A General Elastic Shell Model for Advanced Structures

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Abstract: This work proposes an exact general three-dimensional shell model for the free vibration analysis of advanced structures. The equilibrium equations are written in general orthogonal curvilinear coordinate system valid for plates and shells with constant radii of curvature. The model is developed in layer-wise form in the case of multilayered structures, and a closed form solution is provided supposing simply supported edges. The partial differential equations in z are solved by means of the exponential matrix method which is stable and not expansive from the computational point of view. The equilibrium equations are written for the general case of spherical shells and they automatically degenerate in the cylindrical and flat panel cases considering one or both radii of curvature equals zero, respectively. Results are proposed for single- and multi-layered isotropic, orthotropic, composite and functionally graded structures, and for single- and multi-walled carbon nanotubes.

Key–Words: Exact three-dimensional shell model, shell structures, free vibrations, vibration modes, composites, functionally graded materials, carbon nanotubes.

1 Introduction

Three-dimensional exact models for plates and shells have been developed by several researchers, and they have received considerable attention in the past few decades. Plate and shell elements are fundamental geometries in the analysis of general structures in several engineering fields. Therefore, an accurate validation is fundamental for plate and shell elements. Such validations and checks can be made using three-dimensional exact solutions. These solutions also allow the investigation of details about three-dimensional effects and their importance. In the open literature, papers concerning exact three-dimensional solutions separately analyze the various geometries without giving a general overview for plate and shell elements. The present method tries to fill this gap developing a general formulation for the equations of motion written in general orthogonal curvilinear coordinates. This formulation is valid for square and rectangular plates, cylindrical shell panels, spherical shell panels and cylinders. Different materials can be included in the proposed structural model: isotropic, orthotropic, composite and functionally graded materials, and equivalent elastic properties for carbon nanotube analysis after opportune considerations about the possibility of considering a continuum approach for such discrete structures. All these possibilities give a general overview for those readers interested in both plate and shell analyses. The equations of motion (in general curvilinear orthogonal coordinates including an exact geometry for shell structures without simplifications) are exactly solved.

The most relevant works in the literature about three-dimensional analysis separately consider plate and shell geometries. Examples of three-dimensional shell models are given in [1]-[17], while three-dimensional analyses of plates are usually performed using simpler equations which do not allow the analysis of shell geometries (see works [18]-[40]).

The literature proposed in the previous part about 3D plate and shell analyses shows a large variety of papers concerning plate or shell geometry. Considered topics were static and dynamic analysis, functionally graded, composite and piezoelectric materials, models based on displacement or mixed formulations. This new 3D model proposes a general formulation for several geometry types (square and rectangular plates, cylindrical and spherical shell panels, and cylindrical closed shells). The equations of motion given in general orthogonal curvilinear coordinates use an exact geometry for shells. The obtained system of second order differential equations in z is reduced to a system of first order differential equations in z. After this reduction, the system is solved using the exponential matrix method. This method has been used by Messina [25] in the case of plates in rectilinear orthogonal coordinates, and by Soldatos and Ye [7] for angle-ply laminated cylinders in cylindrical coordinates. The equations of motion here written in orthogonal curvilinear coordinates allow general exact solutions for plates and shell geometries with constant radii of curvature.

The method will be presented in Section 2 and it allows the free vibration analysis for one-layered and multilayered isotropic, orthotropic and composite structures [41]-[44], for one-layered and multilayered functionally graded structures [45], [46], and for single-walled and double-walled carbon nanotubes [47], [48]. Some of these results are proposed and discussed in Section 3, however a complete overview can be found in [41]-[48]. The main conclusions and the possible future developments are given in Section 4.

2 Three-dimensional exact shell model

The equilibrium equations are proposed in a general orthogonal curvilinear coordinate system (α, β, z) valid for plates and shells with constant radii of curvature. These equations can be solved in exact form by means of simply supported boundary conditions and the use of the exponential matrix method for the solution of the differential equations in z. The three differential equations written for the free vibration analysis of multilayered spherical shells (embedding N_{L} layers) with constant radii of curvature R_α and R_β are:

