Passable motions and stick motions of friction-induced oscillator with 2-DOF on a speed-varying belt

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Abstract: In this paper, a two-degree friction-induced oscillator system is presented for passable motions and stick motions. The system consists of two masses moving on a speed-varying traveling belt, which are connected with three linear springs, three dampers and exerted by two periodic excitations. The oscillator system experiences friction between the masses and the traveling belt, and the friction will cause the stick and nonstick motions between the masses and the belt. The dynamical behaviors of passable motions and stick motions of such oscillator system are investigated by using the flow switchability theory of discontinuous dynamical systems. The onset and vanishing conditions for the stick motions between the oscillator and belt are given, and the analytical conditions for the passable motions will also be obtained, from which it can been seen that such oscillator has more complicated and rich dynamical behaviors. There are more theories about such oscillator to be discussed in future.

Key–Words: friction-induced oscillator; two-degree of freedom; discontinuous dynamical system; stick motion

1 Introduction

In mechanical engineering, the friction contact between two surfaces of two bodies is an important connection and friction phenomenon widely exists. In recent years, much research effort in science and engineering has focussed on nonsmooth dynamical systems[1-12]. This problem can go back to the 30's of last century. In 1930, Den Hartog [1] investigated the non-stick periodic motion of the forced linear oscillator with Coulomb and viscous damping. In 1960, Levitan [2] proved the existence of periodic motions in a friction oscillator with the periodically driven base. In 1964, Filippov [3] investigated the motion in the Coulomb friction oscillator and presented differential equation theory with discontinuous right-hand sides. The investigations of such discontinuous differential equations were summarized in Filippov [4]. However, the Filippov's theory mainly focused on the existence and uniqueness of the solutions for non-smooth dynamical systems. Such a differential equation theory with discontinuity is difficult to apply to practical problems. In 2005-2012, Luo [5-11] developed a general theory to define real, imaginary, sink and source flows and to handle the local singularity and flow swtichability in discontinuous dynamical systems. Luo and Gegg [9] presented the force criteria for the stick and nonstick motions for 1-DOF(Degree of Freedom) oscillator moving on the belt with dry friction. Based on this improved model, which consists of two masses moving on the speed-varying traveling belt and the two masses are connected with three linear springs and three dampers and are exerted by two periodic excitations, nonlinear dynamics mechanism of such a 2-DOF oscillator system will be investigated.

In this paper, the main goal is to study the analytical prediction conditions for motion switching and stick motions on the corresponding boundaries in a friction-induced oscillator with 2-DOF on a speed-varying belt by using the theory of discontinuous dynamical systems. Based on the discontinuity, domain partitions and boundaries will be defined. The analytical conditions for the onset and vanishing of the stick motions will be given, and the analytical conditions for passable motions will also be obtained.

2 Preliminaries

For convenience, we give the following concepts(see [10]-[11]). Assume that Ω is a bounded simply connected domain in R^n and its boundary $\partial\Omega \subset R^{n-1}$ is a smooth surface.

Consider a dynamic system consisting of N sub-dynamic systems in a universal domain $\Omega \subset R^n$. The universal domain is divided into N accessible sub-domains $\Omega_\alpha(\alpha \in I)$ and the inaccessible domain Ω_0. The union of all the accessible sub-domains is $\cup_{\alpha\in I}\Omega_\alpha$ and $\Omega = \cup_{\alpha\in I}\Omega_\alpha \bigcup \Omega_0$ is the universal domain. On the αth open sub-domain Ω_α, there is a C^{r_α}-continuous system ($r_\alpha \geq 1$) in form of

$$\dot{\mathbf{x}}^{(\alpha)} \equiv \mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha) \in R^n, \tag{1}$$

$$\mathbf{x}^{(\alpha)} = (x_1^{(\alpha)}, x_2^{(\alpha)}, ..., x_n^{(\alpha)})^{\mathrm{T}} \in \Omega_\alpha. \tag{2}$$

The time is t and $\dot{\mathbf{x}} = d\mathbf{x}/dt$. In an accessible sub-domain Ω_α, the vector field $\mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha)$ with parameter vector $\mathbf{p}_\alpha = (p_\alpha^{(1)}, p_\alpha^{(2)}, ..., p_\alpha^{(l)})^{\mathrm{T}} \in R^l$ is C^{r_α}-continuous ($r_\alpha \geq 1$) in $\mathbf{x} \in \Omega_\alpha$ and for all time t, and the continuous flow in Eqs. (1) and (2) $\mathbf{x}^{(\alpha)}(t) = \mathbf{\Phi}^{(\alpha)}(\mathbf{x}^{(\alpha)}(t_0), t, \mathbf{p}_\alpha)$ with $\mathbf{x}^{(\alpha)}(t_0) = \mathbf{\Phi}^{(\alpha)}(\mathbf{x}^{(\alpha)}(t_0), t_0, \mathbf{p}_\alpha)$ is $C^{r_\alpha+1}$ continuous for time t.

The flow on the boundary $\partial\Omega_{\alpha\beta} = \Omega_\alpha \bigcap \Omega_\beta$ can be determined by

$$\dot{\mathbf{x}}^{(0)} \equiv \mathbf{F}^{(0)}(\mathbf{x}^{(0)}, t, \lambda) \text{ with } \varphi_{ij}(\mathbf{x}^{(0)}, t, \lambda) = 0, \tag{3}$$

where $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)})^{\mathrm{T}}$. With specific initial conditions, one always obtains different flows on $\varphi_{ij}(\mathbf{x}^{(0)}, t, \lambda) = \varphi_{ij}(\mathbf{x}_0^{(0)}, t_0, \lambda)$.

Consider a dynamic system in Eqs. (1) and (2) in domain $\Omega_\alpha(\alpha \in \{i,j\})$ which has a flow $\mathbf{x}^{(\alpha)} = \mathbf{\Phi}^{(\alpha)}(t_0, \mathbf{x}_0^{(\alpha)}, \mathbf{p}_\alpha, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(\alpha)})$, and on the boundary $\partial\Omega_{ij}$, there is an enough smooth flow $\mathbf{x}^{(0)} = \mathbf{\Phi}(t_0, \mathbf{x}_0^{(0)}, \lambda, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(0)})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t-\varepsilon, t)$ or $(t, t+\varepsilon]$ for flow $\mathbf{x}^{(\alpha)}(\alpha \in \{i,j\})$ and the flow $\mathbf{x}_t^{(\alpha)}$ approaches the separation boundary at time t_m(.i.e., $\mathbf{x}_{t_{m\pm}}^{(\alpha)} = \mathbf{x}_m = \mathbf{x}_{t_m}^{(0)}$), where $\mathbf{x}_{t_{m\pm}}^{(\alpha)} = \mathbf{x}^{(\alpha)}(t_m\pm)$, $\mathbf{x}_{t_m}^{(0)} = \mathbf{x}^{(0)}(t_m)$, and $\mathbf{x}_m \in \partial\Omega_{ij}$).

The G-functions $G_{\partial\Omega_{ij}}^{(\alpha)}$ of the flow $\mathbf{x}_t^{(\alpha)}$ to the flow $\mathbf{x}_t^{(0)}$ on the boundary $\partial\Omega_{ij}$ are defined as

$$\begin{aligned} &G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \lambda) \\ &= \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}^{(0)}, t, \lambda) \cdot [\mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha) \\ &\quad -\mathbf{F}^{(0)}(\mathbf{x}^{(0)}, t, \lambda)]|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_{m\pm})}, \end{aligned} \tag{4}$$

where $\mathbf{x}_m^{(0)} = \mathbf{x}^{(0)}(t_m)$, $\mathbf{x}_{m\pm}^{(\alpha)} = \mathbf{x}^{(\alpha)}(t_{m\pm})$, $t_{m\pm} \equiv t_m \pm 0$ is to represent the quantity in the domain rather than on the boundary and $G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \lambda)$ is a time rate of the inner product of displacement difference and the normal direction $\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}, t_m, \lambda)$.

