Spectral and statistical analysis of the internal bore evolution

OXANA KURKINA, EKATERINA ROUVINSKAYA, AYRAT GINIYATULLIN,
ANDREY KURKIN, TATIANA TALIPOVA, EFIM PELINOVSKY
Laboratory of Modeling of Natural and Anthropogenic Disasters
Nizhny Novgorod State Technical University n.a. R.E. Alekseev
603950 Nizhny Novgorod, Minin St., 24
RUSSIAN FEDERATION
oksana.kurkina@mail.ru http://www.nntu.ru

Abstract: Undular internal bores appear in oceans as an intermediate stage of internal tidal wave evolution. We study nonlinear disintegration of long sine wave in the framework of the Gardner equation (extended version of the Korteweg – de Vries equation with both quadratic and cubic nonlinear terms) which is actively applied in physical oceanography. We carried out numerical modeling of long sine wave evolution for different signs of the cubic nonlinear term and different initial amplitudes to demonstrate its principal features. The focus of the study is made on spectral and statistical analysis of generated wave field. If cubic nonlinearity is positive and amplitude of sine wave is large enough, soliton-like impulses of both polarities are generated and their interactions may result in the formation of extreme amplitude waves. Statistical analysis of wave heights in time shows permanent exceedance somewhere in the wave field of the level of two significant wave heights (criterion for freak wave appearance).

Key-Words: Undular bore, Gardner equation, significant wave height, extreme amplitude waves, solitons, exceedance probability distribution, skewness, kurtosis

1 Introduction

Undular bores are very often observed in estuaries and river mouths during a tide circle, when long tidal wave entering shallow waters. Brilliant collection of undular bores' observations can be found in the book [1]. Also, they were observed during the 1983 Japan Sea tsunami [2] and 2004 Indian Ocean tsunami [3]. In general, the criterion of undular bore formation is a relationship between bore height H, measured from the bottom, and unperturbed depth of reservoir h: $H < 1.5h$. Recently, this criterion has been revised in [4].

Undular bores are also very often observed in the stratified ocean as the vertical displacements of the pycnocline lied on depth 50-200 m and manifested on the ocean surface as the slicks of various intensity, see for instance, [5]. Similar phenomenon is observed also in lakes [6]. Sometimes internal undular bore is called as a solibore after [7].

The undular bore is generated in systems with weak dispersion and in the presence of nonlinearity, for example, when the initial disturbance is very long or due to "dam-break" process. Simplified model of such phenomenon is based on the famous Korteweg-de Vries equation with initial condition in the form of the "dam-break".

In the present paper we would like to study the evolution of the long sine wave in the framework of the non-dimensional Gardner equation with different signs of cubic nonlinearity. This problem is of practical interest because degeneration of the long tidal wave is often responsible for generation of intense undular bores, often observed in the mouths and estuaries. These waves contain huge energy, so they are a major source of sediment transport, resuspension as well as turbulent mixing in the water column. Such waves have a significant influence on the propagation of sound in the water column and on the formation of the bottom sound channel. Another interesting aspect of the solibores' studying is the fact that extreme amplitude pulses may be generated in the process of their evolution under certain conditions. Main goal of this study is to carry out spectral and statistical analysis of long sine wave degeneration at various times in the framework of Gardner equation.

2 Theoretical model

We will use the canonical form of the Gardner equation with positive or negative sign of cubic nonlinearity term:

$$\frac{\partial\eta}{\partial t}+6\eta(1\pm\eta)\frac{\partial\eta}{\partial x}+\frac{\partial^3\eta}{\partial x^3}=0\,, \qquad (1)$$

Initial value problem with periodic boundary conditions is solved for this equation:

$$\eta(x,0)=A\cos\left(\frac{2\pi}{L}x\right),\ \eta(x+L,t)=\eta(x,t)\,, \qquad (2)$$

where L – length of the computational domain, and amplitude of the wave A ranged from 0.1 to 3 dimensionless units.