$$H_\beta\frac{\partial\sigma^k_{\alpha\alpha}}{\partial\alpha}+H_\alpha\frac{\partial\sigma^k_{\alpha\beta}}{\partial\beta}+H_\alpha H_\beta\frac{\partial\sigma^k_{\alpha z}}{\partial z}+ \left(\frac{2H_\beta}{R_\alpha}+\frac{H_\alpha}{R_\beta}\right)\sigma^k_{\alpha z}=\rho^k H_\alpha H_\beta \ddot{u}^k, \quad (1)$$

$$H_\beta\frac{\partial\sigma^k_{\alpha\beta}}{\partial\alpha}+H_\alpha\frac{\partial\sigma^k_{\beta\beta}}{\partial\beta}+H_\alpha H_\beta\frac{\partial\sigma^k_{\beta z}}{\partial z}+ \left(\frac{2H_\alpha}{R_\beta}+\frac{H_\beta}{R_\alpha}\right)\sigma^k_{\beta z}=\rho^k H_\alpha H_\beta \ddot{v}^k, \quad (2)$$

$$H_\beta\frac{\partial\sigma^k_{\alpha z}}{\partial\alpha}+H_\alpha\frac{\partial\sigma^k_{\beta z}}{\partial\beta}+H_\alpha H_\beta\frac{\partial\sigma^k_{zz}}{\partial z}- \frac{H_\beta}{R_\alpha}\sigma^k_{\alpha\alpha}-\frac{H_\alpha}{R_\beta}\sigma^k_{\beta\beta}+\left(\frac{H_\beta}{R_\alpha}+\frac{H_\alpha}{R_\beta}\right)\sigma^k_{zz} =\rho^k H_\alpha H_\beta \ddot{w}^k, \quad (3)$$

in above equations, ρ^k indicates the mass density, $(\sigma^k_{\alpha\alpha},\sigma^k_{\beta\beta},\sigma^k_{zz},\sigma^k_{\beta z},\sigma^k_{\alpha z},\sigma^k_{\alpha\beta})$ are the six stress components, $\ddot{u}^k$, $\ddot{v}^k$ and $\ddot{w}^k$ are the second temporal derivatives of the displacements u^k, v^k and w^k considered through α, β and z directions, respectively. Each quantity has a dependence from the k physical layer. R_α and R_β are the radii of curvature evaluated in the mid-surface Ω_0 of the whole multilayered structure. Parametric coefficients H_α and H_β for shells with constant radii of curvature continuously vary through the thickness direction z of the multilayered structure:

$$H_\alpha=(1+\frac{z}{R_\alpha}), H_\beta=(1+\frac{z}{R_\beta}), H_z=1. \quad (4)$$

Shells and plates are considered as simply supported. Therefore, the three displacement components are written in harmonic form:

$$u^j(\alpha,\beta,z,t)=U^j(z)e^{i\omega t}cos(\bar{\alpha}\alpha)sin(\bar{\beta}\beta), \quad (5)$$

$$v^j(\alpha,\beta,z,t)=V^j(z)e^{i\omega t}sin(\bar{\alpha}\alpha)cos(\bar{\beta}\beta), \quad (6)$$

$$w^j(\alpha,\beta,z,t)=W^j(z)e^{i\omega t}sin(\bar{\alpha}\alpha)sin(\bar{\beta}\beta). \quad (7)$$

$U^j(z)$, $V^j(z)$ and $W^j(z)$ are the displacement amplitudes in α, β and z directions, respectively. i is the imaginary unit. $\omega=2\pi f$ is the circular frequency with f as frequency value in Hz. t is the time. $\bar{\alpha}=\frac{m\pi}{a}$ and $\bar{\beta}=\frac{n\pi}{b}$, where m and n are the half-wave numbers and a and b are the shell dimensions in α and β directions, respectively (evaluated at the mid-surface Ω_0). j indicates the mathematical layers used to approximate the curvatures and/or the functionally graded laws in each k physical layer. The substitution of equations (5)-(7) and constitutive and geometrical equations (here omitted but given in details in [45]) in equations (1)-(3) leads to the final system.