The kth-order G-functions of the domain flow $\mathbf{x}_t^{(\alpha)}$ to the boundary flow $\mathbf{x}_t^{(0)}$ in the normal direction of $\partial\Omega_{ij}$ are defined as

$$\begin{aligned} &G_{\partial\Omega_{ij}}^{(k,\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \lambda) \\ &= \Sigma_{s=1}^{k+1} C_{k+1}^s D_0^{k+1-s} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \\ &\quad \cdot [D_\alpha^{s-1}\mathbf{F}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha) \\ &\quad -D_0^{s-1}\mathbf{F}^{(0)}(\mathbf{x}^{(0)}, t, \lambda)]|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_{m\pm})}, \end{aligned} \tag{5}$$

where the total derivative operators are defined as

$$D_0(.) \equiv \frac{\partial(.)}{\partial\mathbf{x}^{(0)}}\dot{\mathbf{x}}^{(0)} + \frac{\partial(.)}{t}, \tag{6}$$

$$D_\alpha(.) \equiv \frac{\partial(.)}{\partial\mathbf{x}^{(\alpha)}}\dot{\mathbf{x}}^{(\alpha)} + \frac{\partial(.)}{t}. \tag{7}$$

For $k = 0$, we have

$$\begin{aligned} &G_{\partial\Omega_{ij}}^{(k,\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \lambda) \\ &\quad = G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \lambda). \end{aligned} \tag{8}$$

For a discontinuous dynamical system in Eqs. (1) and (2), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t-\varepsilon, t)$ and $(t, t+\varepsilon]$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$, if

$$\left.\begin{aligned} &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m-\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m-\varepsilon}^{(0)} - \mathbf{x}_{m-\varepsilon}^{(i)}] > 0, \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}(\mathbf{x}_{m+\varepsilon}^{(0)}) \cdot [\mathbf{x}_{m+\varepsilon}^{(j)} - \mathbf{x}_{m+\varepsilon}^{(0)}] > 0 \end{aligned}\right\} \tag{9}$$

for $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$, then a resultant flow of two flows $\mathbf{x}^{(\alpha)}(t)(\alpha \in \{i,j\})$ is a semi-passable flow from domain Ω_i to Ω_j at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$, where $\mathbf{x}_{m\pm\epsilon}^{(0)} = \mathbf{x}^{(0)}(t_m \pm \varepsilon)$, $\mathbf{x}_{m\pm\varepsilon}^{(\alpha)} = \mathbf{x}^{(\alpha)}(t_m \pm \varepsilon)$.

More detailed theory on the flow switchability such as the definitions or theorems about various flow passability in discontinuous dynamical systems can be referred to [10]-[11].

3 Physical Model

Consider a friction-induced oscillator with two-degree of freedom on the speed-varying traveling belt, as shown in Fig.1. The system consists of two masses $m_\alpha(\alpha = 1, 2)$, which are connected with three linear springs of stiffness k_α $(\alpha = 1, 2, 3)$, and three dampers of coefficient r_α $(\alpha = 1, 2, 3)$. Both of masses move on the belt with varying speed $V(t)$. Two periodic excitations $A_\alpha + B_\alpha \cos \Omega t$ $(\alpha = 1, 2)$ with frequency Ω, amplitudes B_α $(\alpha = 1, 2)$ and constant forces A_α $(\alpha = 1, 2)$ are exerted on the two masses, respectively.

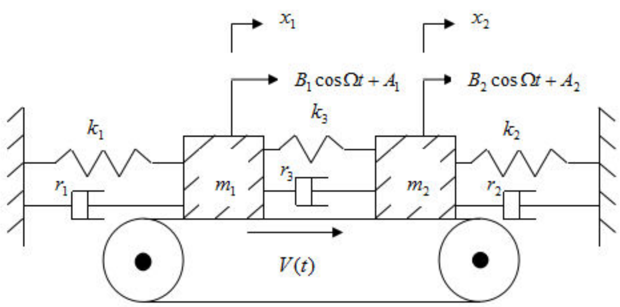

Fig. 1: Physical model

There exist friction forces between the two masses and the belt, so the two masses can move or stay on the surface of the belt. Let $V(t)$ be the speed of the belt and

$$V(t) = V_0 \cos(\Omega t + \beta) + V_1, \tag{10}$$

where Ω is the oscillation frequency of the traveling belt, and V_0 is the oscillation amplitude of the traveling belt, and V_1 is constant.

Further, the friction force shown in Fig. 2 is described by

$$F_f^{(\alpha)}(\dot{x}_\alpha) \begin{cases} = \mu_k F_N^{(\alpha)}, & \dot{x}_\alpha > V(t); \\ \in [-\mu_k F_N^{(\alpha)}, \mu_k F_N^{(\alpha)}], & \dot{x}_\alpha = V(t); \\ = -\mu_k F_N^{(\alpha)}, & \dot{x}_\alpha < V(t), \end{cases} \tag{11}$$

where $\dot{x}_\alpha = dx_\alpha/dt$, μ_k is the coefficient of friction between m_α and the belt, $F_N^{(\alpha)} = m_\alpha g$ $(\alpha = 1, 2)$ and g is the acceleration of gravity. The non-friction force acting on the mass m_α in the x_α-direction is defined

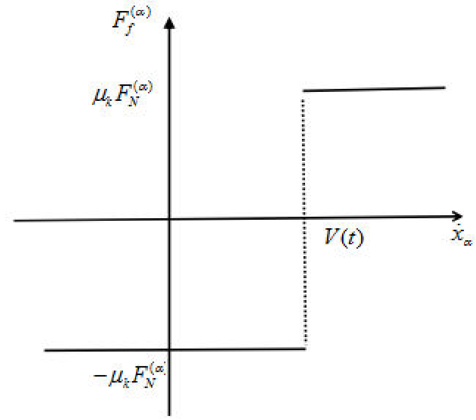

Fig. 2: Force of friction

as

$$F_s^{(\alpha)} = B_\alpha \cos \Omega t + A_\alpha - r_\alpha \dot{x}_\alpha - r_3(\dot{x}_\alpha - \dot{x}_\beta) - k_\alpha x_\alpha - k_3(x_\alpha - x_\beta), \tag{12}$$

where $\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta$. From now on, $F_f^{(\alpha)} = \mu_k \cdot F_N^{(\alpha)}$.

From the previous discussion, there are four cases of motions:

Case I: nonstick motion $(\dot{x}_\alpha \neq V(t))(\alpha = 1, 2)$.

