Selecting a Representative Image from a Collection of Images by Solving a System of Non-Linear Algebraic Equations

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Abstract: - The problem of selecting a best representative image from a group of similar images is an important problem as it can expedite the task of image search and image matching. We solve this problem by first measuring the similarity between every pair of image in the set by a suitable similarity measure, and then transforming the problem to similarity space and finding the corresponding locations of the images in the similarity space. Finally, the image located closest to the center of the preoccupied similarity space is selected as the best representative image. The difficulty in such a problem arises in attempting to find the locations of N images in the similarity space, since this leads to a set of $N(N–1)$ non-linear simultaneous algebraic equations with N^2 unknowns. We solve such a problem by forcing the solution to be in $\Re^{N-1}$. We present a closed-form solution for the cases when N = 3, 4 and 5. We give examples of finding the best representative images for two sets as an application of the method.

Key-Words: - Image search, similarity space, non-linear algebraic equations, image correlation.

1 Introduction

The generation and acquisition of images has never been as widespread as it is today. Worldwide, and on a daily basis, millions of images are easily being generated by inexpensive mobile phones and digital cameras. A simple examination of the number of images posted on social media networks is tremendous; in mid-2014, it was estimated that 1.8 billion photos were being uploaded to the top five social networks on a daily basis [1]. In addition, a vast number of images are being generated on a continuous basis for research, monitoring and measurements. For example, satellite cameras generate a huge amount of images for weather forecasting, military surveillance, mineral exploration, space exploration and urban planning. The generation of such huge quantities of images has raised technical problems that are in need of real-time solutions. Image organization and categorization, as well as efficient image search techniques, are just two problems that have promoted much research.

One sub-problem of interest in image search, is selecting an adequate image representative from a collection of images, based on some similarity criteria. If such an image can be adequately determined, then databases can be searched quickly and efficiently for a given query image, by only comparing it to a finite number of representative images (which are predetermined offline prior to the search), rather than comparing all images in the database.

Many similarity metrics have been developed to measure similarity between images such as image correlation [2], image subtraction [3], mutual information [4], minimizing image intensity co-occurrence [5] and the Hamming distance [6]. With such metrics, the similarity distance between images can be easily calculated. By obtaining the similarity distance between every pair of images for a given set of images, the problem can be transformed to similarity space and one can hope to determine the position of the images in it. Once the location of the images in similarity space is determined, then the center of the preoccupied similarity space can be determined. The closest image to the center of the space can then be selected as the best representative image of the set.

However, the difficulty lies in determining the positions of the images in similarity space, since this produces a set of non-linear algebraic equations of the form,

$$\sum_{k=1}^{N-1} (x_i^{<k>} - x_j^{<k>})^2 = r_{i,j}^2 \qquad i = 1, \dots, N-1, \; j = i+1, \dots, N \tag{1}$$

where $x^{<k>}$ is the k^{th} dimensional variable, i.e., $x^{<1>}= x$, $x^{<2>}= y$, $x^{<3>}= z$, … , etc. In general, such a system of equations is difficult to solve analytically. However, under certain conditions and assumptions, an analytical solution can be found as we show in this paper. We present a closed form solution to the problem of finding the coordinates of N images in $\Re^{N\text{-}1}$ similarity space for the cases when N = 3, 4 and 5.

This paper is organized as follows; section 2 briefly discusses the difficulties associated with non-linear algebraic equations and finding closed form solutions to them. Section 3 presents the main theme of the paper: the problem addressed and our approach to solving the problem. Section 4 presents two examples on the application of the proposed solution. Section 5 finally concludes our work and discusses where our research is headed,

2 Non-Linear Algebraic Equations

Most non-linear algebraic equations are difficult to solve analytically and only a small fraction of them have an analytical solution. The difficulty is amplified when there is more than one variable to solve; in this case a system of simultaneous non-linear algebraic equations are produced that need to be solved.