The compact form of this system is (all the omitted details can be found in [45]):

$$\boldsymbol{D}^j\frac{\partial\boldsymbol{U}^j}{\partial z}=\boldsymbol{A}^j\boldsymbol{U}^j\,, \quad (8)$$

where $\frac{\partial\boldsymbol{U}^j}{\partial z}=\boldsymbol{U}^{j\prime}$ and $\boldsymbol{U}^j=[U^j\ V^j\ W^j\ U^{j\prime}\ V^{j\prime}\ W^{j\prime}]$. Equation (8) can be rewritten as:

$$\boldsymbol{D}^j\boldsymbol{U}^{j\prime}=\boldsymbol{A}^j\boldsymbol{U}^j\,, \quad (9)$$

$$\boldsymbol{U}^{j\prime}=\boldsymbol{D}^{j^{-1}}\boldsymbol{A}^j\ \boldsymbol{U}^j\,, \quad (10)$$

$$\boldsymbol{U}^{j\prime}=\boldsymbol{A}^{j*}\ \boldsymbol{U}^j\,, \quad (11)$$

with $\boldsymbol{A}^{j^*} = \boldsymbol{D}^{j^{-1}}\boldsymbol{A}^j$.

Equation (11) can be solved, as already done in [45], by means of the exponential matrix method:

$$\boldsymbol{U}^j(z^j) = exp(\boldsymbol{A}^{j^*}z^j)\boldsymbol{U}^j(0) \quad \text{with} \quad z^j \in [0, h^j] \,, \tag{12}$$

z^j is the thickness coordinate for each j layer. It goes from 0 at the bottom to h^j at the top. The exponential matrix must be calculated for the solution of the system using $z^j = h^j$ for each j layer:

$$\boldsymbol{A}^{j^{**}} = exp(\boldsymbol{A}^{j^*}h^j) = \boldsymbol{I} + \boldsymbol{A}^{j^*}h^j + \frac{\boldsymbol{A}^{j^{*2}}}{2!}h^{j^2} + \frac{\boldsymbol{A}^{j^{*3}}}{3!}h^{j^3} + \ldots + \frac{\boldsymbol{A}^{j^{*N}}}{N!}h^{j^N} \,, \tag{13}$$

where $\boldsymbol{I}$ is the 6×6 identity matrix. This expansion has a fast convergence and it is not time consuming from the computational point of view as demonstrated in the past author's works [41] -[48].

$j = M$ mathematical/fictitious layers must be used to approximate the shell curvature and/or the functionally graded law. $M - 1$ transfer matrices are calculated by means of the interlaminar continuity conditions of displacements and transverse shear/normal stresses. Simply supported plates or shells must be considered to obtain a closed form solution. Such structures are also considered as free stresses at the top and at the bottom surfaces. Using all these conditions, the following final system is obtained:

$$\boldsymbol{E}\,\boldsymbol{U}^1(0) = \boldsymbol{0} \,, \tag{14}$$

matrix $\boldsymbol{E}$ has (6×6) dimension, independently from the number of layers M and from the use of a layer-wise method. $\boldsymbol{U}^1(0)$ is defined as $\boldsymbol{U}$ written at the bottom of the whole multilayered structure (first layer 1 with $z^1 = 0$). All the algebraic steps not written in this work can be found in [41] -[48] where this 3D exact solution has been developed for the first time and applied to different benchmarks. The model has been extensively validated in [41] -[48], and it can be now used with confidence for the frequency comparisons shown in Section 3 about plates, cylinders, cylindrical shells, spherical shells and carbon nanotubes.

The free vibration analysis means the nontrivial solution of $\boldsymbol{U}^1(0)$ in equation (14). Therefore, the determinant of matrix $\boldsymbol{E}$ is imposed equal to zero:

$$det[\boldsymbol{E}] = 0 \,. \tag{15}$$

The above evaluation allows to calculate the roots of an higher order polynomial in $\lambda = \omega^2$. A couple of half-wave numbers (m, n) is imposed to obtain a number of circular frequency values $\omega = 2\pi f$ (theoretically, from I to ∞). This number of frequencies depends on the order N for the exponential matrix $\boldsymbol{A}^{j^{**}}$ and on the number M of fictitious/mathematical layers used for the approximation of shell curvatures and FGM laws. In all the cases proposed in [41] -[48], $N = 3$ for the exponential matrix and $M = 100$ or $M = 102$ for the mathematical layers (the choice depends on the number of physical layers embedded in the structure in order to opportunely divide the thickness and physical layers of the considered structure) have been used for the analysis of one-layered and multi-layered isotropic, orthotropic, composite and functionally graded structures. In the case of carbon nanotubes, $N = 3$ for the exponential matrix and $M = 228$ for mathematical layers are used. The higher M value proposed for carbon nanotubes is due to the higher stiffness and higher thickness of such equivalent continuum structures. The N and M values here discussed are always sufficient to calculate the correct results in Section 3 for each geometry, lamination sequence, number of layers, material, thickness ratio, vibration mode, frequency order and half-wave numbers.

