Bessel models of linear viscoelasticity

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Abstract: In this paper we briefly discuss the origin and derivation of the Bessel models of linear viscoelasticity, which where first introduced by Colombaro, Giusti and Mainardi in *Meccanica*, **2017**, 52, 825–832.

Key–Words: Fractional calculus, linear viscoelasticity, Bessel functions

1 Introduction

Linear viscoelasticity has proven itself to be one of the fundamental playgrounds for fraction calculus [8, 14, 18, 20] and the theory of completely monotone functions [7, 18, 19, 20].

Among the many recent proposals for viscoelastic models, mostly introduced for geophysical and mechanical purposes, it is worth stressing the role of the Maxwell-Prabhakar model [6, 11], also known as the Giusti-Colombaro model, and of the Bessel models [3, 4]. The latter, in particular, seem to play an important role in the mathematical description of fluid-filled elastic tubes [12], that justifies a special interest for this topic from a theoretical biology perspective.

The paper is therefore organized as follows:

In Section 2 we review the physical origin of the Bessel models. In particular, we briefly summarize the model for arterial pulse propagation whose viscoelastic analogy leads to the simplest realization of a model of the Bessel class.

In Section 3, we present a generalization of the viscoelastic model obtained in Section 2 and present some generalities on its physical properties.

Then, in Section 4, we present a study of the propagation of transient waves in a Bessel body by means of the Buchen-Mainardi algorithm [2].

Finally, in Section 5 we discuss an electrical ladder network dual to a viscoelastic model of the Bessel type.

2 Fluid-filled elastic tubes

One of the most interesting question that one could pose in hemodynamics, form a mathematical perspective, is: what are the effects that blood viscosity induces on the propagation of a pulse?

This problem can be formulated as follows: let us consider the propagation of a single pulse as it propagates within a uniform, semi-infinite, elastic tube and let us also assume the validity of Womersley's model for pulsatile flow (see [12] for further details).

Then one can write down, in cylindrical coordinates, the Navier-Stokes equations for the blood, which can be thought of as an incompressible Newtonian fluid of density ρ and kinematic viscosity ν. If one further neglects the motion along the circumferential direction (*i.e.* the fluid is not allowed to rotate, for sake of simplicity) then the general evolution equation for the system is given by

$$Y_{tt}(t,x) = c_0^2 \left[1 - \Phi(t) \ast\right] Y_{xx}(t,x) \,, \qquad (1)$$

where t is the time, x is the axial direction (along the axis of symmetry of the tube), $\ast$ represent the Laplace convolution integral and $Y(t,x) = \{U,\, A,\, p\}$ with U the averaged velocity of the fluid on the cross-sectional area A of the tube and p the pressure of the fluid. Furthermore, in Eq. (1) we also have two important quantities such as $c_0^2 = (A_0/\rho)\, \frac{\mathrm{d}p}{\mathrm{d}A}|_{A_0}$, where A_0 the unperturbed area of the tube, and what we shall call the memory function $\Phi(t)$ (or relaxation rate), which can be expressed in the Laplace domain as

$$\widetilde{\Phi}(\tau\, s) = \frac{2}{\sqrt{s\,\tau}} \, \frac{I_1(\sqrt{s\,\tau})}{I_0(\sqrt{s\,\tau})} \, , \tag{2}$$

where $\tau = A_0^2/\pi\,\nu$ is the so called relaxation time and I_0, I_1 denote the modified Bessel functions of order 0, 1, respectively.

Now, inspecting Eq. (2), one can easily notice that it strongly resembles a form of a wave equation that is generally found in linear viscoelasticity, according to the relaxation representation. Therefore, we can immediately conclude that the system is characterized by a peculiar memory effect which is due to the viscosity of the fluid.

If we decide to pursue this viscoelastic analogy, we shall recall that a wave equation, in linear viscoelasticity, can also be expressed in the so called Creep representation. Namely,

$$[1 + \Psi(t)\,*]\, Y_{tt}(t,x) = c_0^2\, Y_{xx}(t,x)\, , \tag{3}$$

where $\Psi(t)$ is the so called creep rate, that turns out to be related to the relaxation one $\Phi(t)$ by

$$1 + \widetilde{\Psi}(\tau\, s) = [1 - \widetilde{\Phi}(\tau\, s)]^{-1} \, ,$$

in the Laplace domain. Specifically, for this model we find that

$$\widetilde{\Psi}(\tau\, s) = \frac{2}{\sqrt{s\,\tau}} \, \frac{I_1(\sqrt{s\,\tau})}{I_2(\sqrt{s\,\tau})} \, . \tag{4}$$

For sake of simplicity, from now on we will set the relaxation time $\tau = 1$.

