number of intersecting beams. Then finite element based matrix methods will be used to determine the exact shape, fixed end forces and stiffness matrices of beam elements resting on elastic foundations. These individual element matrices will be used to form the system load and stiffness matrices for plates.

3 Derivation of equivalent nodal loads

For equivalent nodal force vectors firstly it is necessary to derive shape functions for beam elements. By equating $q(x)$=0; the homogeneous form of Equation 3 the shape functions obtained and the necessary evaluations as previously done [13] the s functions for one dimensional elements resting on Winkler foundation can be derived.

The bending shape functions are directly affected by the foundation parameter. It is possible to define them in non-dimensional forms for comparing the functions with the corresponding Hermitian polynomials.

$$\xi = \frac{x}{L} \qquad \text{for} \qquad 0 \le x \le L \tag{4}$$

and

$$p = \lambda L = \sqrt[4]{\frac{k_1}{4EI}}\,L \tag{5}$$

where L is the length of the beam. Note that both p and ξ are non-dimensional quantities. Since the torsional shape functions are not affected, then only the non-dimensional forms of the bending shape functions will be considered as follows;

$$\psi_1 = \left(1 - \frac{x}{L}\right) \tag{6.a}$$

$$\frac{\psi_2}{L} = \left(\frac{\begin{array}{l} \sin[p\xi]\cosh[p(2-\xi)] \\ -\cosh[p\xi]\sin[p\xi]+ \\ \cos[p(2-\xi)]\sinh[p\xi] \\ -\cos[p\xi]\sinh[p\xi] \end{array}}{p(-2+\cos[2p]+\cosh[2p]} \right) \tag{6.b}$$

$$\psi_3 = \left(\frac{\begin{array}{l} \cos[p\xi]\cosh[p(2-\xi)] \\ +\cosh[p\xi]\cos[p(2-\xi)] \\ -2\cos[p\xi]\cosh[p\xi] \\ +\sin[p\xi]\sinh[p(2-\xi)] \\ -\sinh[p\xi]\sin[p(2-\xi)] \end{array}}{2-(\cos[2p]+\cosh[2p])} \right) \tag{6.c}$$

$$\psi_4 = \left(\frac{x}{L}\right) \tag{6.d}$$

$$\frac{\psi_5}{L} = \left(\frac{\begin{array}{l} \sin[p(1-\xi)]\cosh[p(1-\xi)] \\ -\cosh[p(1+\xi)]\sin[p(1-\xi)] \\ +\cos[p(1-\xi)]\sinh[p(1-\xi)] \\ -\cos[p(1+\xi)]\sinh[p(1-\xi)] \end{array}}{p(-2+\cos[2p]+\cosh[2p]} \right) \tag{6.e}$$

$$\psi_6 = \left(\frac{\begin{array}{l} -2\cos[p(1-\xi)]\cosh[p(1-\xi)] \\ +\cosh[p(1-\xi)]\cos[p(1+\xi)] \\ +\cos[p(1-\xi)]\cosh[p(1+\xi)] \\ -\sinh[p(1-\xi)]\sin[p(1+\xi)] \\ +\sin[p(1-\xi)]\sinh[p(1+\xi)] \end{array}}{2-(\cos[2p]+\cosh[2p])} \right) \tag{6.f}$$

The shape functions [N] for beam elements resting on one-parameter elastic foundation are the main tool for fixed end moments and forces. The conventional cases are not valid for beam elements resting on elastic foundations as seen in Figure 2.a. It is obvious that the foundation reaction will affect the fixed end bending moments and forces. In some cases influence of foundation have a great importance. The nodal load vector corresponding to the loading function, q(x), acting on the span L is given by;

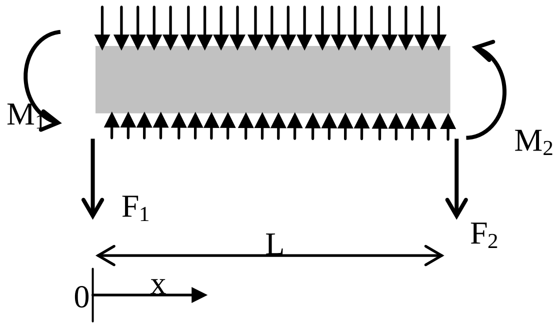
$$\{\underline{P}\} = \int_0^L [\underline{N}]q(x)dx \tag{7}$$

where [N] is the shape functions for beam elements resting on one-parameter elastic foundation.