Because of the difficulty in solving such problems most solutions are usually numerical and not analytical. Many numerical methods are available that can be used to solve these equations such as, the Secant Method, Newton's method, Muller's method, and others [7] [8] [9]. Ongoing research in the last several decades has concentrated primarily on improving these methods, e.g. [10] [11] [12]. Nevertheless, all numerical methods require an initial guess that is "close" to the solution for the solution to converge. If the initial guess is "bad", then convergence to a solution -most likely- will not occur. Techniques are available that can aid in finding a solution, such as sketching the equation and bracketing the solution. However, when the equations become multi-dimension in more than 3 coordinates, then such techniques become extremely difficult -if not impossible to solve. Needless to say, as the number of unknowns in the problem increases, finding a numerical solution becomes even more of a challenge. However, despite such difficulties, under certain conditions, closed form solutions have been found for many cases, e.g. [13] and [14].

3 Problem Formulation

Given a set of N images whose similarity distance matrix **r** is given, we are interested in determining the best representative image from among the images, based on the similarity distance employed. This is equivalent to the problem of finding the coordinates of N points, P_i for i = 1, …, N, given the distance between them $r_{i,j}$.

3.1 A Closed Form Solution

We are particularly interested in finding a closed form solution to the problem, i.e. finding the relative coordinates of the points, $P_i = (x_i, y_i, z_i, \ldots)$ in multi-dimensional space. Let,

$$\begin{aligned} |P_1 - P_2| &= r_{1,2} \\ |P_1 - P_3| &= r_{1,3} \\ &\vdots \\ |P_1 - P_N| &= r_{1,N} \\ |P_2 - P_3| &= r_{2,3} \\ &\vdots \\ |P_{N\text{-}1} - P_N| &= r_{N\text{-}1,N} \end{aligned} \tag{2}$$

These equations can be rewritten as,

$$\begin{aligned} &(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + \ldots = r^2_{1,2} \\ &(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 + \ldots = r^2_{1,3} \\ &\vdots \\ &(x_{N\text{-}1} - x_N)^2 + (y_{N\text{-}1} - y_N)^2 + (z_{N\text{-}1} - z_N)^2 + \ldots \\ &\qquad = r^2_{N\text{-}1,N} \end{aligned} \tag{3}$$

This is a system of $N(N–1)/2$ non-linear algebraic equations with N^2 unknowns. Thus, the system is underdetermined.

Geometrically, this system of equations represent the intersections of N hyper-spheres in N–1 space, and the intersection of the hyper-spheres at common points represent the solution to these equations (i.e. the coordinates of P_i).

Since the relative positions are sought here and not the absolute coordinates of P_i (which are non-recoverable), it can be shown that the solution space can be reduced to N–1 instead of N space (and perhaps even a lower space in some cases). This reduces the number of unknowns to $N(N–1)$, but still leaves the system underdetermined by $N(N–1)/2$ (with infinite solutions). Hence, we have the freedom of placing $N(N–1)/2$ constraints. Let,

$$| P_i - P_j | = r_{i,j}, \quad i = 1, \dots, N-1, \; j = i+1, \dots, N \tag{4}$$

This can be written compactly as (1). Expanding (1),

$$\begin{aligned}
&(x_1^{<1>} - x_2^{<1>})^2 + (x_1^{<2>} - x_2^{<2>})^2 + \cdots + (x_1^{<N-1>} - x_2^{<N-1>})^2 = r_{1,2}^2 \\
&\vdots \\
&(x_1^{<1>} - x_3^{<1>})^2 + (x_1^{<2>} - x_3^{<2>})^2 + \cdots + (x_1^{<N-1>} - x_3^{<N-1>})^2 = r_{1,2}^2 \\
&(x_{N-1}^{<1>} - x_N^{<1>})^2 + (x_{N-1}^{<2>} - x_N^{<2>})^2 + \cdots + (x_{N-1}^{<N-1>} - x_N^{<N-1>})^2 = r_{N-1,N}^2
\end{aligned} \tag{5}$$