When $F_s^{(\alpha)}$ can overcome the static friction force $F_f^{(\alpha)}$ (i.e. $|F_s^{(\alpha)}| > |F_f^{(\alpha)}|, \alpha = 1, 2)$, the mass m_α has relative motion to the belt, i.e.

$$\dot{x}_\alpha \neq V(t),\ (\alpha = 1, 2). \tag{13}$$

For the nonstick motion of the mass $m_\alpha(\alpha = 1, 2)$, the total force acting on the mass m_α is

$$\begin{aligned} F^{(\alpha)} &= F_s^{(\alpha)} - F_f^{(\alpha)} sgn(\dot{x}_\alpha - V(t)) \\ &= B_\alpha \cos \Omega t + A_\alpha - r_\alpha \dot{x}_\alpha - r_3(\dot{x}_\alpha - \dot{x}_\beta) \\ &\quad -k_\alpha x_\alpha - k_3(x_\alpha - x_\beta) - F_f^{(\alpha)} sgn(\dot{x}_\alpha - V(t)), \end{aligned} \tag{14}$$

and the equations of non-stick motion for the 2-DOF dry friction induced oscillator are

$$m_\alpha \ddot{x}_\alpha + r_\alpha \dot{x}_\alpha + r_3(\dot{x}_\alpha - \dot{x}_\beta) + k_\alpha x_\alpha + k_3(x_\alpha - x_\beta) = B_\alpha \cos \Omega t + A_\alpha - F_f^{(\alpha)} sgn(\dot{x}_\alpha - V(t)), \tag{15}$$

where $\alpha, \beta \in \{1, 2\}, \alpha \neq \beta$.

Case II: single stick motion($\dot{x}_1 = V(t), \dot{x}_2 \neq V(t)$).

When $F_s^{(1)}$ can't overcome the static friction force $F_f^{(1)}$ (i.e. $|F_s^{(1)}| \leq |F_f^{(1)}|$), mass m_1 don't have any relative motion to the belt, i.e.

$$\dot{x}_1 = V(t), \ddot{x}_1 = \dot{V}(t) = -V_0\Omega\sin(\Omega t + \beta), \quad (16)$$

meanwhile $F_s^{(2)}$ can overcome the static friction force $F_f^{(2)}$(i.e. $|F_s^{(2)}| > |F_f^{(2)}|$), the mass m_2 has relative motion to the belt, i.e.

$$\dot{x}_2 \neq V(t), \quad (17)$$

$$m_2\ddot{x}_2 + r_2\dot{x}_2 + r_3(\dot{x}_2 - \dot{x}_1) + k_2x_2 + k_3(x_2 - x_1) = B_2\cos\Omega t + A_2 - F_f^{(2)} sgn(\dot{x}_2 - V(t)). \quad (18)$$

Case III: single stick motion($\dot{x}_2 = V(t), \dot{x}_1 \neq V(t)$).

When $F_s^{(2)}$ can't overcome the static friction force $F_f^{(2)}$ (i.e.$|F_s^{(2)}| \leq |F_f^{(2)}|$), mass m_2 don't have any relative motion to the belt, i.e.

$$\dot{x}_2 = V(t), \ddot{x}_2 = \dot{V}(t) = -V_0\Omega\sin(\Omega t + \beta), \quad (19)$$

meanwhile $F_s^{(1)}$ can overcome the static friction force $F_f^{(1)}(i.e.|F_s^{(1)}| > |F_f^{(1)}|)$, mass m_1 has relative motion to the belt, i.e.

$$\dot{x}_1 \neq V(t), \quad (20)$$

$$m_1\ddot{x}_1 + r_1\dot{x}_1 + r_3(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_3(x_1 - x_2) = B_1\cos\Omega t + A_1 - F_f^{(1)} sgn(\dot{x}_1 - V(t)). \quad (21)$$

Case IV: double stick motions $(\dot{x}_\alpha = V(t))(\alpha = 1, 2)$.

When $F_s^{(\alpha)}$ can't overcome the static friction force $F_f^{(\alpha)}$ (i.e.$|F_s^{(\alpha)}| \leq |F_f^{(\alpha)}|$), mass m_α don't have any relative motion to the belt, i.e.

$$\dot{x}_\alpha = V(t), \ddot{x}_\alpha = \dot{V}(t) = -V_0\Omega\sin(\Omega t + \beta). \quad (22)$$

Integrating Eq. (10) leads to the displacement of the belt:

$$X(t) = \frac{V_0}{\Omega}[\sin(\Omega t + \beta) - \sin(\Omega t_i + \beta)] + V_1(t - t_i) + X_{t_i} \quad (23)$$

where $t > t_i$ and $X_{t_i} = X(t_i)$.

4 Domains and boundaries

Due to frictions between the mass m_α $(\alpha = 1, 2)$ and the traveling belt, the motions become discontinuous and more complicated. The phase space of the discontinuous dynamical system is divided into four 4-dimensional domains.

The state variables and vector fields are introduced by

$$\mathbf{x} = (x_1, \dot{x}_1, x_2, \dot{x}_2)^{\mathrm{T}} = (x_1, y_1, x_2, y_2)^{\mathrm{T}}, \quad (24)$$

$$\mathbf{F} = (y_1, F_1, y_2, F_2)^{\mathrm{T}}. \quad (25)$$

By the state variables, the domains are defined as

$$\left.\begin{aligned} \Omega_1 &= \{(x_1, y_1, x_2, y_2) \mid y_1 > V(t), y_2 > V(t)\}, \\ \Omega_2 &= \{(x_1, y_1, x_2, y_2) \mid y_1 > V(t), y_2 < V(t)\}, \\ \Omega_3 &= \{(x_1, y_1, x_2, y_2) \mid y_1 < V(t), y_2 < V(t)\}, \\ \Omega_4 &= \{(x_1, y_1, x_2, y_2) \mid y_1 < V(t), y_2 > V(t)\} \end{aligned}\right\} \quad (26)$$

and the corresponding boundaries are defined as

$$\left.\begin{aligned} \partial\Omega_{12} &= \partial\Omega_{21} \\ &= \{(x_1, y_1, x_2, y_2) \mid \varphi_{12} = \varphi_{21} = y_2 - V(t) = 0, y_1 \geq V(t)\}, \\ \partial\Omega_{23} &= \partial\Omega_{32} \\ &= \{(x_1, y_1, x_2, y_2) \mid \varphi_{23} = \varphi_{32} = y_1 - V(t) = 0, y_2 \leq V(t)\}, \\ \partial\Omega_{34} &= \partial\Omega_{43} \\ &= \{(x_1, y_1, x_2, y_2) \mid \varphi_{34} = \varphi_{43} = y_2 - V(t) = 0, y_1 \leq V(t)\}, \\ \partial\Omega_{14} &= \partial\Omega_{41} \\ &= \{(x_1, y_1, x_2, y_2) \mid \varphi_{12} = \varphi_{21} = y_1 - V(t) = 0, y_2 \geq V(t)\}. \end{aligned}\right\} \quad (27)$$

The phase plane of m_α is shown in Fig. 3.