3 Some results

Several benchmarks, which can be solved by means of the 3D exact shell model above presented, are here given. A complete overview will be given at the conference and can also be found in [41] for one-layered plates and shells, in [42]-[44] for multilayered composite and sandwich structures, in [45], [46] for functionally graded plates, cylinders, cylindrical and spherical shells, and in [47] and [48] for single-walled and double-walled (also including van derWaals interaction) carbon nanotubes, respectively.

The first presented benchmark considers the free vibration analysis of a composite plate with thickness ratio $a/h = 10$ ($a = b = 10m$). Each composite layer has Young modulus components $E_1 = 25.1 \times 10^6 psi$, $E_2 = 4.8 \times 10^6 psi$ and $E_3 = 0.75 \times 10^6 psi$, shear modulus components $G_{12} = 1.36 \times 10^6 psi$, $G_{13} = 1.2 \times 10^6 psi$ and $G_{23} = 0.47 \times 10^6 psi$, Poisson ratio components $\nu_{12} = 0.036$, $\nu_{13} = 0.25$ and $\nu_{23} = 0.171$. The mass density value is $\rho = 0.054191\, lb/in^3$. The first three vibration frequencies are proposed in Table 1 in non-dimensional form $\bar{\omega} = \omega h\sqrt{\rho/E_2}$ for half-wave numbers (m,n) equals (1,1), (1,2), (2,1) and (2,2). Two-layered, three-layered and four-layered composite plates are analyzed for lamination sequences (0°/90°), (0°/90°/0°) and (0°/90°/0°/90°) (each layer has the same thickness). The first three modes (indicated as I, II and III) are calculated for each (m,n) couple. The proposed three-dimensional results are very similar to those calculated by Messina [25] for each imposed half-wave number, lamination sequence and considered mode. This validation is important because the proposed three-dimensional solution extended to general orthogonal curvilinear coordinates uses the same methodology applied by Messina [25] to plates with orthogonal rectilinear coordinates.

	Mode I	Mode II	Mode III	(m,n)
		0°/90°		
3D[25]	0.060274	0.52994	0.58275	(1,1)
Present 3D	0.060274	0.52994	0.58275	(1,1)
3D[25]	0.14539	0.62352	0.95652	(1,2)
Present 3D	0.14538	0.62352	0.95652	(1,2)
3D[25]	0.14539	0.62352	0.95652	(2,1)
Present 3D	0.14538	0.62352	0.95652	(2,1)
3D[25]	0.20229	0.95796	1.0300	(2,2)
Present 3D	0.20229	0.95796	1.0300	(2,2)
		0°/90°/0°		
3D[25]	0.067147	0.50349	0.63775	(1,1)
Present 3D	0.067147	0.50349	0.63775	(1,1)
3D[25]	0.12811	0.6888	0.95017	(1,2)
Present 3D	0.12811	0.6888	0.95017	(1,2)
3D[25]	0.17217	0.58366	1.1780	(2,1)
Present 3D	0.17217	0.58366	1.1780	(2,1)
3D[25]	0.20798	0.97517	1.2034	(2,2)
Present 3D	0.20798	0.97517	1.2034	(2,2)
		0°/90°/0°/90°		
3D[25]	0.066210	0.54596	0.59996	(1,1)
Present 3D	0.066210	0.54596	0.59995	(1,1)
3D[25]	0.15194	0.63875	1.0761	(1,2)
Present 3D	0.15194	0.63875	1.0761	(1,2)
3D[25]	0.15194	0.63875	1.0761	(2,1)
Present 3D	0.15194	0.63875	1.0761	(2,1)
3D[25]	0.20841	1.0623	1.1557	(2,2)
Present 3D	0.20841	1.0623	1.1557	(2,2)

Table 1: First case: simply supported multilayered composite square plate.