3 Bessel models

A straightforward generalization of the previous model can be obtained by replacing the creep and relaxation rates, $\Psi(t)$ and $\Phi(t)$, with the two functions

$$\widetilde{\Phi}_\nu(s) := \frac{2\,(\nu+1)}{\sqrt{s}} \, \frac{I_{\nu+1}(\sqrt{s})}{I_\nu(\sqrt{s})} \, , \tag{5}$$

$$\widetilde{\Psi}_\nu(s) := \frac{2\,(\nu+1)}{\sqrt{s}} \, \frac{I_{\nu+1}(\sqrt{s})}{I_{\nu+2}(\sqrt{s})} \, , \tag{6}$$

defined in the Laplace domain, provided that $\nu \in \mathbb{R}$ and $\nu > -1$.

A viscoelastic model featuring creep and relaxation rates of this kind is then defined to be a model of the Bessel class [4, 10]. In particular, if we set $\nu = 0$ we recover the model discussed in Section 2.

Inverting $\widetilde{\Phi}_\nu(s)$ and $\widetilde{\Psi}_\nu(s)$ back to the time domain (see [4]) one finds

$$\Phi_\nu(t) = 4\,(\nu+1) \sum_{k=1}^{\infty} \exp\left(-j_{\nu,\,k}^2\, t\right) \, , \tag{7}$$

$$\begin{aligned} \Psi_\nu(t) \;&=\; 4\,(\nu+1)(\nu+2) + \\ &+ 4\,(\nu+1) \sum_{k=1}^{\infty} \exp\left(-j_{\nu+2,\,k}^2\, t\right) \, , \end{aligned} \tag{8}$$

where $j_{\nu,\,k}$ and $j_{\nu+2,\,k}$ are the kth positive real root of the Bessel functions of the first kind of order ν and $\nu+2$, respectively.

In the following we provide a graphical representation of both $\Phi_\nu(t)$ and $\Psi_\nu(t)$, respectively in Figure 1 and 2, for $\nu = 1/4$.

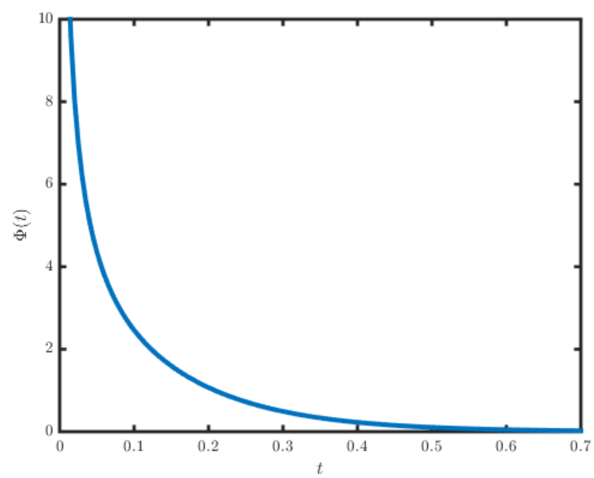

Figure 1: Rate of relaxation for a Bessel model of order $\nu = 1/4$ as a function of time.

By means of the Tauberian theorem, one can easily compute the asymptotic behaviour for these models. Specifically, for the rate of relaxation one finds

$$\Phi_\nu(t) \sim 2\,(\nu+1)/\sqrt{\pi\, t}\, , \;\text{ for } t \to 0\, , \tag{9}$$

$$\Phi_\nu(t) \sim 4\,(\nu+1)\, \exp\left(-j_{\nu,\,1}^2\, t\right) \, , \;\text{ for } t \to \infty\, , \tag{10}$$

and, for the rate of creep

$$\Psi_\nu(t) \sim 2\,(\nu+1)/\sqrt{\pi\, t}\, , \;\text{ for } t \to 0\, , \tag{11}$$

$$\Psi_\nu(t) \sim 4\,(\nu+1)(\nu+2)\, , \;\text{ for } t \to \infty\, . \tag{12}$$

From Figure 3 and Figure 4 one can easily appreciate the clear matching of the rates of creep and relaxation with the asymptotic behaviors computed above.