The 2-dimensional edges of the 3-dimensional boundaries are defined by

$$\angle\Omega_{\alpha_1\alpha_2\alpha_3} = \partial\Omega_{\alpha_1\alpha_2} \bigcap \partial\Omega_{\alpha_2\alpha_3} = \bigcap_{i=1}^{3} \Omega_{\alpha_i} \quad (28)$$

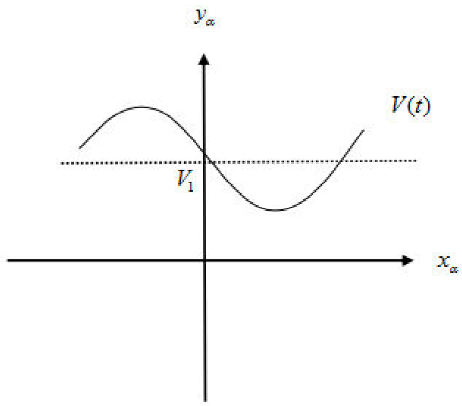

Fig. 3: Phase plane of m_α

for ($\alpha_i \in \{1,2,3,4\}, i = 1,2,3; \alpha_1,\alpha_2,\alpha_3$ is not equal to each other without repeating) and the intersection of four 2-dimensional edges is

$$\begin{aligned}\angle\Omega_{1234} &= \cap\angle\Omega_{\alpha_1\alpha_2\alpha_3} \\ &= \{(x_1,y_1,x_2,y_2) \mid \\ &\quad \varphi_{12} = \varphi_{34} = y_2 - V(t) = 0, \\ &\quad \varphi_{23} = \varphi_{14} = y_1 - V(t) = 0\}. \end{aligned} \tag{29}$$

From the above discussion, the motion equations of the oscillator described in Section 3 in absolute coordinates are

$$\left.\begin{aligned} &\dot{\mathbf{x}}^{(\alpha)} = \mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)},t) \quad \text{in } \Omega_\alpha, \\ &\dot{\mathbf{x}}^{(\alpha_1\alpha_2)} = \mathbf{F}^{(\alpha_1\alpha_2)}(\mathbf{x}^{(\alpha_1\alpha_2)},t) \text{ on } \partial\Omega_{\alpha_1\alpha_2}, \\ &\dot{\mathbf{x}}^{(\alpha_1\alpha_2\alpha_3)} \\ &\quad = \mathbf{F}^{(\alpha_1\alpha_2\alpha_3)}(\mathbf{x}^{(\alpha_1\alpha_2\alpha_3)},t) \text{ on } \partial\Omega_{\alpha_1\alpha_2\alpha_3} \end{aligned}\right\} \tag{30}$$

and

$$\left.\begin{aligned} &\mathbf{x}^{(\alpha)} = \mathbf{x}^{(\alpha_1\alpha_2)} = \mathbf{x}^{(\alpha_1\alpha_2\alpha_3)} \\ &\quad = (x_1,y_1,x_2,y_2)^{\mathrm{T}}, \\ &\mathbf{F}^{(\alpha)} = (y_1, F_1^{(\alpha)}, y_2, F_2^{(\alpha)})^{\mathrm{T}}, \\ &\mathbf{F}^{(\alpha_1\alpha_2)} = (y_1, F_1^{(\alpha_1\alpha_2)}, y_2, F_2^{(\alpha_1\alpha_2)})^{\mathrm{T}}, \\ &\mathbf{F}^{(\alpha_1\alpha_2\alpha_3)} \\ &\quad = (y_1, F_1^{(\alpha_1\alpha_2\alpha_3)}, y_2, F_2^{(\alpha_1\alpha_2\alpha_3)})^{\mathrm{T}}, \end{aligned}\right\} \tag{31}$$

where the forces of unit mass for the 2-DOF friction induced oscillator in the domain Ω_α ($\alpha \in \{1,2,3,4\}$) are

$$\left.\begin{aligned} F_1^{(1)} &= F_1^{(2)} \\ &= b_1\cos\Omega t + a_1 - c_1y_1 - p_1(y_1-y_2) \\ &\quad -d_1x_1 - q_1(x_1-x_2) - f_1, \\ F_1^{(3)} &= F_1^{(4)} \\ &= b_1\cos\Omega t + a_1 - c_1y_1 - p_1(y_1-y_2) \\ &\quad -d_1x_1 - q_1(x_1-x_2) + f_1, \\ F_2^{(1)} &= F_2^{(4)} \\ &= b_2\cos\Omega t + a_2 - c_2y_2 - p_2(y_2-y_1) \\ &\quad -d_2x_2 - q_2(x_2-x_1) - f_2, \\ F_2^{(2)} &= F_2^{(3)} \\ &= b_2\cos\Omega t + a_2 - c_2y_2 - p_2(y_2-y_1) \\ &\quad -d_2x_2 - q_2(x_2-x_1) + f_2, \end{aligned}\right\} \tag{32}$$

here

$$a_\alpha = \frac{A_\alpha}{m_\alpha},\ b_\alpha = \frac{B_\alpha}{m_\alpha},\ c_\alpha = \frac{r_\alpha}{m_\alpha},\ d_\alpha = \frac{k_\alpha}{m_\alpha},$$

$$p_\alpha = \frac{r_3}{m_\alpha},\ q_\alpha = \frac{k_3}{m_\alpha},\ f_\alpha = \frac{F_f^{(\alpha)}}{m_\alpha}, \alpha \in \{1,2\},$$

and the forces of unit mass of the oscillator on the boundary $\partial\Omega_{\alpha_1\alpha_2}$ are

$$\left.\begin{aligned} F_1^{(12)} &\equiv b_1\cos\Omega t + a_1 - c_1y_1 - p_1(y_1-y_2) \\ &\quad -d_1x_1 - q_1(x_1-x_2) - f_1, \\ F_2^{(12)} &= 0 \quad \text{for stick on } \partial\Omega_{12}, \\ F_2^{(12)} &\in [F_2^{(1)}, F_2^{(2)}] \text{ for nonstick on } \partial\Omega_{12}; \end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} F_2^{(23)} &\equiv b_2\cos\Omega t + a_2 - c_2y_2 - p_2(y_2-y_1) \\ &\quad -d_2x_2 - q_2(x_2-x_1) + f_2, \\ F_1^{(23)} &= 0 \quad \text{for stick on } \partial\Omega_{23}, \\ F_1^{(23)} &\in [F_1^{(2)}, F_1^{(3)}] \text{ for nonstick on } \partial\Omega_{23}; \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} F_1^{(34)} &\equiv b_1\cos\Omega t + a_1 - c_1y_1 - p_1(y_1-y_2) \\ &\quad -d_1x_1 - q_1(x_1-x_2) + f_1, \\ F_2^{(34)} &= 0 \quad \text{for stick on } \partial\Omega_{34}, \\ F_2^{(34)} &\in [F_2^{(4)}, F_2^{(3)}] \text{ for nonstick on } \partial\Omega_{34}; \end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned} F_2^{(14)} &\equiv b_2\cos\Omega t + a_2 - c_2y_2 - p_2(y_2-y_1) \\ &\quad -d_2x_2 - q_2(x_2-x_1) - f_2, \\ F_1^{(14)} &= 0 \quad \text{for stick on } \partial\Omega_{14}, \\ F_1^{(14)} &\in [F_1^{(1)}, F_1^{(4)}] \text{ for nonstick on } \partial\Omega_{14}. \end{aligned}\right\} \tag{36}$$

The forces of unit mass of the oscillator on the boundary $\partial\Omega_{\alpha_1\alpha_2\alpha_3}$ for ($\alpha_i \in \{1,2,3,4\}, i = 1,2,3; \alpha_1,\alpha_2,\alpha_3$ is not equal to each other without repeating) are

$$\left.\begin{aligned} &F_\alpha^{(\alpha_1\alpha_2\alpha_3)} \in (F_\alpha^{(\alpha_1\alpha_2)}, F_\alpha^{(\alpha_2\alpha_3)}), \alpha \in \{1,2\} \\ &\qquad \text{for nonstick on } \partial\Omega_{\alpha_1\alpha_2\alpha_3}; \\ &F_\alpha^{(\alpha_1\alpha_2\alpha_3)} = 0, \alpha \in \{1,2\} \\ &\qquad \text{for full stick on } \partial\Omega_{\alpha_1\alpha_2\alpha_3}. \end{aligned}\right\}$$