To solve the problem (1), (2) we use a numerical code based on the implicit pseudo-spectral method [8], which allows us to save the integrals defined by expressions (it is obvious that limits of integration are the computational domain boundaries in numerical model):

$$M = \int_{-\infty}^{\infty} \eta dx\,, \quad E = \int_{-\infty}^{\infty} \eta^2 dx\,. \qquad (3)$$

Numerical code, that we use, repeatedly was verified in simulation of wave processes of different nature (see, eg, [9, 10]).

The periodic solutions ("cnoidal" waves) of the Gardner equation can be found in [11]. The solitons of this equation are very good studied also [12 – 14] and their properties depends on values of nonlinear and dispersion coefficients.

In the present paper we eliminate the real values of the coefficients of the Gardner equation using appropriated scaling. But in context of the various physical applications it can be necessary to introduce they into practice.

To move on from the canonical form of the Gardner equation (1) to the dimensional Gardner equation, it is necessary to carry out the change of variables:

$$x = \frac{\hat{x}}{L}, t = \frac{\hat{t}}{T}, \eta = \frac{\xi}{\xi_0}, \xi_0 = \frac{\alpha}{|\alpha_1|}, \qquad (4)$$
$$L = \sqrt{\frac{6|\alpha_1|\beta}{\alpha^2}}, T = \sqrt{\beta}\left(\frac{6|\alpha_1|}{\alpha^2}\right)^{3/2},$$

where α – coefficient of quadratic nonlinearity, α_1 – coefficient of cubic nonlinearity, β – coefficient of dispersion.

The analytical one-soliton solution of Gardner equation is well known:

$$\xi(x,t) = \frac{A}{1 + B\cosh(\gamma(x - Vt))}. \qquad (5)$$

The soliton nonlinear velocity $V = \beta\gamma^2$ is expressed through the inverse width of the soliton γ. The parameters A and B depend on the coefficients of Gardner equation and determine the soliton amplitude a as the extreme value of the function $\xi(x, t)$:

$$a = \frac{A}{1+B}, \ A = \frac{6\beta\gamma^2}{\alpha}, \ B^2 = 1 + \frac{6\alpha_1\beta\gamma^2}{\alpha^2}. \qquad (6)$$

The parameters of the family of solutions can also be expressed through its amplitude a:

$$\gamma^2 = a\frac{2\alpha + a\alpha_1}{6\beta}, \ A = a\left(2 + a\frac{\alpha_1}{\alpha}\right) \ B = 1 + a\frac{\alpha_1}{\alpha}. \qquad (7)$$

There are different branches of the soliton solutions depending on the signs of coefficients at the nonlinear terms, see Fig. 1.

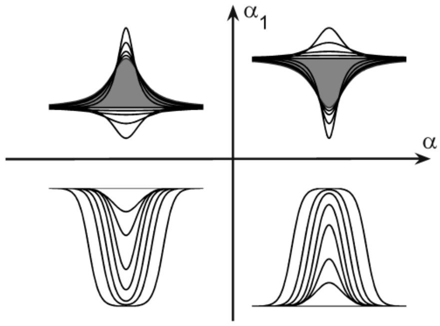

Fig. 1. Shapes of soliton solutions to Gardner equation for different combinations of the signs of coefficients at its nonlinear terms (idea of representation by R.Grimshaw, E. Pelinovsky & T. Talipova)

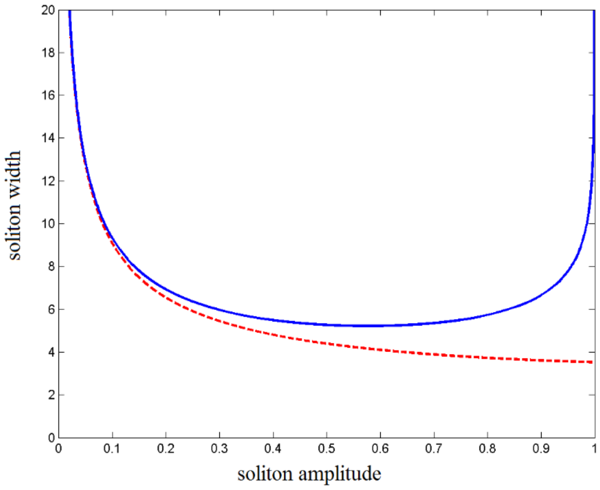

Fig. 2. The soliton length versus its amplitude: solid line – Gardner equation, dashed line – Korteweg – de Vries equation