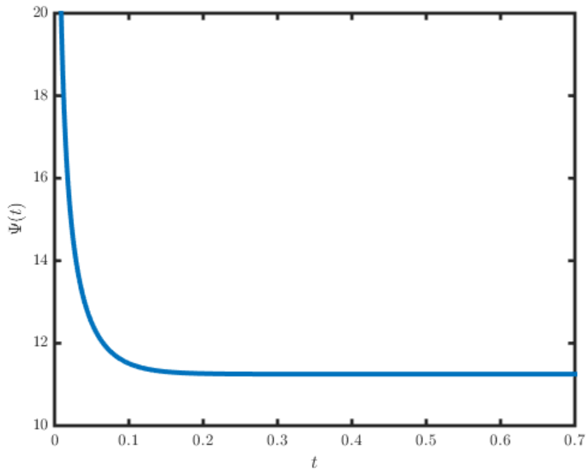

Figure 2: Rate of creep for a Bessel model of order $\nu = 1/4$ as a function of time.

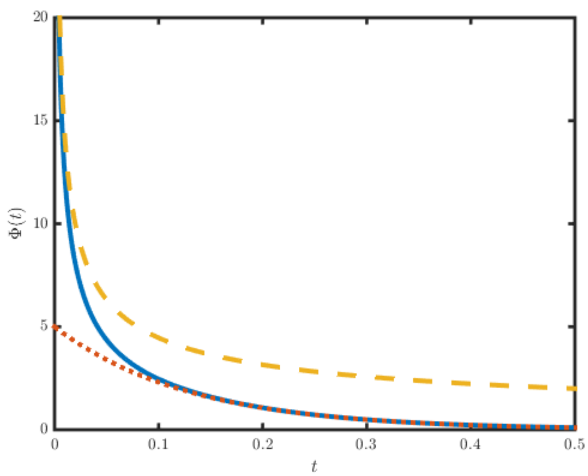

Figure 3: Rate of relaxation for a Bessel model of order $\nu = 1/4$ as a function of time. Matching with the asymptotic behaviors.

From these asymptotics we can infer a very important property of the Bessel models, namely they are characterized by a continuous transition from a fractional Maxwell-like behaviour, for short times, to an ordinary Maxwell-like behaviour for late time (see [4, 5]).

It is also important to remark that one can prove that the constitutive equation for a given Bessel body only involves ordinary infinite order differential operators and that it is formally equivalent to an infinite network of ordinary springs and dash-pots. For further details, we invite the intersted reader to refer to [10].

From a physical perspective, it is also worth re-

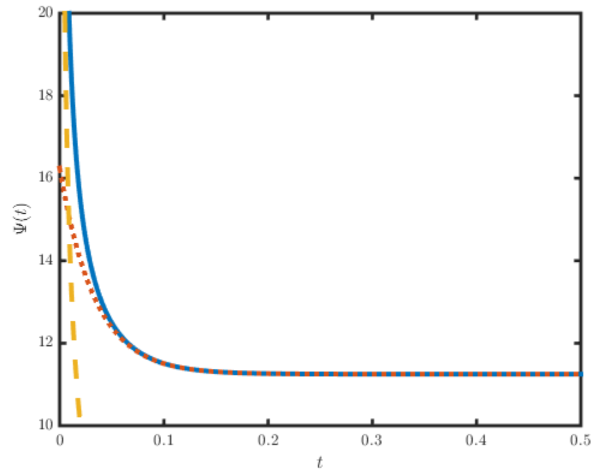

Figure 4: Rate of creep for a Bessel model of order $\nu = 1/4$ as a function of time. Matching with the asymptotic behaviors.

calling the form of the material functions $\mathtt{G}(t)$ and $\mathtt{J}(t)$, respectively known as the *relaxation modulus* and the *creep compliance* of the material. Now, from the general theory of linear viscoelasticity, one has that

$$\mathtt{G}(t) = \mathtt{G}(0^+) \left[1 - \int_0^t \Phi(t')\, dt' \right] , \tag{13}$$

$$\mathtt{J}(t) = \mathtt{J}(0^+) \left[1 + \int_0^t \Psi(t')\, dt' \right] . \tag{14}$$