We place the following $N(N-1)/2$ constraints:

$$x_i^{<k>} = 0, \quad i = 1, \dots, N-1, \; k = 1, \dots, N-I \tag{6}$$

This implies,

$$\begin{aligned}
&x_1^{<k>} = 0, \; k = 1, \dots, N-1 \\
&x_2^{<k>} = 0, \; k = 1, \dots, N-2 \\
&\vdots \\
&x_{N-1}^{<k>} = 0, \; k = 1
\end{aligned} \tag{7}$$

This forces the solution coordinate matrix **x**, which is not square, but rather $N \times K$ (i.e. $N \times N-1$), to be arranged such that it is lower triangular of the form,

$$\mathbf{x} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \iddots & x_2^{<k>} \\ 0 & 0 & \iddots & \vdots \\ 0 & x_{N-1}^{<2>} & \cdots & x_{N-1}^{<k>} \\ x_N^{<1>} & x_N^{<2>} & \cdots & x_N^{<k>} \end{bmatrix} \tag{8}$$

In other words, $\mathbf{P_1} = (0, 0, \dots, 0)$ is forced to be at the origin, $\mathbf{P_2} = (0, 0, \dots, 0, x_2^{<k>})$ is forced to be on the k^{th} axis, $\mathbf{P_3} = (0, 0, \dots, 0, x_3^{<k-1>}, x_3^{<k>})$ is forced to be on the $(k-1)^{th}$-k^{th} plane, and so on. This will produce 2^{N-1} solutions exhibiting a symmetric pattern; each successive point placed on a given coordinate axis produces two solutions for the next point, exhibiting reflection about this axis. The constraints can be incorporated into (5) as,

$$\sum_{k=1}^{N-1} (\alpha_{i,k} \cdot x_i^{<k>} - \alpha_{j,k} \cdot x_j^{<k>})^2 = r_{i,j}^2, \quad i = 1, \dots, N-1, \; j = i+1, \dots, N \tag{9}$$

where,

$$\alpha_{i,j} = \begin{cases} 1, & if\ (i + j) \geq (N+1) \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

Hence, $\boldsymbol{\alpha}$ is of the form,

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \iddots & 1 \\ 0 & 0 & \iddots & \vdots \\ 0 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{11}$$

3.2 Existence of Other Solutions

As earlier stated, there are an infinite number of solutions as can be easily verified by multiplying **x** by a *transformation* matrix, **T**, with any translational or rotational value, producing a new coordinate matrix **x'**,

$$\mathbf{x'} = \mathbf{T}\,\mathbf{x} \tag{12}$$

T is $N \times N$,

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{H} \\ \mathbf{D} & S \end{bmatrix} \tag{13}$$

R is a *rotational matix* of size $(N-1) \times (N-1)$, **D** is a *translational vector* of size $1 \times (N-1)$, **H** is a *shear vector* of size $(N-1) \times 1$, and S is a *scaling* value. Here, $\mathbf{H} = [0\ 0\ \dots]$ is a null vector and $S = 1$. As an example, for $N = 3$, then **R** is 2×2 and **D** is 1×2:

$$\mathbf{R} = \mathbf{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{14}$$

$$\mathbf{D} = [\Delta x \;\; \Delta y] \tag{15}$$

where θ is the rotation angle, and $(\Delta x, \Delta y)$ are the translational offset.

3.3 Closed-Form Solutions

In this section we present closed-form solutions for the cases when $N = 3$, 4 and 5.

3.3.1 Closed-Form Solution for $N = 3$

For 3 points, the solution space can be found in $\Re^2$. This is because the location of any 3 points lie on a 2D plane. The equations are,