Let us briefly describe the properties of solitary waves for positive sign of the quadratic nonlinear term (In opposite case, the polarity of soliton should be changed). If cubic nonlinear term is negative, the solitons have positive polarity. The soliton height has the limited value

$$a_{\lim} = -\frac{\alpha}{\alpha_1}, \qquad (8)$$

which is equal to 1 for canonical form (1), and it presents the table soliton. The soliton width is varied non-monotonically with amplitude increase (Fig. 2). The soliton with amplitude 0.5 has a minimal width. For amplitude less than this value the Korteweg – de Vries equation is a good model to describe soliton parameters. Solitons with height exceeded 0.8 - 0.9 can be considered as “table” solitons.

In the case of positive cubic nonlinear term there are two branches of solitary waves. The first branch has the polarity determined by sign of α, and its amplitude can be arbitrary with no limited amplitude (within the applicability of Gardner equation). The second branch describes the solitons of alternative polarity. The soliton amplitude of this branch exceeds the minimal value corresponded the so called algebraic soliton amplitude (which is equal to -2 for canonical Gardner equation (1))

$$a_{\mathrm{alg}} = -\frac{2\alpha}{\alpha_1}. \tag{9}$$

3 Nonlinear disintegration of sine wave

The scenario for evolution of small-amplitude long harmonic waves in the case of negative as well as positive values of cubic nonlinear term in the Gardner equation has many features in common with the process of disintegration such an impulse in the framework of the Korteweg-de Vries equation, see, for instance [15, 16]. The snapshots of evolution of wave (2) with amplitude A = 0.1 dimensionless units and negative cubic nonlinearity are shown in Fig. 3.

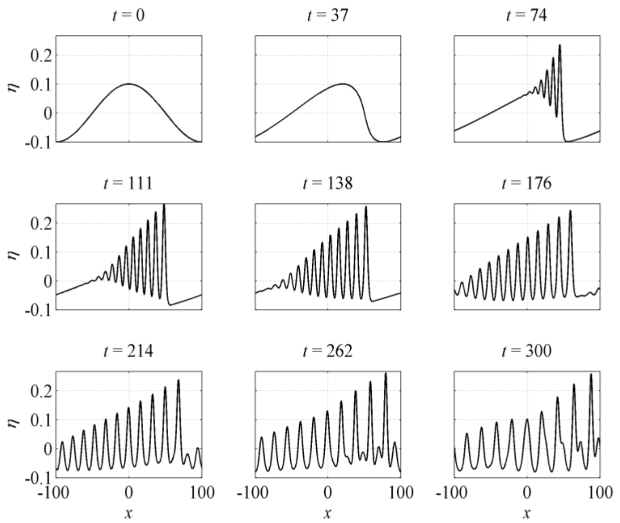

Fig. 3 Snapshots of wave dynamics with A = 0.1 and negative cubic nonlinear term for the Gardner equation

After a time, one of the fronts becomes steeper due to nonlinearity, and cnoidal waves of variable, decreasing linearly amplitudes are generated on it. These waves interact with each other because of the periodicity of the boundary condition. These interactions lead to a negative phase shift and decreasing of waves’ velocity as in the case of two-soliton interaction such as overtaking. But amplitudes of the resulting impulse in the moment of the interaction is less than the amplitude of cnoidal wave with greater amplitude.