Hence, for a general Bessel model of order ν one can easily infer that

$$\mathtt{G}(t\,;\,\nu) = 4(\nu+1) \sum_{n=1}^{\infty} \frac{1}{j_{\nu,\,n}^2} \exp\left(-j_{\nu,\,n}^2\, t\right) , \tag{15}$$

$$\begin{aligned} \mathtt{J}(t\,;\,\nu) \;&=\; 2\left(\frac{\nu+2}{\nu+3}\right) + 4(\nu+1)(\nu+2)t \\ &-\; 4(\nu+1) \sum_{n=1}^{\infty} \frac{1}{j_{\nu+2,\,n}^2} e^{-j_{\nu+2,\,n}^2\, t} . \end{aligned} \tag{16}$$

In Figure 5 and 6 we show some plots, in loglog scale, depicting the behavior of the material functions for $\nu = 1/4$.

One can also compute the asymptotic behavior for the material functions, which are then given by

$$\mathtt{G}(t\,;\,\nu) \sim \begin{cases} 1 - \frac{4(\nu+1)}{\sqrt{\pi}}\, t^{1/2} , & t \to 0 , \\ 4(\nu+1) \frac{1}{j_{\nu,1}^2} \exp(-j_{\nu,1}^2 t) , & t \to \infty , \end{cases} \tag{17}$$

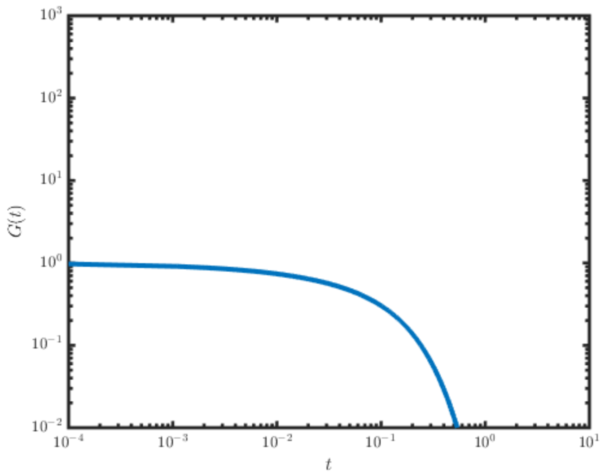

Figure 5: Relaxation modulus for a Bessel model of order $\nu = 1/4$ as a function of time.

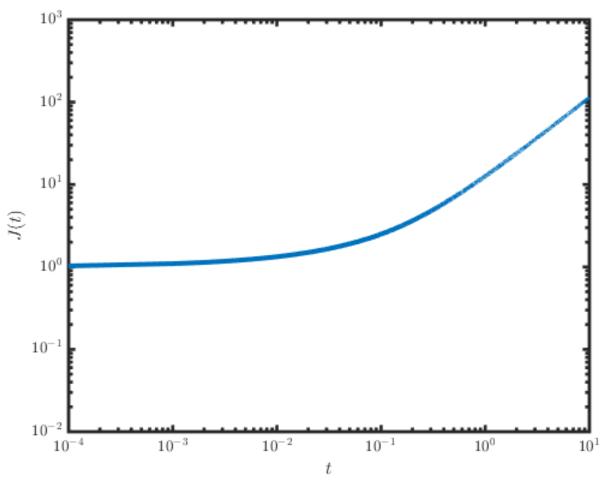

Figure 6: Creep compliance for a Bessel model of order $\nu = 1/4$ as a function of time.

$$\mathtt{J}(t\,;\,\nu) \sim \begin{cases} 1 + \frac{4(\nu+1)}{\sqrt{\pi}}\, t^{1/2}\,, & t \to 0, \\ 2\frac{(\nu+2)}{(\nu+3)} + 4(\nu+1)(\nu+2)t\,, & t \to \infty, \end{cases} \tag{18}$$

Again, for sake of completeness, in Figure 7 and 8 we provide some plots, in linear scale, showing the matching of the analytic expressions for the material functions with their asymptotic behaviors, for both short and long time.