The second case considers a simply supported cylindrical shell panel. The shell has the two in-plane dimensions $a = b$. The thickness ratio is a/h equals 5. Two different radii of curvature R_α are proposed: a/R_α equals 0.5 or 1. The radius of curvature R_β is infinite. The shell is one-layered and it is made of a functionally graded material (FGM) which is full metallic at the bottom and full ceramic at the top. Young modulus and mass density continuously vary through the thickness direction z in accordance with:

$$E(z) = E_m + (E_c - E_m)V_c \, , \qquad (16)$$

$$\rho(z) = \rho_m + (\rho_c - \rho_m)V_c \, , \qquad (17)$$

where the volume fraction of the ceramic phase V_c is:

$$V_c = (0.5 + z/h)^p \, . \qquad (18)$$

The metallic (m) material has Young modulus $E_m = 70GPa$, mass density $\rho_m = 2702kg/m^3$ and Poisson ratio $\nu_m = 0.3$. The ceramic (c) material has Young modulus $E_c = 380GPa$, mass density $\rho_c = 3800kg/m^3$ and Poisson ratio $\nu_c = 0.3$. These material data can also be found in Zahedinejad et al. [49], where a three-dimensional differential quadrature method was proposed for the free vibration analysis of the FGM cylindrical panel for imposed half-wave numbers $m = n = 1$ and for several exponent values p in the volume fraction. The results are given as non-dimensional circular frequencies $\bar{\omega} = \omega h \sqrt{\frac{\rho_c}{E_c}}$. Table 2 shows that the present three-dimensional exact model gives similar results to those obtained using the method proposed by Zahedinejad et al. [49]. The minor differences are due to the fact that the present 3D model is given in exact form while the 3D model in [49] is proposed using a numerical method such as the differential quadrature method.

p	0.5	1.0	4.0	10
		$a/R_\alpha = 0.5$		
Numerical 3D [49]	0.1814	0.1639	0.1367	0.1271
Present 3D	0.1817	0.1638	0.1374	0.1296
		$a/R_\alpha = 1.0$		
Numerical 3D [49]	0.1852	0.1676	0.1394	0.1286
Present 3D	0.1848	0.1671	0.1392	0.1300

Table 2: Second case: simply supported one-layered FGM cylindrical shell panel.

The third benchmark is proposed in Table 3 where a simply supported spherical shell panel is investigated. The shell has constant radii of curvature $R_\alpha = R_\beta = 10m$, thickness $h = 0.2m$ and dimensions $a = b = 2m$. The multilayered structure includes N_L layers, and each layer has the same thickness. The lamination sequence is (0°/90°/0°/90°/...). The results are calculated and proposed as non-dimensional circular frequency $\bar{\omega} = \omega R_\beta \sqrt{\rho/E_0}$. Each composite layer has Young modulus components $E_1 = 25E_0$ and $E_2 = E_3 = E_0$, shear modulus components $G_{12} = G_{13} = 0.5E_0$ and $G_{23} = 0.2E_0$, and Poisson ratio components $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$. The mass density is $\rho = 1500kg/m^3$. Table 3 shows the first fundamental mode for different imposed half-wave numbers (m,n). The three-dimensional solution proposed by Huang [4] is coincident with the present three-dimensional analysis for each couple of half-wave numbers (m,n) and for each considered number of layers (N_L).

m,n	mode	3D[4]	Present 3D
		$N_L = 2$	
1,1	I	4.6238	4.6240
1,2	I	10.753	10.753
1,3	I	19.130	19.130
2,1	I	10.864	10.864
2,2	I	14.909	14.909
2,3	I	21.961	21.961
3,1	I	19.315	19.315
3,2	I	22.053	22.053
3,3	I	27.483	27.483
		$N_L = 4$	
1,1	I	5.8070	5.8070
1,2	I	12.134	12.134
1,3	I	19.846	19.845
2,1	I	12.188	12.188
2,2	I	16.298	16.298
2,3	I	22.719	22.719
3,1	I	19.932	19.931
3,2	I	22.757	22.757
3,3	I	27.790	27.790
		$N_L = 10$	
1,1	I	6.2293	6.2293
1,2	I	13.050	13.050
1,3	I	21.042	21.042
2,1	I	13.076	13.076
2,2	I	17.432	17.432
2,3	I	24.027	24.027
3,1	I	21.082	21.081
3,2	I	24.045	24.045
3,3	I	29.189	29.189

Table 3: Third case: simply supported multilayered composite spherical shell panel.