The Fourier spectra of the evolving wave in terms of coefficients C_j:

$$C_j = \frac{2}{N}\sum_{k=1}^{N}\eta(k)w_N^{(k-1)(j-1)},\ w_N = \exp\left(\frac{-2\pi i}{N}\right), \tag{10}$$

(we use discrete set of N harmonics $\eta(k)$) are presented in Fig. 4. Due to nonlinear steepness of initial sine wave, the spectrum on small time has the breaking asymptotic $j^{-4/3}$ for approximately 20 harmonics which is common features of nonlinear hyperbolic systems with weak dispersion [17, 18]. Then, forming of undular bore leads to generation of several spectral peaks in range 10-100 harmonics downshifting with time. The energy of the basic harmonics is decreased transferring the energy in high harmonics. The variable amplitude cnoidal-like structure of undular bore is not strongly periodic that leads to the wide overlapping peaks.

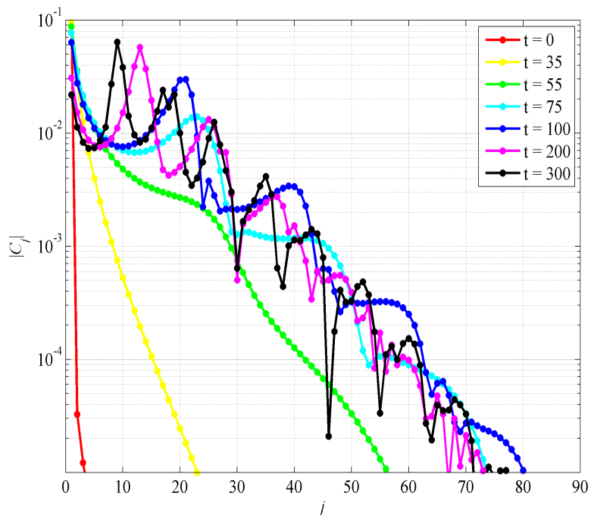

Fig. 4. Spectrum of wave records for A = 0.1 and negative cubic nonlinearity at different times.

In more detail the evolution of small-amplitude sine waves is analyzed in [19].

Cubic nonlinear effects become noticeable with increasing amplitude. When the amplitude of initial wave amounts to 0.5 dimensionless units and the cubic nonlinear term is negative, the “breaking” point shifts to the trough. If cubic nonlinear term is positive and A = 0.5 the “breaking” point shifts to the wave crest (Fig. 5). In both cases envelope of wave crests becomes parabolic. It is worth noting that there are many nonlinear interactions of waves, that are similar to "overtaking", if cubic nonlinearity is negative. But if cubic nonlinearity is positive, scenario of "exchange" takes place.

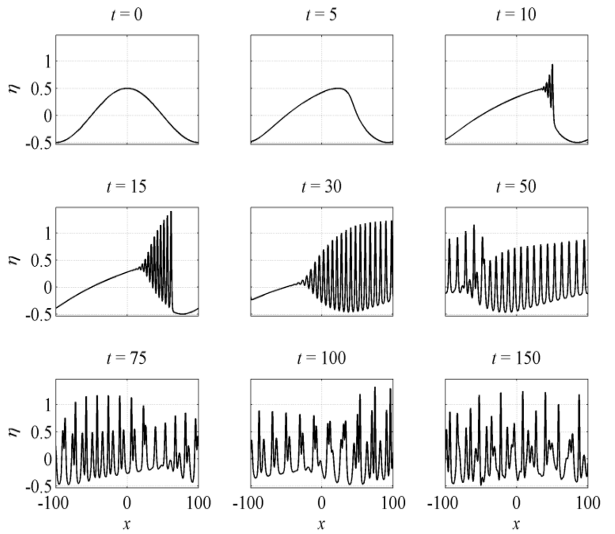

Fig. 5 Snapshots of wave dynamics with A = 0.5 and positive cubic nonlinear term for the Gardner equation

Spectra of sine wave evolution have much in common for such amplitude and negative or positive cubic nonlinearity, therefore only graph for the latter case is given (Fig. 6). The spectra for this run are wider due to increased nonlinearity. Positive cubic nonlinearity accelerates the generation of higher harmonics in comparison with negative cubic nonlinearity. But again qualitatively, the shape of spectra are the same as in previous case with spectral peaks downshifting with time. The spectra after t = 10 are equidistant (with peaks on harmonics with multiple numbers).