For a distributed moment m(x) acting along the element as shown in Figure 2.b, the load vector can be rewritten as;

$$\{\underline{P}\} = \int_0^L \frac{d[\underline{N}]}{dx} m(x)dx \tag{8}$$

The above equations can be used to determine the load vectors for many common loading types. In this study the plate will be represented by a discrete number of intersecting beams. Since beam elements can be accepted as infinitesimal elements of plates, many types of loading can be represented with uniformly distributed loads or point loads applied at the nodes. Therefore, the nodal load vector will be derived only for (q(x) = q0) uniformly distributed loading of the beam elements.



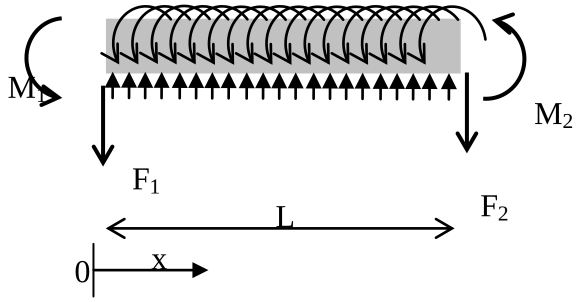


Fig 2. Nodal Forces due to Uniform Loading of a Beam Element Resting on One-Parameter (Winkler) Foundation.

Referring to Figure 2.a for uniform distributed loading, q_0, the equivalent nodal loads can be obtained by rewriting Equation (7) as;

$$\{\underline{P}\} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \int_0^L q_0 \begin{Bmatrix} N_2 \\ N_3 \\ N_5 \\ N_6 \end{Bmatrix} dx \qquad (9)$$

Inserting the corresponding shape functions from Equation (6) into Equation (9), the nodal loads obtained as;

$$F_1 = F_2 = \frac{q_0(\cosh[\lambda L] - \cos[\lambda L])}{\lambda(\sin[\lambda L] + \sinh[\lambda L])} \qquad (10.a)$$

$$M_1 = -M_2 = \frac{q_0(\sinh[\lambda L] - \sin[\lambda L])}{2\lambda^2(\sin[\lambda L] + \sinh[\lambda L])} \qquad (10.b)$$

It is obvious that when foundation parameter k1 tends to zero, the terms in Equations (10.a) and (10.b) will reduce to the conventional beam fixed end forces obtained by Hermitian functions. The well known terms are obtained as;

$$F_i = \lim_{\lambda \to 0} \frac{q_0(\cosh[\lambda L] - \cos[\lambda L])}{\lambda(\sin[\lambda L] + \sinh[\lambda L])} = \frac{q_0 L}{2} \qquad (11.a)$$

$$M_i = \lim_{\lambda \to 0} \frac{q_0(\sinh[\lambda L] - \sin[\lambda L])}{2\lambda^2(\sin[\lambda L] + \sinh[\lambda L])} = \frac{q_0 L^2}{12} \qquad (11.b)$$

In order to compare the influence of the foundation parameter k1 on fixed end forces, the normalized terms of Equation (10) with those of Equation (11) are portrayed in Figures 3 and Figure 4.

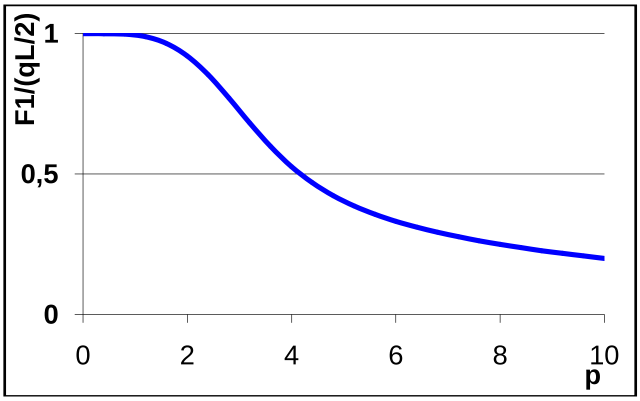


Fig 2. Normalized Nodal Force F1 due to Continuous Loading of a Beam Element Resting on One-Parameter (Winkler) Foundation.