4 Transient waves in Bessel media

The study of wave propagation involving fractional wave equations has been a central research topic of

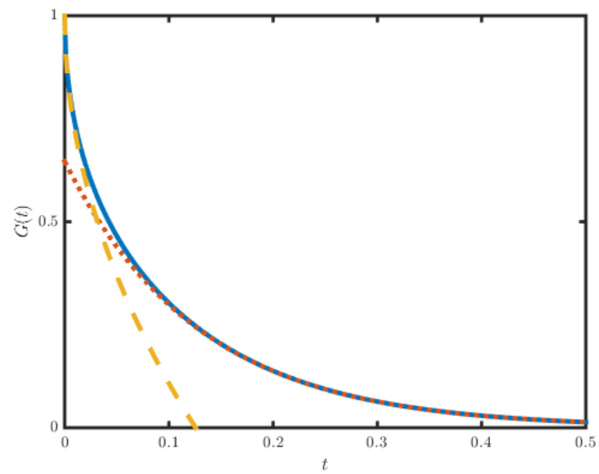

Figure 7: Linear scale plot of the relaxation modulus for a Bessel model of order $\nu = 1/4$ as a function of time. Matching with the asymptotic behaviors.

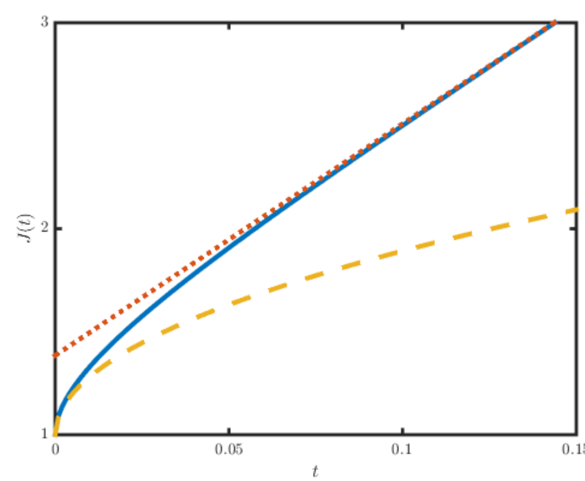

Figure 8: Linear scale plot of the creep compliance for a Bessel model of order $\nu = 1/4$ as a function of time. Matching with the asymptotic behaviors.

the last two decades for the community of fractional calculus (see *e.g.* [2, 9, 18].

One of the most interesting phenomena in the mathematical theory of wave propagation in viscoelasticity is the emergence of transient effects.

In this section we discuss the problem of the propagation of transient pulses within a semi-infinite Bessel medium. Such a problem has a very well known formal solution in the Laplace domain [1, 18], however inverting back to the time domain is often very difficult even for simple viscoelastic models.

To do that we employed the Buchen-Mainardi algorithm [1, 2, 3] that allowed us to explicitly compute the response function $r(t,x)$ of a Bessel body resulting form an impulsive input, $r_0(t) = r(t,0) = \delta(t)$, applied to the free end of the system at $t = 0$. More explicitly, approaching the wave-front, *i.e.* $t \to x^+$ one finds that

$$\begin{aligned} r(t,x) \;\sim\; & \exp\left[\frac{(\nu+1)^2}{2}\,x\right] \times \\ & \times \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\left\{A_{k,\ell}\,\frac{x^\ell}{\ell!}\,(t-x)^{(k-2)/2} \times \right. \\ & \left. \times F_{1/2}\left(\frac{(\nu+1)\,x}{(t-x)^{1/2}},\,\frac{k-2}{2}\right)\right\} \qquad (19) \end{aligned}$$

where $F_{1/2}\left(z,\,\frac{k}{2}\right) = 2^k\,\mathcal{I}^k\mathtt{Erfc}(z/2)$, with $\mathcal{I}^k$ representing the kth repeated integral, whereas the coefficients $A_{k,\ell}$ are obtained as a result of the Buchen-Mainardi algorithm, for further details see [3].