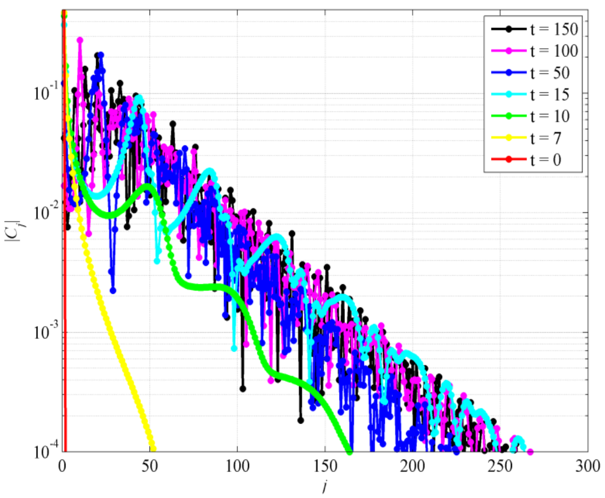

Fig. 6. Spectrum of wave records for A = 0.5 and positive cubic nonlinearity at different times.

Exceedance probability distribution of wave heights over time is shown in Fig. 7 separately for positive and negative parts of the wavefield. Significant wave height, which is defined as:

$$H_s = 4\,\sigma, \tag{11}$$

where σ is the standard deviation of ordinates η, H_s is indicated by the black line. Substantial asymmetry of negative and positive values of η is demonstrated by this plot.

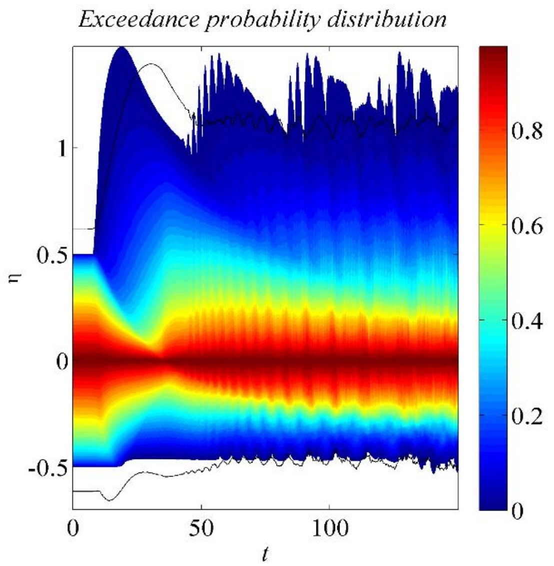

Fig. 7. Exceedance probability distribution of ordinates for A = 0.5 and positive cubic nonlinearity at various moments in time. Black line – significant height.

Further amplitude increasing in the case of negative cubic nonlinearity causes appearance of second breaking point and generation of one table-top soliton with a group of solitary-like waves at the leading edge of the bore. Small solitons run on the crest of the table-top soliton and change their polarity. This process is described in detail in paper [19].

If the coefficient of cubic nonlinearity is positive and amplitude of initial wave amounts to A = 1.5 dimensionless units, a second "breaking" point appears and the pulses of both positive and negative polarity are generated. This process is demonstrated in Fig 8. Interactions of waves of opposite polarities result in an increase of the maximum amplitude of the wave field.

The spectra are qualitatively similar to those shown above but are significantly wider (Fig. 9). After t = 2 they contain almost equidistant peaks corresponding to multiple numbers of harmonics.

Statistical analysis of the wave field is shown in Fig. 10. Even in the case of A = 1.5 the wave interactions cause appearance of impulses with amplitude greater than significant wave height.