The fourth and last case considers a simply supported Single-Walled CarbonNanotube (SWCNT). The equivalent elastic cylinder has properties as calculated in Simsek [50]. The equivalent Young modulus is $E = 1TPa$ with Poisson ratio $\nu = 0.3$. The relative effective thickness considered for this Young modulus value is $h = 0.35nm$. The mass density is $\rho = 2300kg/m^3$. The external diameter of the cylinder is $d_e = 1nm$. This value allows a ratio d_e/h=2.86 which usually requests the use of beam models. In fact, some difficulties may arise when classical 2D shell models are employed for the analysis of these cylinder types. The use of very refined 2D shell models or 3D exact shell models is mandatory for these problems. The radius of curvature in α direction, referred to the mid-surface, is $R_\alpha = d_e/2 - h/2 = 0.325nm$. The dimension in α direction is $a = 2\pi R_\alpha$ and the b dimension can be $L = 5nm$, $10nm$, $20nm$, $50nm$ and $100nm$ for ratios L/d_e = 5, 10, 20, 50 and 100, respectively. Table 4 gives the first three circular frequency values $\bar{\omega} = \omega L^2\sqrt{\frac{\rho A}{EI}}$ (where A is the area of the ring and I is the moment of inertia of the ring) for short and long simply supported cylinders changing L/d_e ratios. The first three non-dimensional circular frequency values are calculated for half-wave number in α direction p=2 and half-wave numbers q in β direction set to 1, 2 and 3. Beam models correctly work for long and moderately long cylinders. On the contrary, shell models calculate correct results for both long and short cylinders. The Euler-Bernoulli Beam Model (EBM) in Table 4 was proposed in Simsek [50] and Aydogdu [51] for the first three frequencies when L/d_e=10, 20 and 50. The same cases were also calculated in [51] using the Timoshenko Beam Model (TBM). TBM gives more accurate results than EBM because it includes the effects of transverse shear deformation and rotary inertia. However, TBM shows some problems for second and third frequency in the case of short SWCNTs (L/d_e=10). The 3D shell model gives satisfactory results for both long and short SWCNTs and it also allows a correct free vibration analysis of cylinders with small diameter/thickness ratios. For these small ratios, classical 2D shell models could exhibit some difficulties. Table 4 shows that the TBM gives similar results to the 3D shell model, while the EBM has larger differences. The TBM has some difficulties for short SWCNTs. Additional results using the 3D shell model are included in Table 4 for very short and very long SWCNTs (these results were not obtained in [50] and [51] via beam models). These additional results give a complete overview of the SWCNT behavior and they can be used as further benchmarks for the validation of future 1D beam and 2D classical and refined shell models. Scientists involved in beam and shell model analyses of SWCNTs can try to complete this table using their models.

mode (p,q)	3D Shell*	EBM[50],[51]	TBM[51]
		$L/d_e = 5$	
I (2,1)	9.3481	...	...
II (2,2)	32.917	...	...
III (2,3)	63.917	...	...
		$L/d_e = 10$	
I (2,1)	9.7295	9.8696	9.7443
II (2,2)	37.392	39.478	36.841
III (2,3)	79.361	88.826	57.450
		$L/d_e = 20$	
I (2,1)	9.8356	9.8696	9.8381
II (2,2)	38.918	39.478	38.964
III (2,3)	86.072	88.826	85.748
		$L/d_e = 50$	
I (2,1)	9.8638	9.8696	9.8645
II (2,2)	39.392	39.478	39.398
III (2,3)	88.375	88.826	88.415
		$L/d_e = 100$	
I (2,1)	9.8487	...	...
II (2,2)	39.488	...	...
III (2,3)	88.752	...	...

Table 4: Fourth case: simply supported single-walled carbon nanotube. * Present 3D.

4 Conclusions

The paper has proposed a general three-dimensional exact shell model based on the differential equations of equilibrium written in general orthogonal curvilinear coordinates. These exact equations are valid for simply supported spherical shells and they automatically degenerate in exact equations for simply supported cylinders, cylindrical panels and plates. A layer wise approach is employed in the case of multilayered structures and the system of partial differential equations is solved using the exponential matrix method. Such equations allow the free vibration analysis of several geometries including different materials such as isotopic, orthotropic, composite and functionally graded ones. Single-walled and double-walled carbon nanotubes can also be analyzed after an opportune identification of the equivalent continuum elastic properties. Future developments will consider the static analysis with the appropriate calculations of the three-dimensional displacement, stress and strain states.

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