For simplicity, the relative displacement, velocity and acceleration between the mass m_α ($\alpha = 1,2$) and the traveling belt are defined as

$$\left.\begin{aligned} z_\alpha &= x_\alpha - X(t), \\ v_\alpha &= \dot{x}_\alpha - V(t), \\ \ddot{z}_\alpha &= \ddot{x}_\alpha - \dot{V}(t). \end{aligned}\right\} \tag{37}$$

The domains and boundaries in relative coordinates are defined as

$$\left.\begin{aligned} \Omega_1 &= \{(z_1,v_1,z_2,v_2) \mid v_1 > 0, v_2 > 0\}, \\ \Omega_2 &= \{(z_1,v_1,z_2,v_2) \mid v_1 > 0, v_2 < 0\}, \\ \Omega_3 &= \{(z_1,v_1,z_2,v_2) \mid v_1 < 0, v_2 < 0\}, \\ \Omega_4 &= \{(z_1,v_1,z_2,v_2) \mid v_1 < 0, v_2 > 0\}, \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} &\partial\Omega_{12} = \partial\Omega_{21} \\ &\quad = \{(z_1,v_1,z_2,v_2) \mid \varphi_{12} = \varphi_{21} \\ &\qquad\qquad = v_2 = 0, v_1 \geq 0\}, \\ &\partial\Omega_{23} = \partial\Omega_{32} \\ &\quad = \{(z_1,v_1,z_2,v_2) \mid \varphi_{23} = \varphi_{32} \\ &\qquad\qquad = v_1 = 0, v_2 \leq 0\}, \\ &\partial\Omega_{34} = \partial\Omega_{43} \\ &\quad = \{(z_1,v_1,z_2,v_2) \mid \varphi_{34} = \varphi_{43} \\ &\qquad\qquad = v_2 = 0, v_1 \leq 0\}, \\ &\partial\Omega_{14} = \partial\Omega_{41} \\ &\quad = \{(z_1,v_1,z_2,v_2) \mid \varphi_{14} = \varphi_{41} \\ &\qquad\qquad = v_1 = 0, v_2 \geq 0\}, \end{aligned}\right\} \tag{39}$$

$$\angle\Omega_{\alpha_1\alpha_2\alpha_3} = \partial\Omega_{\alpha_1\alpha_2} \cap \partial\Omega_{\alpha_2\alpha_3} = \cap_{i=1}^{3}\Omega_{\alpha_i} \tag{40}$$

for ($\alpha_i \in \{1,2,3,4\}, i = 1,2,3; \alpha_1,\alpha_2,\alpha_3$ is not equal to each other without repeating) and the intersection of four 2-dimensional edges is

$$\begin{aligned} \angle\Omega_{1234} = \cap\angle\Omega_{\alpha_1\alpha_2\alpha_3} &= \{(z_1,v_1,z_2,v_2) \mid \varphi_{12} \\ &= \varphi_{34} = v_2 = 0, \varphi_{23} = \varphi_{14} = v_1 = 0\}. \end{aligned} \tag{41}$$

The domain partitions and boundaries in relative coordinates are shown in Fig. 4.

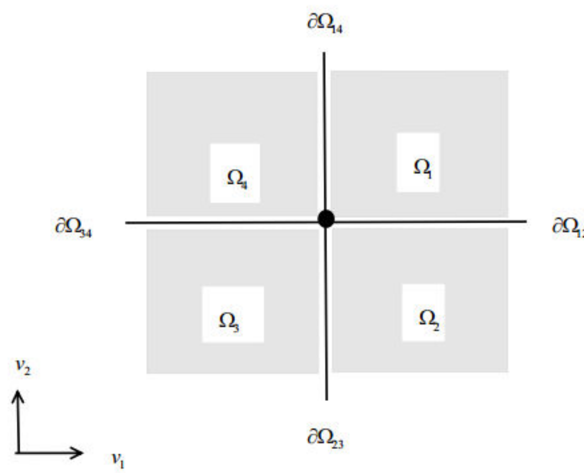

Fig. 4: Relative domains and boundaries

From the foregoing equations, the motion equations in relative coordinates are as follows

$$\left.\begin{aligned} &\dot{\mathbf{z}}^{(\alpha)} = \mathbf{g}^{(\alpha)}(\mathbf{z}^{(\alpha)}, \mathbf{x}^{(\alpha)}, t) \qquad \text{in } \Omega_\alpha, \\ &\dot{\mathbf{z}}^{(\alpha_1\alpha_2)} \\ &\quad = \mathbf{g}^{(\alpha_1\alpha_2)}(\mathbf{z}^{(\alpha_1\alpha_2)}, \mathbf{x}^{(\alpha_1\alpha_2)}, t) \ \text{ on } \partial\Omega_{\alpha_1\alpha_2}, \\ &\dot{\mathbf{z}}^{(\alpha_1\alpha_2\alpha_3)} \\ &\quad = \mathbf{g}^{(\alpha_1\alpha_2\alpha_3)}(\mathbf{z}^{(\alpha_1\alpha_2\alpha_3)}, \mathbf{x}^{(\alpha_1\alpha_2\alpha_3)}, t) \\ &\qquad\qquad\qquad \text{on } \partial\Omega_{\alpha_1\alpha_2\alpha_3}, \end{aligned}\right\} \tag{42}$$

where

$$\left.\begin{aligned} \mathbf{z}^{(\alpha)} &= \mathbf{z}^{(\alpha_1\alpha_2)} = \mathbf{z}^{(\alpha_1\alpha_2\alpha_3)} \\ &= (z_1, \dot{z}_1, z_2, \dot{z}_2)^{\mathrm{T}} \\ &= (z_1, v_1, z_2, v_2)^{\mathrm{T}}, \\ \mathbf{g}^{(\alpha)} &= (\dot{z}_1, g_1^{(\alpha)}, \dot{z}_2, g_2^{(\alpha)})^{\mathrm{T}} \\ &= (v_1, g_1^{(\alpha)}, v_2, g_2^{(\alpha)})^{\mathrm{T}}, \\ \mathbf{g}^{(\alpha_1\alpha_2)} &= (\dot{z}_1, g_1^{(\alpha_1\alpha_2)}, \dot{z}_2, g_2^{(\alpha_1\alpha_2)})^{\mathrm{T}} \\ &= (v_1, g_1^{(\alpha_1\alpha_2)}, v_2, g_2^{(\alpha_1\alpha_2)})^{\mathrm{T}}, \\ \mathbf{g}&^{(\alpha_1\alpha_2\alpha_3)} \\ &= (\dot{z}_1, g_1^{(\alpha_1\alpha_2\alpha_3)}, \dot{z}_2, g_2^{(\alpha_1\alpha_2\alpha_3)})^{\mathrm{T}} \\ &= (v_1, g_1^{(\alpha_1\alpha_2\alpha_3)}, v_2, g_2^{(\alpha_1\alpha_2\alpha_3)})^{\mathrm{T}}. \end{aligned}\right\} \tag{43}$$