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = r_{1,2}^2 \tag{16}$$

$$(x_1 - x_3)^2 + (y_1 - y_3)^2 = r_{1,3}^2 \quad (17)$$

$$(x_2 - x_3)^2 + (y_2 - y_3)^2 = r_{2,3}^2 \quad (18)$$

Applying (6) produces,

$$x_1 = 0,\ y_1 = 0 \quad (19)$$

$$x_2 = 0 \quad (20)$$

Substituting the constraints into equations (16) – (18), reduces these equations to the following,

$$(y_2)^2 = r_{1,2}^2 \quad (21)$$

$$(x_3)^2 + (y_3)^2 = r_{1,3}^2 \quad (22)$$

$$(x_3)^2 + (y_2 - y_3)^2 = r_{2,3}^2 \quad (23)$$

The solutions to these equations are,

$$y_2 = r_{1,2} \quad (24)$$

$$y_3 = \frac{1}{2r_{1,2}}(r_{1,2}^2 + r_{1,3}^2 - r_{2,3}^2) \quad (25)$$

$$x_3 = \sqrt{r_{1,3}^2 - \left(\frac{1}{2r_{1,2}}(r_{1,2}^2 + r_{1,3}^2 - r_{2,3}^2)\right)^2} \quad (26)$$

x_3 and y_2 each have two solutions: one negative and one positive. Hence 4 solutions are possible. The coordinates of the points P_1 – P_3 for all four solutions are:

1. $P_1 = (0,0)$, $P_2 = (0, r_{1,2})$, $P_3 = (|x_3|,|y_3|)$
2. $P_1 = (0,0)$, $P_2 = (0, r_{1,2})$, $P_3 = (-|x_3|,|y_3|)$
3. $P_1 = (0,0)$, $P_2 = (0, -r_{1,2})$, $P_3 = (|x_3|,-|y_3|)$
4. $P_1 = (0,0)$, $P_2 = (0, -r_{1,2})$, $P_3 = (-|x_3|,-|y_3|)$

Fig. 1 shows the four possible solutions of placing P_2 on the y axis a distaance $r_{1,2}$ from the origin. Eq (21) represents a circle about P_1 with raduis $r_{1,2}$ reaching P_2 (not shown in the figure). The circle about P_1 with raduis $r_{1,3}$ represents Eq (22) and the circle about P_2 with raduis $r_{2,3}$ represents Eq (23). The solution to the problem is found by finding the intersection of these two circles. This produces two solutions when $P_2 = (0,y_2) = (0,+r_{1,2})$ and two solutions when $P_2 = (0,-r_{1,2})$.

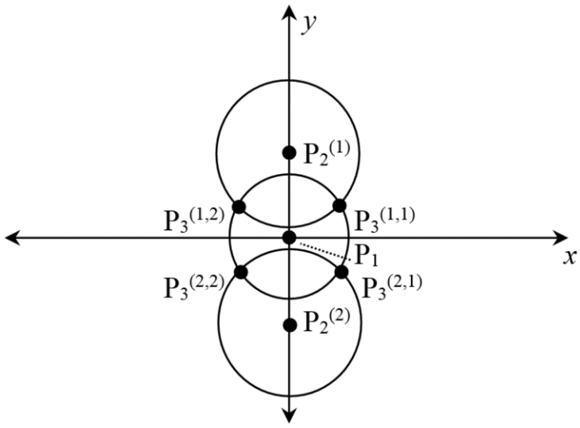

Fig. 1. The four solutions located about the origin when P_2 is placed a distance $\pm r_{1,2}$ from the origin on the y-axis for $N = 3$. Two solutions for $P_2^{(1)} = (0,y_2) = (0,+r_{1,2})$: $P_3^{(1,1)}$ and $P_3^{(1,2)}$, and two solutions for $P_2^{(2)} = (0,y_2) = (0, -r_{1,2})$: $P_3^{(2,1)}$ and $P_3^{(2,2)}$.

3.3.2 Closed-Form Solution for $N = 4$

For $N = 4$, the solution space can be found in $\Re^3$ ($K = 3$). In this case there are 6 equations with the 12 unknowns: $P_i = (x_i, y_i, z_i)$, $i = 1 \ldots 4$. The governing equations are,