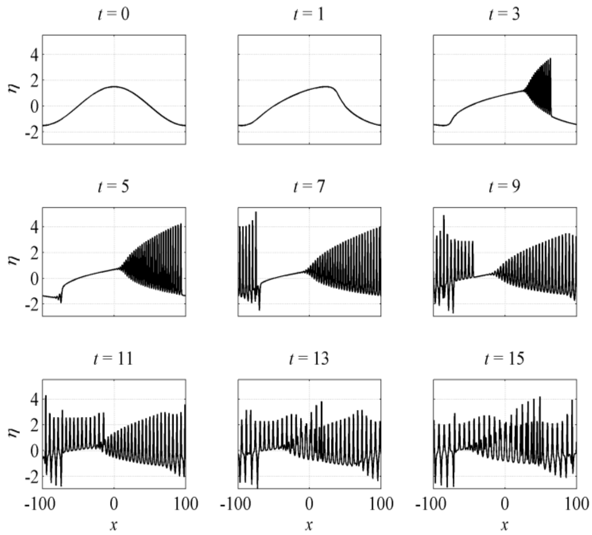

Fig. 8 Snapshots of wave dynamics with A = 1.5 and positive cubic nonlinear term for the Gardner equation

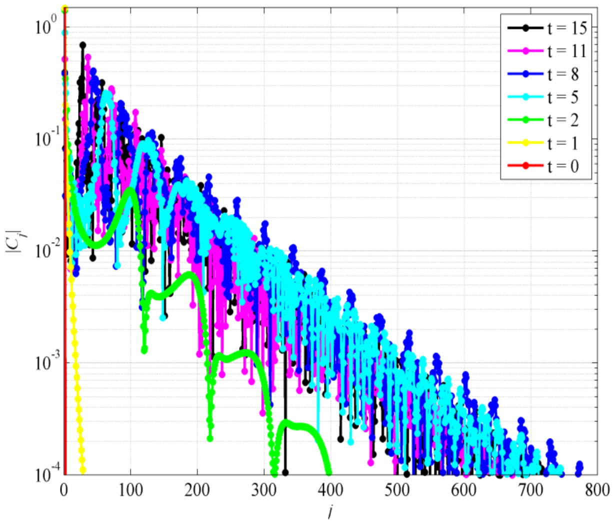

Fig. 9. Spectrum of wave records for A = 1.5 and positive cubic nonlinearity at different times.

Nonlinear interactions of pulses of opposite polarities are more intense with increasing of initial sine wave amplitude. So to study the possible mechanisms of generating of extreme waves in the canonical Gardner equation with a positive cubic nonlinearity we increase the amplitude of sine impulse to a value A = 3 dimensionless units. The process of undular bore development and generation of wave field, which is represented interactions of ensembles of positive and negative polarity solitons, is shown in Fig. 11 for this initial amplitude.

One can see in Fig. 12 for exceedance probability distribution of wave heights, that large values of η are observed when paired collisions of different polarities solitons began after t = 1.6. Amplitudes of such impulses are four times greater than initial sine wave amplitude and often twice more than H_s.

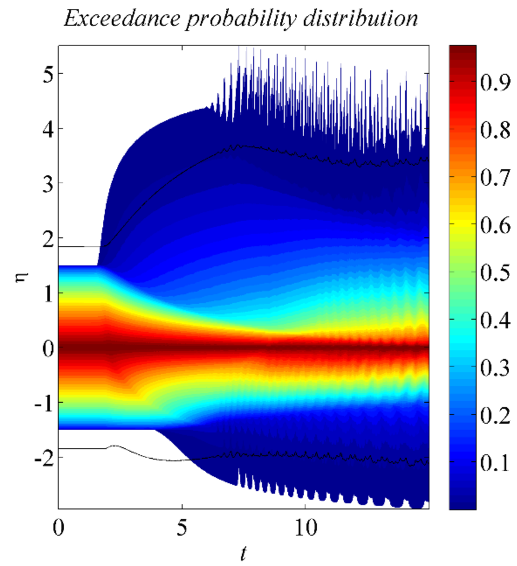

Fig. 10. Exceedance probability distribution of ordinates for A = 1.5 and positive cubic nonlinearity at various moments in time. Black line – significant wave height.