The forces of unit mass for the 2-DOF friction induced oscillator in the domain $\Omega_\alpha (\alpha \in \{1,2,3,4\})$

in relative coordinates are

$$\left.\begin{aligned}
g_1^{(1)} &= g_1^{(2)} \\
&= b_1\cos\Omega t + a_1 - c_1 v_1 - p_1(v_1 - v_2) \\
&\quad -d_1 z_1 - q_1(z_1 - z_2) - c_1 V(t) \\
&\quad -d_1 X(t) - f_1 - \dot{V}(t), \\
g_1^{(3)} &= g_1^{(4)} \\
&= b_1\cos\Omega t + a_1 - c_1 v_1 - p_1(v_1 - v_2) \\
&\quad -d_1 z_1 - q_1(z_1 - z_2) - c_1 V(t) \\
&\quad -d_1 X(t) + f_1 - \dot{V}(t); \\
g_2^{(1)} &= g_2^{(4)} \\
&= b_2\cos\Omega t + a_2 - c_2 v_2 - p_2(v_2 - v_1) \\
&\quad -d_2 z_2 - q_2(z_2 - z_1) - c_2 V(t) \\
&\quad -d_2 X(t) - f_2 - \dot{V}(t), \\
g_2^{(2)} &= g_2^{(3)} \\
&= b_2\cos\Omega t + a_2 - c_2 v_2 - p_2(v_2 - v_1) \\
&\quad -d_2 z_2 - q_2(z_2 - z_1) - c_2 V(t) \\
&\quad -d_2 X(t) + f_2 - \dot{V}(t).
\end{aligned}\right\} \quad (44)$$

The forces of unit mass of the friction induced oscillator on the boundary $\partial\Omega_{\alpha_1\alpha_2}$ in relative coordinates are

$$\left.\begin{aligned}
g_1^{(12)} &\equiv b_1\cos\Omega t + a_1 - c_1 v_1 - p_1(v_1 - v_2) \\
&\quad -d_1 z_1 - q_1(z_1 - z_2) - c_1 V(t) \\
&\quad -d_1 X(t) - f_1 - \dot{V}(t), \\
g_2^{(12)} &= 0 \qquad \text{for stick on } \partial\Omega_{12}, \\
g_2^{(12)} &\in [g_2^{(1)}, g_2^{(2)}] \quad \text{for nonstick on } \partial\Omega_{12};
\end{aligned}\right\} \quad (45)$$

$$\left.\begin{aligned}
g_2^{(23)} &\equiv b_2\cos\Omega t + a_2 - c_2 v_2 - p_2(v_2 - v_1) \\
&\quad -d_2 z_2 - q_2(z_2 - z_1) - c_2 V(t) \\
&\quad -d_2 X(t) + f_2 - \dot{V}(t), \\
g_1^{(23)} &= 0 \qquad \text{for stick on } \partial\Omega_{23}, \\
g_1^{(23)} &\in [g_1^{(2)}, g_1^{(3)}] \quad \text{for nonstick on } \partial\Omega_{23};
\end{aligned}\right\} \quad (46)$$

$$\left.\begin{aligned}
g_1^{(34)} &\equiv b_1\cos\Omega t + a_1 - c_1 v_1 - p_1(v_1 - v_2) \\
&\quad -d_1 z_1 - q_1(z_1 - z_2) - c_1 V(t) \\
&\quad -d_1 X(t) + f_1 - \dot{V}(t), \\
g_2^{(34)} &= 0 \qquad \text{for stick on } \partial\Omega_{34}, \\
g_2^{(34)} &\in [g_2^{(4)}, g_2^{(3)}] \quad \text{for nonstick on } \partial\Omega_{34};
\end{aligned}\right\} \quad (47)$$

$$\left.\begin{aligned}
g_2^{(14)} &\equiv b_2\cos\Omega t + a_2 - c_2 v_2 - p_2(v_2 - v_1) \\
&\quad -d_2 z_2 - q_2(z_2 - z_1) - c_2 V(t) \\
&\quad -d_2 X(t) - f_2 - \dot{V}(t), \\
g_1^{(14)} &= 0 \qquad \text{for stick on } \partial\Omega_{14}, \\
g_1^{(14)} &\in [g_1^{(1)}, g_1^{(4)}] \quad \text{for nonstick on } \partial\Omega_{14}.
\end{aligned}\right\} \quad (48)$$

The forces of unit mass of the oscillator on the boundary $\partial\Omega_{\alpha_1\alpha_2\alpha_3}$ for ($\alpha_i \in \{1,2,3,4\}, i = 1,2,3; \alpha_1, \alpha_2, \alpha_3$ is not equal to each other without repeating) are

$$\left.\begin{aligned}
&g_\alpha^{(\alpha_1\alpha_2\alpha_3)} \in (g_\alpha^{(\alpha_1\alpha_2)}, g_\alpha^{(\alpha_2\alpha_3)}), \alpha \in \{1,2\} \\
&\qquad\qquad \text{for nonstick on } \partial\Omega_{\alpha_1\alpha_2\alpha_3}; \\
&g_\alpha^{(\alpha_1\alpha_2\alpha_3)} = 0, \alpha \in \{1,2\} \\
&\qquad\qquad \text{for full stick on } \partial\Omega_{\alpha_1\alpha_2\alpha_3}.
\end{aligned}\right\}$$

5 Analytical conditions

Using the absolute coordinates, it is very difficult to develop the analytical conditions for the complex motions of the oscillator described in Section 3 because the boundaries are dependent on time, thus the relative coordinates are needed herein for simplicity.

From Eqs. (4) and (5) in Section 2, we have

$$\begin{aligned}
&G^{(0,\alpha_1)}(\mathbf{z}^{(\alpha)}, \mathbf{x}^{(\alpha)}, t_{m\pm}) \\
&= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1)}(\mathbf{z}^{(\alpha)}, \mathbf{x}^{(\alpha)}, t_{m\pm}),
\end{aligned} \quad (49)$$

$$\begin{aligned}
&G^{(1,\alpha_1)}(\mathbf{z}^{(\alpha)}, \mathbf{x}^{(\alpha)}, t_{m\pm}) \\
&= 2D\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot [\mathbf{g}^{(\alpha_1)}(t_{m\pm}) - \mathbf{g}^{(\alpha_1\alpha_2)}(t_m)] \\
&+\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot [D\mathbf{g}^{(\alpha_1)}(t_{m\pm}) - D\mathbf{g}^{(\alpha_1\alpha_2)}(t_m)].
\end{aligned} \quad (50)$$

In relative coordinates, the boundary $\partial\Omega_{\alpha_1\alpha_2}$ is independent on t, so $D\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} = 0$. Because of

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1\alpha_2)} = 0,$$

therefore

$$D\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1\alpha_2)} + \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha_1\alpha_2)} = 0,$$

thus

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha_1\alpha_2)} = 0.$$

Eq. (50) is simplified as

$$G^{(1,\alpha_1)}(\mathbf{z}_\alpha, \mathbf{x}_\alpha, t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha_1)}(t_{m\pm}). \tag{51}$$

The t_m represents the time for the motion on the velocity boundary and $t_{m\pm} = t_m \pm 0$ reflects the responses in the domain rather than on the boundary.