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = r_{1,2}^2 \quad (27)$$

$$(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 = r_{1,3}^2 \quad (28)$$

$$(x_1 - x_4)^2 + (y_1 - y_4)^2 + (z_1 - z_4)^2 = r_{1,4}^2 \quad (29)$$

$$(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = r_{2,3}^2 \quad (30)$$

$$(x_2 - x_4)^2 + (y_2 - y_4)^2 + (z_2 - z_4)^2 = r_{2,4}^2 \quad (31)$$

$$(x_3 - x_4)^2 + (y_3 - y_4)^2 + (z_3 - z_4)^2 = r_{3,4}^2 \quad (32)$$

The 6 coordinate constraints are,

$$x_1 = 0,\ y_1 = 0,\ z_1 = 0 \quad (33)$$

$$x_2 = 0,\ y_2 = 0 \quad (34)$$

$$x_3 = 0 \quad (35)$$

This reduces the number of unknowns to 6. Applying the constraints to equations (27) – (32) produces,

$$(z_2)^2 = r_{1,2}^2 \quad (36)$$

$$(y_3)^2 + (z_3)^2 = r_{1,3}^2 \quad (37)$$

$$(x_4)^2 + (y_4)^2 + (z_4)^2 = r_{1,4}^2 \quad (38)$$

$$(y_3)^2 + (z_2 - z_3)^2 = r_{2,3}^2 \quad (39)$$

$$(x_4)^2 + (y_4)^2 + (z_2 - z_4)^2 = r_{2,4}^2 \quad (40)$$

$$(x_4)^2 + (y_3 - y_4)^2 + (z_3 - z_4)^2 = r_{3,4}^2 \quad (41)$$

Solving for the system of equations (36) – (41) by back-substitution produces,

$$z_2 = r_{1,2} \quad (42)$$

$$z_3 = \frac{1}{2z_2}(z_2^2 + r_{1,3}^2 - r_{2,3}^2) \quad (43)$$

$$y_3 = \sqrt{r_{1,3}^2 - z_3^2} \quad (44)$$

$$z_4 = \frac{1}{2z_2}(z_2^2 + r_{1,4}^2 - r_{2,4}^2) \quad (45)$$

$$y_4 = \frac{1}{2y_3}(y_3^2 - r_{3,4}^2 + r_{1,4}^2 - {z_4}^2 + (z_3 - z_4)^2) \quad (46)$$

$$x_4 = \sqrt{r_{1,4}^2 - y_4^2 - z_4^2} \quad (47)$$

Note that z_2, y_3 and x_4 each have two solutions: one negative and one positive. Hence there are 8 solutions:

1. $P_1 = (0,0,0)$, $P_2 = (0,0,z_2)$, $P_3 = (0,y_3,z_3)$, $P_4 = (x_4,y_4,z_4)$
2. $P_1 = (0,0,0)$, $P_2 = (0,0,z_2)$, $P_3 = (0,y_3,z_3)$, $P_4 = (-x_4,y_4,z_4)$
3. $P_1 = (0,0,0)$, $P_2 = (0,0,z_2)$, $P_3 = (0,-y_3,z_3)$, $P_4 = (x_4,-y_4,z_4)$
4. $P_1 = (0,0,0)$, $P_2 = (0,0,z_2)$, $P_3 = (0,-y_3,z_3)$, $P_4 = (-x_4,-y_4,z_4)$
5. $P_1 = (0,0,0)$, $P_2 = (0,0,-z_2)$, $P_3 = (0,y_3,-z_3)$, $P_4 = (x_4,y_4,-z_4)$
6. $P_1 = (0,0,0)$, $P_2 = (0,0,-z_2)$, $P_3 = (0,y_3,-z_3)$, $P_4 = (-x_4,y_4,-z_4)$
7. $P_1 = (0,0,0)$, $P_2 = (0,0,-z_2)$, $P_3 = (0,-y_3,-z_3)$, $P_4 = (x_4,-y_4,-z_4)$
8. $P_1 = (0,0,0)$, $P_2 = (0,0,-z_2)$, $P_3 = (0,-y_3,-z_3)$, $P_4 = (-x_4,-y_4,-z_4)$