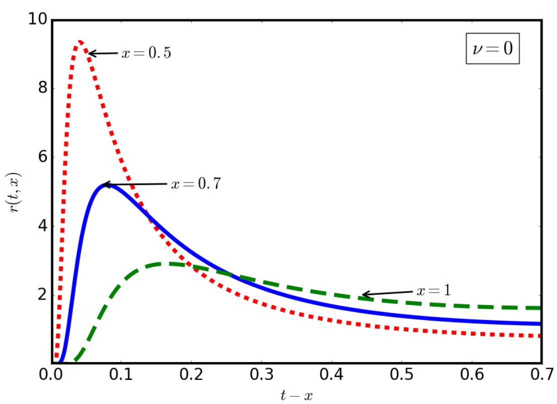

Figure 9: The impulse response for the Bessel model of order $\nu = 0$ depicted versus $t - x$ for some fixed values of x.

The model with $\nu = 0$ is particularly interesting because of its connection with hemodynamics. In particular, Figure 1 tells us that if a static observer "sits" at a given position x of the artery, then the peak of the response function will reach x with a time delay $\Delta = t_{peak} - x$, due to the viscosity of the fluid. Furthermore, one can also infer that the viscosity of the blood will also be responsible for the dissipation of part of the energy carried by the pulse, that leads to a damping of the peak as the pulse propagates within the artery.

5 Electro-Mechanical Analogy

In 1956, B. Gross and R. M. Fuoss first introduced in the literature a formal correspondence between the analytical description of electrical ladder structures and viscoelastic systems (see [15, 16, 17]).

The key idea resulting from the argument presented by Gross and Fuoss was that one can obtain an electrical system which is formally equivalent to a given mechanical system by implementing the following (formal) identifications

$$\left.\begin{array}{l} \sigma \text{ stress} \\ \varepsilon \text{ strain} \\ E \text{ elastic modulus} \\ \eta \text{ viscosity} \end{array}\right\} \iff \left\{\begin{array}{l} I \text{ current} \\ V \text{ potential} \\ 1/R \text{ conductance} \\ C \text{ capacitance} \end{array}\right.$$

Moreover, if one starts off with a general viscoelastic system, its corresponding analogous electrical system would not be described simply in terms of a single electrical component. Indeed, a viscoelastic system would formally correspond to a class of electrical ladder networks resulting from the following (formal) duality

$$\left.\begin{array}{l} \text{Spring} \\ \text{Dashpot} \end{array}\right\} \iff \left\{\begin{array}{l} \text{Resistor} \\ \text{Capacitor} \end{array}\right.$$

Now, let us first recall the general stress-strain relation in the time domain and in non-dimensional form is given by

$$\sigma(t) = \varepsilon(t) + (\dot{\mathsf{G}} * \varepsilon)(t)\,, \qquad (20)$$

where the dot is to be intended ad the (time) derivative with respect of the argument of G while $*$ represents the Laplace convolution product.

Then, if we implement the formal electro-mechanical analogy discussed above one finds that

$$I(t) = V(t) + (\dot{\mathsf{G}} * V)(t)\,, \qquad (21)$$

which represents a characteristic equation for a certain electrical ladder networks dual to a given viscoelastic model determined by the choice of the relaxation modulus $\mathsf{G}(t)$.

Hence, if we now plug into Eq. (21) the relaxation modulus of a given Bessel model of order ν [13], one can easily see the Eq. (21) turns into

$$I(t) = V(t) + [\dot{\mathsf{G}}(\,\cdot\,,\,\nu) * V](t)\,. \qquad (22)$$

From a physical perspective, it is often useful to study what is the response in current of an electrical ladder when the system is subject to a step potential, namely $V(t) = \Theta(t)$ with Θ denoting the Heaviside step function. Thus, plugging the input potential into Eq. (22) one gets

$$I(t) = \mathsf{G}(t\,,\,\nu) = 4(\nu+1)\sum_{n=1}^{\infty}\frac{\exp\left(-j_{\nu,n}^2 t\right)}{j_{\nu,n}^2}\,, \qquad (23)$$

which clearly describes a completely monotone relaxation process.

Furthermore, it is also worth noting that

$$\frac{dI(t)}{dt} = \frac{d\mathbb{G}(t\,,\,\nu)}{dt} = -\Phi_\nu(t) \tag{24}$$

that for $t \to 0^+$ behaves like

$$\frac{dI(t)}{dt} = -\Phi_\nu(t) \sim 2(\nu+1)\,\frac{t^{-1/2}}{\sqrt{\pi}}\,, \tag{25}$$

which is a peculiar feature of our electro-mechanical model.

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