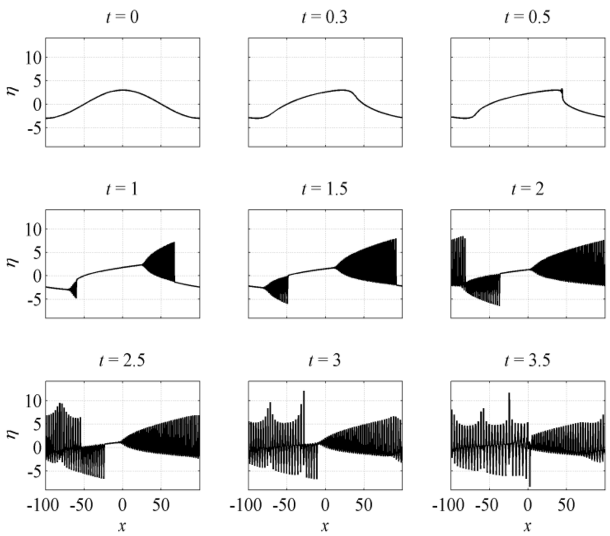

Fig. 11 Snapshots of wave dynamics with A = 3 and positive cubic nonlinear term for the Gardner equation

Graphs of kurtosis and skewness (Fig. 13) are also characterized by presence of peaks at times of higher probability of large-amplitude waves.

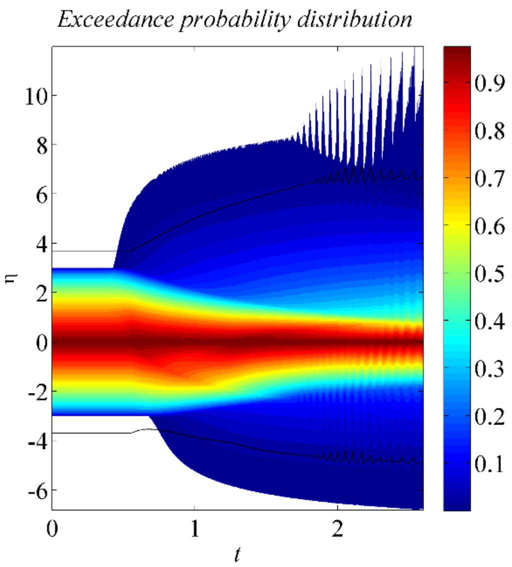

Fig. 12. Exceedance probability distribution of ordinates for A = 3 and positive cubic nonlinearity at various moments in time. Black line – significant height.

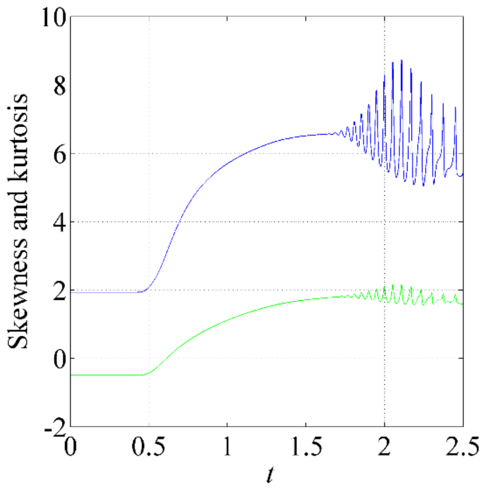

Fig. 13. Skewness (green line) and kurtosis (blue line) of η for different points in time for A = 3 and positive cubic nonlinearity

4 Conclusion

The present paper studied the process of long sine wave disintegration in weakly nonlinear and weakly dispersive media within the Gardner equation with positive and negative cubic nonlinearity. We made a series of numerical computations to demonstrate the features of undular bores development for different signs of the cubic nonlinear term. If the cubic nonlinear term is positive and the wave amplitude is large, the solitons of both polarities appear. These waves interact and extreme amplitude waves can generate as a result of such collisions. This process is demonstrated well in exceedance probability distribution graphs for wave height as ordinates substantially exceeding significant amplitude H_s. Nonlinear interactions lead to the generation of higher harmonics forming the breaking asymptotic $j^{-4/3}$ for small times (as it was predicted within the dispersionless Gardner equation). For larger times the spectral peaks appear, they downshift with time.