Fig. 2 shows four of the possible eight solutions of placing P_2 on the z axis a distance $|z_2|$ from the origin. When $P_2^{(1)} = (0,0,+z_2)$, two solutions for P_3 are possibile: one at $P_3^{(1,1)} = (0,+y_3,+z_2)$ and the other at $P_3^{(1,2)} = (0,-y_3,+z_2)$. Each solution for P_3 also produces two solutions for P_4: $P_3^{(1,1)}$ produces $P_4^{(1,1,1)} = (+x_4,+y_4,+z_4)$ and $P_4^{(1,1,2)} = (-x_4,+y_4,+z_4)$. Alternatively, $P_3^{(1,2)}$ produces $P_4^{(1,2,1)} = (+x_4,-y_4,+z_4)$ and $P_4^{(1,2,2)} = (-x_4,-y_4,+z_4)$. The remaining 4 solutions –not shown in the figure– follow a similar argument for $P_2^{(2)} = (0,0,-z_2)$. It can be seen that each solution pair is a reflection across the z-y plane.

3.3.3 Closed-Form Solution for N = 5

For N = 5 points, the solution space can be found in $\Re^4$; K = 4. In this case there are 10 equations with 20 unknowns, $P_i = (x_i, y_i, z_i, u_i)$, i = 1, …, 5. The governing equations are,

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (u_1 - u_2)^2 = r_{1,2}^2 \quad (48)$$

$$(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 + (u_1 - u_3)^2 = r_{1,3}^2 \quad (49)$$

$$(x_1 - x_4)^2 + (y_1 - y_4)^2 + (z_1 - z_4)^2 + (u_1 - u_4)^2 = r_{1,4}^2 \quad (50)$$

$$(x_1 - x_5)^2 + (y_1 - y_5)^2 + (z_1 - z_5)^2 + (u_1 - u_5)^2 = r_{1,5}^2 \quad (51)$$

$$(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 + (u_2 - u_3)^2 = r_{2,3}^2 \quad (52)$$

$$(x_2 - x_4)^2 + (y_2 - y_4)^2 + (z_2 - z_4)^2 + (u_2 - u_4)^2 = r_{2,4}^2 \quad (53)$$

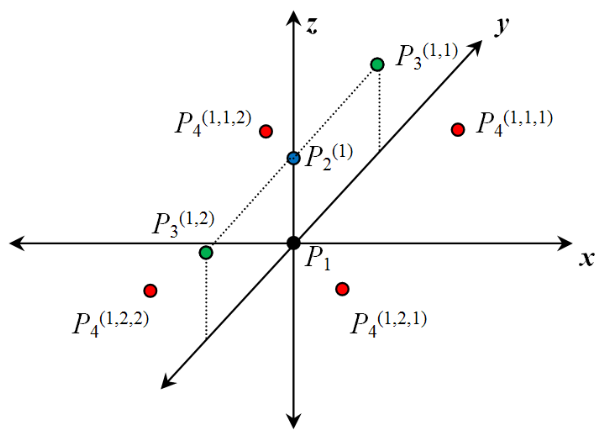

Fig. 2. The four solutions of placing P_2 a distance $+r_{1,2}$ from the origin on the y-axis for N = 4. (i) Two solutions for $P_3^{(1,1)} = (0,y_3,z_3) = (0,\ y_3,+r_{1,2})$: $P_4^{(1,1,1)}$ and $P_4^{(1,1,2)}$. (ii) Two solutions for $P_3^{(1,1)} = (0,y_3,z_3) = (0,\ -y_3,+r_{1,2})$: $P_4^{(1,2,1)}$ and $P_4^{(1,2,2)}$.

$$(x_2 - x_5)^2 + (y_2 - y_5)^2 + (z_2 - z_5)^2 + (u_2 - u_5)^2 = r_{2,5}^2 \quad (54)$$

$$(x_3 - x_4)^2 + (y_3 - y_4)^2 + (z_3 - z_4)^2 + (u_3 - u_4)^2 = r_{3,4}^2 \quad (55)$$

$$(x_3 - x_5)^2 + (y_3 - y_5)^2 + (z_3 - z_5)^2 + (u_3 - u_5)^2 = r_{3,5}^2 \quad (56)$$

$$(x_4 - x_5)^2 + (y_4 - y_5)^2 + (z_4 - z_5)^2 + (u_4 - u_5)^2 = r_{4,5}^2 \quad (57)$$