From the previous descriptions for the system, the normal vector of the boundary $\partial\Omega_{\alpha_1\alpha_2}$ in the relative coordinates is

$$\mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}} = (\frac{\partial\varphi_{\alpha_1\alpha_2}}{\partial z_1}, \frac{\partial\varphi_{\alpha_1\alpha_2}}{\partial v_1}, \frac{\partial\varphi_{\alpha_1\alpha_2}}{\partial z_2}, \frac{\partial\varphi_{\alpha_1\alpha_2}}{\partial v_2})^{\mathrm{T}}. \tag{52}$$

With Eqs. (39) and (52), we have

$$\left.\begin{array}{l} \mathbf{n}_{\partial\Omega_{23}} = \mathbf{n}_{\partial\Omega_{14}} = (0,1,0,0)^{\mathrm{T}}, \\ \mathbf{n}_{\partial\Omega_{12}} = \mathbf{n}_{\partial\Omega_{34}} = (0,0,0,1)^{\mathrm{T}}. \end{array}\right\} \tag{53}$$

Theorem 1 *For the 2-DOF friction induced oscillator described in Section 3, the non-stick motion (or called passable motion to boundary) on $\mathbf{x}_m \in \partial\Omega_{\alpha_1\alpha_2}$ at time t_m appears iff*

$$(a)\ \ \alpha_1 = 2, \alpha_2 = 1: \quad \left.\begin{array}{l} g_2^{(2)}(t_{m-}) > 0, \\ g_2^{(1)}(t_{m+}) > 0 \end{array}\right\} \text{ from } \Omega_2 \to \Omega_1; \tag{54}$$

$$(b)\ \ \alpha_1 = 1, \alpha_2 = 2: \quad \left.\begin{array}{l} g_2^{(1)}(t_{m-}) < 0, \\ g_2^{(2)}(t_{m+}) < 0 \end{array}\right\} \text{ from } \Omega_1 \to \Omega_2; \tag{55}$$

$$(c)\ \ \alpha_1 = 3, \alpha_2 = 4: \quad \left.\begin{array}{l} g_2^{(3)}(t_{m-}) > 0, \\ g_2^{(4)}(t_{m+}) > 0 \end{array}\right\} \text{ from } \Omega_3 \to \Omega_4; \tag{56}$$

$$(d)\ \ \alpha_1 = 4, \alpha_2 = 3: \quad \left.\begin{array}{l} g_2^{(4)}(t_{m-}) < 0, \\ g_2^{(3)}(t_{m+}) < 0 \end{array}\right\} \text{ from } \Omega_4 \to \Omega_3; \tag{57}$$

$$(e)\ \ \alpha_1 = 2, \alpha_2 = 3: \quad \left.\begin{array}{l} g_1^{(2)}(t_{m-}) < 0, \\ g_1^{(3)}(t_{m+}) < 0 \end{array}\right\} \text{ from } \Omega_2 \to \Omega_3; \tag{58}$$

$$(f)\ \ \alpha_1 = 3, \alpha_2 = 2: \quad \left.\begin{array}{l} g_1^{(3)}(t_{m-}) > 0, \\ g_1^{(2)}(t_{m+}) > 0 \end{array}\right\} \text{ from } \Omega_3 \to \Omega_2; \tag{59}$$

$$(g)\ \ \alpha_1 = 4, \alpha_2 = 1: \quad \left.\begin{array}{l} g_1^{(4)}(t_{m-}) > 0, \\ g_1^{(1)}(t_{m+}) > 0 \end{array}\right\} \text{ from } \Omega_4 \to \Omega_1; \tag{60}$$

$$(h)\ \ \alpha_1 = 1, \alpha_2 = 4: \quad \left.\begin{array}{l} g_1^{(1)}(t_{m-}) < 0, \\ g_1^{(4)}(t_{m+}) < 0 \end{array}\right\} \text{ from } \Omega_1 \to \Omega_4. \tag{61}$$

Proof: By Theorem 2.1 in [10], the passable motion for a flow from domain Ω_{α_1} to Ω_{α_2} on the boundary $\partial\Omega_{\alpha_1\alpha_2}$ at time t_m appears iff for $\mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}} \to \Omega_{\alpha_1}$

$$\left.\begin{array}{l} G^{(0,\alpha_1)}(t_{m-}) \\ \quad = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1)}(t_{m-}) \\ \quad < 0, \\ G^{(0,\alpha_2)}(t_{m+}) \\ \quad = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_2)}(t_{m+}) \\ \quad < 0 \end{array}\right\} \tag{62}$$

From (53) and $\mathbf{g}^{(\alpha)} = (v_1, g_1^{(\alpha)}, v_2, g_2^{(\alpha)})$, we have

$$\begin{array}{ll} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(\alpha)}(t_{m\pm}) = g_2^{(\alpha)}(t_{m\pm}) & (\alpha = 1,2), \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{34}} \cdot \mathbf{g}^{(\alpha)}(t_{m\pm}) = g_2^{(\alpha)}(t_{m\pm}) & (\alpha = 3,4), \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{23}} \cdot \mathbf{g}^{(\alpha)}(t_{m\pm}) = g_1^{(\alpha)}(t_{m\pm}) & (\alpha = 2,3), \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{14}} \cdot \mathbf{g}^{(\alpha)}(t_{m\pm}) = g_1^{(\alpha)}(t_{m\pm}) & (\alpha = 1,4). \end{array} \tag{63}$$

Substitute the first formula of (63) into (62), we have

$$\left.\begin{array}{l} G^{(0,1)}(t_{m-}) \\ = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(1)}(t_{m-}) \\ = g_2^{(1)}(t_{m-}) < 0, \\ G^{(0,2)}(t_{m+}) \\ = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(2)}(t_{m+}) \\ = g_2^{(2)}(t_{m+}) < 0 \end{array}\right\} \text{ from } \Omega_1 \to \Omega_2, \tag{64}$$

$$\left.\begin{aligned}&G^{(0,2)}(t_{m-})\\&=\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}}\cdot\mathbf{g}^{(2)}(t_{m-})\\&=g_2^{(2)}(t_{m-})>0,\\&G^{(0,1)}(t_{m+})\\&=\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}}\cdot\mathbf{g}^{(1)}(t_{m+})\\&=g_2^{(1)}(t_{m+})>0\end{aligned}\right\}\text{ from }\Omega_2\to\Omega_1. \quad (65)$$

So, (a) and (b) hold. Similarly, $(c)-(h)$ can be proved. □

By Theorem 2.4, Theorem 3.15 in [10] and Theorem 2.15 in [11], we can easily obtain the following theorems.

Theorem 2 *For the 2-DOF friction induced oscillator described in Section 3, the stick motion in physics (or called the sliding motion in mathematics) to the boundary* $\partial\Omega_{\alpha_1\alpha_2}$ *is guaranteed iff*

$$\begin{aligned}&g_2^{(2)}(t_{m-})>0, g_2^{(1)}(t_{m-})<0 \quad \text{on } \partial\Omega_{12};\\&g_2^{(3)}(t_{m-})>0, g_2^{(4)}(t_{m-})<0 \quad \text{on } \partial\Omega_{34};\\&g_1^{(4)}(t_{m-})>0, g_1^{(1)}(t_{m-})<0 \quad \text{on } \partial\Omega_{14};\\&g_1^{(3)}(t_{m-})>0, g_1^{(2)}(t_{m-})<0 \quad \text{on } \partial\Omega_{23}.\end{aligned} \quad (66)$$