Acknowledgements

The presented results were obtained with the financial support of the grants of the President of the Russian Federation for state support of leading scientific schools of the Russian Federation (NSH-6637.2016.5) and the young Russian scientists (MK-5208.2016.5).

References:

[1] Chanson H., *Tidal Bores, Aegir, Eagre, Mascaret, Pororoca: Theory and Observations*, World Scientific, 2011.

[2] Tsuji Y., Yanuma T., Murata I., Fujiwara C., Tsunami ascending in rivers as an undular bore, *Natural Hazards*, Vol. 4, 1991, pp. 257-266.

[3] Constantin A., Johnson R.S., Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis, *Fluid Dynamics Research*, Vol. 40, 2008, pp. 175-211.

[4] Pelinovsky E.N., Shurgalina E.G., Rodin A.A., Criteria for the transition from a breaking bore to an undular bore, *Izvestiya, Atmospheric and Oceanic Physics*, Vol. 51, No. 5, 2015, pp. 530-533.

[5] Jackson C.R., Apel J.R., *An atlas of internal solitary-like waves and their properties*, Global Ocean Associates, 2004.

[6] Hutter K., *Nonlinear Internal Waves in Lakes*, Springer, 2012.

[7] Apel J.R., A new analytical model for internal solitons in the ocean, *J. Phys. Oceanogr.*, Vol. 33, 2003, pp. 2247-2269.

[8] Fornberg, B.A. *Practical Guide to Pseudospectral Methods*, Cambridge University Press, 1998.

[9] Kurkin A.A., Polukhina O.E., Numerical experiments on the propagation of Rossby waves in the ocean, *Proceedings of the Academy of Engineering Sciences. Applied Mathematics and Mechanics*, Vol. 4, 2003, pp. 99-116.

[10] Pelinovsky E. et al., *Solitary Waves in Fluids*, WIT Press, 2007.

[11] Kamchatnov A.M., Kuo Y.-H., Liu T.-C., Horng T.-L., Gou S.-C., Clift R., El G.A., Grimshaw R.H.J., Undular bore theory for the

Gardner equation, *Phys. Rev. E*, Vol. 86, 2012, pp. 036605.
[12] Ostrovsky L., Pelinovsky E., Shrira V., Stepanyants Y., Beyond the KDV: post-explosion development, *Chaos*, Vol. 25, No. 9, 2015, pp. 097620.
[13] Grimshaw R., Pelinovsky E., Talipova T., Solitary wave transformation in a medium with sign-variable quadratic nonlinearity and cubic nonlinearity, *Physica D*, Vol. 132, 1999, pp. 40-62.
[14] Grimshaw R., Pelinovsky E., Talipova T., Modeling internal solitary waves in the coastal ocean, *Survey in Geophysics*, Vol. 28, No. 4, 2007, pp. 273-298.
[15] Salupere A., Pederson P., Engelbrecht J., Long-time behaviour of soliton ensembles. Part I. Emergence of ensembles, *Chaos, Solitons and Fractals*, Vol. 14, 2002, pp. 1413-1424.
[16] Salupere A., Pederson P., Engelbrecht J., Long-time behaviour of soliton ensembles. Part II. Periodical patterns of trajectories, *Chaos, Solitons and Fractals*, vol. 15, 2003, pp. 29-40.
[17] Kartashova E., Pelinovsky E., Talipova T., Fourier spectrum and shape evolution of an internal Riemann wave of moderate amplitude, *Nonlinear Processes in Geophysics*, Vol. 20, 2013, pp. 571-580.
[18] Pelinovsky D., Pelinovsky E., Kartashova E., Talipova T., Giniyatullin A., Universal power law for the energy spectrum of breaking Riemann waves, *JETP Letters*, Vol. 98, 2013, pp. 237-241.
[19] Kurkina O., Rouvinskaya E., Talipova T., Kurkin A., Pelinovsky E., Nonlinear disintegration of sine wave in the framework of the Gardner equation, *Physica D: Nonlinear Phenomena*, 2015, http://dx.doi.org/10.1016/j.physd.2015.12.007.