The constraints are,

$$x_1 = 0,\ y_1 = 0,\ z_1 = 0,\ u_1 = 0 \quad (58)$$

$$x_2 = 0,\ y_2 = 0,\ z_2 = 0 \quad (59)$$

$$x_3 = 0,\ y_3 = 0 \quad (60)$$

$$x_4 = 0 \quad (61)$$

This reduces the number of unknowns to 10. Applying the constraints simplifies equations (48) – (57) to,

$$u_2^2 = r_{1,2}^2 \quad (62)$$

$$(z_3)^2 + (u_3)^2 = r_{1,3}^2 \quad (63)$$

$$(y_4)^2 + (z_4)^2 + (u_4)^2 = r_{1,4}^2 \quad (64)$$

$$(x_5)^2 + (y_5)^2 + (z_5)^2 + (u_5)^2 = r_{1,5}^2 \quad (65)$$

$$(z_3)^2 + (u_2 - u_3)^2 = r_{2,3}^2 \quad (66)$$

$$(y_4)^2 + (z_4)^2 + (u_2 - u_4)^2 = r_{2,4}^2 \quad (67)$$

$$(x_5)^2 + (y_5)^2 + (z_5)^2 + (u_2 - u_5)^2 = r_{2,5}^2 \quad (68)$$

$$(y_4)^2 + (z_3 - z_4)^2 + (u_3 - u_4)^2 = r_{3,4}^2 \quad (69)$$

$$(x_5)^2 + (y_5)^2 + (z_3 - z_5)^2 + (u_3 - u_5)^2 = r_{3,5}^2 \quad (70)$$

$$(x_5)^2 + (y_4 - y_5)^2 + (z_4 - z_5)^2 + (u_4 - u_5)^2 = r_{4,5}^2 \quad (71)$$

Solving for the system of equations (62) – (71),

$$u_2 = r_{1,2} \quad (72)$$

$$u_3 = \frac{1}{2u_2}(u_2^2 + r_{1,3}^2 - r_{2,3}^2) \quad (73)$$

$$z_3 = \sqrt{r_{1,3}^2 - u_3^2} \quad (74)$$

$$u_4 = \frac{1}{2u_2}(u_2^2 + r_{1,4}^2 - r_{2,4}^2) \quad (75)$$

$$z_4 = \frac{1}{2z_3}(z_3^2 - r_{3,4}^2 + r_{1,4}^2 - {u_4}^2 + (u_3 - u_4)^2) \quad (76)$$

$$y_4 = \sqrt{r_{1,4}^2 - z_4^2 - u_4^2} \quad (77)$$

$$u_5 = \frac{1}{2u_2}(u_2^2 + r_{1,5}^2 - r_{2,5}^2) \quad (78)$$

$$z_5 = \frac{1}{2z_3}(z_3^2 - r_{3,5}^2 + r_{1,5}^2 - {u_5}^2 + (u_5 - u_3)^2) \quad (79)$$

$$y_5 = \frac{1}{2y_4}(y_4^2 - r_{4,5}^2 + r_{1,5}^2 + (u_5 - u_4)^2 - (z_5^2 - (z_5 - z_4)^2 + u_5^2)) \quad (80)$$

$$x_5 = \sqrt{r_{1,5}^2 - (y_5^2 + z_5^2 + u_5^2)} \quad (81)$$

There are 16 different solutions since u_2, z_3, y_4 and x_5 each have two solutions: one negative and one positive.

3.3 Selecting a Solution

As previously mentioned, the proposed strategy results in 2^{N-1} solutions symmetrically placed about the origin. For our specific purpose of finding a solution of the best representative image, all solutions result in the same representative image, as our particular problem is a relative one.