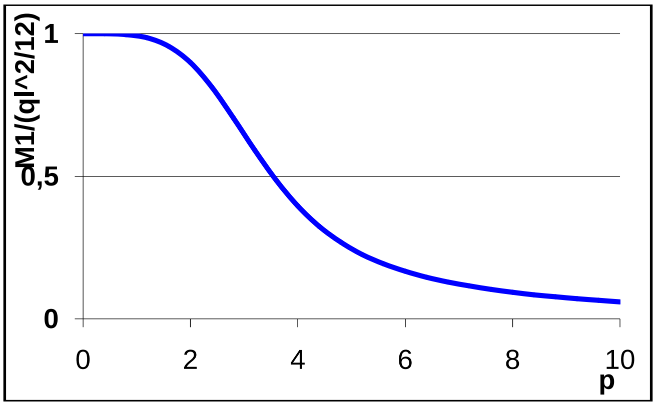


Fig 4. Normalized Nodal Force M_1 due to Continuous Loading for a Beam Element Resting on One-Parameter (Winkler) Foundation.

The figures represent behavior of the soil underneath one dimensional beam element as a part of continuous plate element. The figures indicate that the foundation rigidity has a great importance on nodal loads as fixed end forces.

4 Conclusion

The exact fixed end forces as work equivalent forces and shape functions for one-dimensional beam elements resting on one parameter foundation are the tools to solve complicate plate problems. closed form solutions even for conventional plate analysis can only be applied to the problems with simple geometry, load and boundary conditions. For plates supported elastic foundations the solution is usually much too complex and there is apparently no analytical solution other than for simple cases. A combination of finite element method, lattice analogy and matrix displacement analysis of grid works was used to obtain a finite grid solution. In this method the due to the shape functions nodal forces derived to solve the conventional beam elements and beam elements resting on Winkler foundation are valuable tools to solve plate bending problems.

References:

[1] M. M. Filonenko-Borodich, Some approximate theories of the elastic foundation, Achene Zapiski Moskowskgo Gosudertuennogo Universiteta Mechanica, U.S.S.R., vol. 46, 1940, pp. 3-18.

[2] I.E. Avramidis and K. Morfidis, "Bending of Beams on Three-Parameter Elastic Foundation, International Journal of Solids and Structures, Vol.43, No. 2, 357–375, 2006.

[3] P. L. Pasternak, On a new method of analysis of an elastic foundation by means of the foundation constants, Gosudarstvennoe Izdatelstvo Literaturi po Stroittelstvu Arkhitekture, 1954.

[4] M. A. El-Sayad, and A. M. Farag, Semi-Analytical solution based on strip method for buckling and vibration of isotropic plate, Journal of Applied Mathematics, vol. 2013, 2013, pp. 1-10.

[5] M. H. Huang, and D. P. Thambiratnam, Analysis of plate resting on elastic supports and elastic foundation by finite strip method, Computers and Structures, vol. 79, 2001, pp. 2547-2557.

[6] Y. K. Cheung, F. T. K. Au, and D. Y. Zheng, Finite strip method for the free vibration and buckling analysis of plates with abrupt changes in thickness and complex support conditions, Thin-Walled Structures, vol. 36, 2000, pp. 89-110.

[7] P. Gagnon, C. Gosselin, and L. Cloutier, A finite strip element for the analysis of variable thickness rectangular thick plates, Computers and Structures, vol. 63, no. 2, 1997, pp. 349-362.

[8] R. Szilard, Theory and analysis of plates: classical and numerical methods, Prentice Hall, Englewood Cliffs, New Jersey, 1974.

[9] K. Frydrýšek, R. Jančo, and H. Gondek, Solutions of Beams, Frames and 3D Structures on Elastic Foundation Using FEM, International Journal of Mechanics, Vol. 7, No. 4, pp., 362-369, 2013.

[10] M.K. Singha, T. Prakash, and M. Ganapathi, Finite Element Analysis of Functionally Graded Plates Under Transverse Load, Finite Elements in Analysis and Design, Vol. 47, No. 4, pp. 453–460, 2011.

[11] J. Park, H. Bai, and T. S. Jang, A Numerical approach to static deflection analysis of an infinite beam on a nonlinear elastic foundation: one-way spring model, Journal of Applied Mathematics, vol. 2013, 2013, pp.1-10.

[12] A. Karasin, M.E. Oncu, M. Suer, Extension the Consistent Mass Matrices of Beam Elements for Vibration Problems of Rectangular Plates on Winkler Foundation, 17th International Conference on Mathematical and Computational Methods in Science and Engineering, 23-25 Nisan 2015, Kuala Lumpur, Malaysia, pp.101-106, 2015.

[13] A. Karasin, Extension The Matrices of one Dimensional Beam Elements for Solution of Rectangular Plates Resting on Elastic Foundation Problems, International Journal of Mechanics, Vol.9, pp.189-197, 2015.