Theorem 3 *For the 2-DOF friction induced oscillator described in Section 3, the analytical conditions for vanishing of the stick motion from* $\partial\Omega_{\alpha_1\alpha_2}$ *and entering domain* Ω_{α_1} *are*

$$\left.\begin{aligned}&g_2^{(2)}(t_{m-})>0,\\&g_2^{(1)}(t_{m\pm})=0,\\&Dg_2^{(1)}(t_{m\pm})>0\end{aligned}\right\}\text{ from }\partial\Omega_{12}\to\Omega_1, \quad (67)$$

$$\left.\begin{aligned}&g_2^{(2)}(t_{m\pm})=0,\\&g_2^{(1)}(t_{m-})<0,\\&Dg_2^{(2)}(t_{m\pm})<0\end{aligned}\right\}\text{ from }\partial\Omega_{12}\to\Omega_2; \quad (68)$$

$$\left.\begin{aligned}&g_2^{(3)}(t_{m-})>0,\\&g_2^{(4)}(t_{m\pm})=0,\\&Dg_2^{(4)}(t_{m\pm})>0\end{aligned}\right\}\text{ from }\partial\Omega_{34}\to\Omega_4, \quad (69)$$

$$\left.\begin{aligned}&g_2^{(3)}(t_{m\pm})=0,\\&g_2^{(4)}(t_{m-})<0,\\&Dg_2^{(3)}(t_{m\pm})<0\end{aligned}\right\}\text{ from }\partial\Omega_{34}\to\Omega_3; \quad (70)$$

$$\left.\begin{aligned}&g_1^{(4)}(t_{m-})>0,\\&g_1^{(1)}(t_{m\pm})=0,\\&Dg_1^{(1)}(t_{m\pm})>0\end{aligned}\right\}\text{ from }\partial\Omega_{14}\to\Omega_1, \quad (71)$$

$$\left.\begin{aligned}&g_1^{(4)}(t_{m\pm})=0,\\&g_1^{(1)}(t_{m-})<0,\\&Dg_1^{(4)}(t_{m\pm})<0\end{aligned}\right\}\text{ from }\partial\Omega_{14}\to\Omega_4; \quad (72)$$

$$\left.\begin{aligned}&g_1^{(3)}(t_{m-})>0,\\&g_1^{(2)}(t_{m\pm})=0,\\&Dg_1^{(2)}(t_{m\pm})>0\end{aligned}\right\}\text{ from }\partial\Omega_{23}\to\Omega_2, \quad (73)$$

$$\left.\begin{aligned}&g_1^{(3)}(t_{m\pm})=0,\\&g_1^{(2)}(t_{m-})<0,\\&Dg_1^{(3)}(t_{m\pm})<0\end{aligned}\right\}\text{ from }\partial\Omega_{23}\to\Omega_3. \quad (74)$$

Theorem 4 *For the 2-DOF friction induced oscillator described in Section 3, the stick motion on the boundary* $\partial\Omega_{\alpha_1\alpha_2}$ *appears iff*

$$\left.\begin{aligned}&g_2^{(2)}(t_{m-})>0,\\&g_2^{(1)}(t_{m\pm})=0,\\&Dg_2^{(1)}(t_{m\pm})<0\end{aligned}\right\}\text{ from }\Omega_1\text{ to }\partial\Omega_{12}, \quad (75)$$

$$\left.\begin{aligned}&g_2^{(2)}(t_{m\pm})=0,\\&g_2^{(1)}(t_{m-})<0,\\&Dg_2^{(2)}(t_{m\pm})>0\end{aligned}\right\}\text{ from }\Omega_2\text{ to }\partial\Omega_{12}; \quad (76)$$

$$\left.\begin{aligned}&g_2^{(3)}(t_{m-})>0,\\&g_2^{(4)}(t_{m\pm})=0,\\&Dg_2^{(4)}(t_{m\pm})<0\end{aligned}\right\}\text{ from }\Omega_4\text{ to }\partial\Omega_{34}, \quad (77)$$

$$\left.\begin{aligned}&g_2^{(3)}(t_{m\pm})=0,\\&g_2^{(4)}(t_{m-})<0,\\&Dg_2^{(3)}(t_{m\pm})>0\end{aligned}\right\}\text{ from }\Omega_3\text{ to }\partial\Omega_{34}; \quad (78)$$

$$\left.\begin{aligned}&g_1^{(4)}(t_{m-})>0,\\&g_1^{(1)}(t_{m\pm})=0\\&Dg_1^{(1)}(t_{m\pm})<0\end{aligned}\right\}\text{ from }\Omega_1\text{ to }\partial\Omega_{14}, \quad (79)$$

$$\left.\begin{array}{l} g_1^{(4)}(t_{m\pm}) = 0, \\ g_1^{(1)}(t_{m-}) < 0, \\ Dg_1^{(4)}(t_{m\pm}) > 0 \end{array}\right\} \text{ from } \Omega_4 \text{ to } \partial\Omega_{14}; \quad (80)$$

$$\left.\begin{array}{l} g_1^{(3)}(t_{m-}) > 0, \\ g_1^{(2)}(t_{m\pm}) = 0, \\ Dg_1^{(2)}(t_{m\pm}) < 0 \end{array}\right\} \text{ from } \Omega_2 \text{ to } \partial\Omega_{23}, \quad (81)$$

$$\left.\begin{array}{l} g_1^{(3)}(t_{m\pm}) = 0, \\ g_1^{(2)}(t_{m-}) < 0, \\ Dg_1^{(3)}(t_{m\pm}) > 0 \end{array}\right\} \text{ from } \Omega_3 \text{ to } \partial\Omega_{23}. \quad (82)$$

6 Conclusion

In this paper, passable motions and stick motions of 2-DOF friction-induced oscillator with two harmonically external excitations on a speed-varying traveling belt were investigated by using the theory of flow switchability for discontinuous dynamical systems. Different domains and boundaries for such system in the absolute space and relative space were defined according to the friction discontinuity, respectively. The analytical conditions for the passable motions and the stick motions of such 2-DOF friction-induced oscillator were presented, from which it can been seen that such oscillator has more complicated and rich dynamical behaviors. There are more theories about such oscillator to be discussed in future.

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References:

[1] J.P.D. Hartog, Forced Vibrations with Coulomb and Viscous Damping, *Transactions of the American Society of Mechanical Engineers.* 53, 1930, pp. 107-115.

[2] E.S. Levitan, Forced Oscillation of a Spring-mass System Having Combined Coulomb and Viscous Damping, *Journal of the Acoustical Society of America.* 32, 1960, pp. 1265-1269.

[3] A.F. Filippov, Differential Equations with Discontinuous Right-hand Side, *American Mathematical Society Translations.* 2:42,1964,pp. 199-231.

[4] A.F. Filippov, Differential Equations with Discontinuous Right-hand Sides, *Dordrecht:Kluwer Academic Publishers.* 1988.

[5] A.C.J. Luo, A Theory for Non-smooth Dynamical Systems on Connectable Domains, *Communication in Nonlinear Science and Numerical Simulation.* 10, 2005,pp. 1-55.

[6] A.C.J. Luo, Imaginary, Sink and Source Flows in the Vicinity of the Separatrix of Nonsmooth Dynamic System, *Journal of Sound and Vibration.* 285, 2005, pp. 443-456.

[7] A.C.J. Luo, Singularity and Dynamics on Discontinuous Vector Fields, *Amsterdam: Elsevier.* 2006

[8] A.C.J. Luo, A Theory for Flow Swtichability in Discontinuous Dynamical Systems, *Nonlinear Analysis: Hybrid Systems.* 2, 2008, pp. 1030-1061.

[9] A.C.J. Luo, and B.C. Gegg, On the Mechanism of Stick and Non-stick Periodic Motion in a Periodically Forced,Linear Oscillator with Dry-friction, *ASME Journal of Vibration and Acoustics.* 128, 2006, pp. 97-105.

[10] A.C.J. Luo, Discontinuous Dynamical Systems, *Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg.* 2012.

[11] A.C.J. Luo, Discontinuous Dynamical Systems on Time-varying Domains, *Beijing,Higher Education Press.* 2009.

[12] Ge Chen, and Jinjun Fan, Analysis of Dynamical Behaviors of a Double Belt Friction-oscillator Model, *WSEAS TRANSACTIONS on MATHEMATICS.* 15, 2016, pp. 357-373.