3.4 Singularities

Singularities can appear in the solution if the solution space is reducible, i.e. $K < N–1$ (e.g. if $z_3 = 0$ in (76)); the proposed solution strategy then breaks down. In this case, at least one of the unknowns ($x_i^{<k>}$) –other than those constrained to be zero– is also zero. Since the solution space is reduced by 1, the number of unknowns are then $N(N – 2)$. Hence, N unknowns have been eliminated and the number of freedom in constraints is also reduced by the same amount. Hence, α does not have the form as (11). As a result, this leads to a different system of non-linear equations that cannot be solved using the proposed strategy.

4 Application

In this section we give two examples on the application of the proposed method.

4.1 Example #1

As a first example, we are interested in selecting the best representative image from the five binary *Highway* images shown in Fig. 3. Clearly, a great amount of similarity exists between these images, and obtaing a best representative image for the set for matching purposes will eliminate the neeed to compare any query image to all images in the set. Image corelation is used for the similatary metric. The co-corelation distance matrix is,

$$\mathbf{r} := \begin{pmatrix} 0 & 0.477 & 0.757 & 0.776 & 0.664 \\ 0.477 & 0 & 0.567 & 0.591 & 0.606 \\ 0.757 & 0.567 & 0 & 0.833 & 0.754 \\ 0.776 & 0.591 & 0.833 & 0 & 0.775 \\ 0.664 & 0.606 & 0.754 & 0.775 & 0 \end{pmatrix}$$

Solving for the coordinates of the points in similarity space using (62) - (81), produces,

$$P_1 = (0,0,0,0)$$

$$P_2 = (0,0,0,\ 0.477)$$

$$P_3 = (0,0,\ 0.567,\ 0.501)$$

$$P_4 = (0,\ 0.590,\ 0.022,\ 0.504)$$

$$P_5 = (0.562,\ 0.109,\ 0.113,\ 0.316)$$

as one of the possible 16 solutions for the location of the points in similarity space. The center of the data is then at,

$$P_\mu = (0.112,\ 0.140,\ 0.132,\ 0.359)$$

Comparing the distance between the centroid and each point, results in P_2 as the closest point to the centroid. Hence, the second image is the best representative image of this collection of images.

4.2 Example #2

As a second example, we are interested in selecting the best representative image from the five grayscale *Cameraman* images shown in Fig. 4. The distance matrix is,

Fig. 3. Five *Highway images* with great similarity. From top to bottom I_1 to I_5.

$$r := \begin{pmatrix} 0 & 0.967 & 0.875 & 0.983 & 0.993 \\ 0.967 & 0 & 0.831 & 0.95 & 0.962 \\ 0.875 & 0.831 & 0 & 0.882 & 0.875 \\ 0.983 & 0.95 & 0.882 & 0 & 0.977 \\ 0.993 & 0.962 & 0.875 & 0.977 & 0 \end{pmatrix}$$

Solving for the coordinates of the points in similarity space and selecting one of the possible solutions produces:

$P_1 = (0, 0, 0, 0)$

$P_2 = (0, 0, 0, 0.967)$

$P_3 = (0, 0, 0.702, 0.523)$

$P_4 = (0, 0.783, 0.295, 0.516)$

$P_5 = (0.767, 0.178, 0.319, 0.514)$

The center of the data is at,

$P_\mu = (0.153, 0.192, 0.263, 0.504)$

As a result, P_3 is the closest point to the centroid. Hence, the third image is the best representative image of this set.

5 Conclusion

In this paper we have presented a method to find the best representative image from a group of images. Using a similarity metric, the distance between each pair of images is calculated. The location of the images in similarity space is then solved by solving a set of non-linear simultaneous algebraic equations. Since the equations are undetermined, we solve such a problem by applying rigid constraints forcing the solution to be in $\Re^{N-1}$, where N is the number of images in the set. We present a closed-form solution to the problem when N = 3, 4 and 7. We give examples of finding the best representative images for two sets as an application of the method.

Our research on this topic is continuous and we have been able to derive a general solution for any number of images N, which is being tested with excellent results, but not yet finalized. We hope to report on the general solution in a future article.

Fig. 4. Five *Cameraman* images of the same scene. From top to bottom I_1 to I_5.

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