Numerical Implementations for Tridiagonal Kernel Enhanced Multivariance Products Representation (TKEMPR) Method: Bivariate Case

AYLA OKAN
İstanbul Technical University
Informatics Institute
Maslak, 34469, İstanbul
TURKEY (TÜRKİYE)
ayla.okan.ozturk@gmail.com.tr

METİN DEMİRALP
İstanbul Technical University
Informatics Institute
Maslak, 34469, İstanbul
TURKEY (TÜRKİYE)
metin.demiralp@be.itu.edu.tr

Abstract: In this paper, a decomposition method Tridiagonal Kernel Enhanced Multivariance Products Representation (TKEMPR) is presented with numerical implementations. These method is based on bivariate square integrable functions over the rectangular hyperprism. This research is also can be considered as the extension to our works whose papers were given in the very recent international conferences. The contribution of this new paper is to apply the method not only to the functions that are as the product of two given univariate functions, but also to any bivariate analytical functions. A number of numerical examples are used to show the efficiency of the method given in this paper.

Key–Words: Enhanced Multivariance Products Representation (EMPR), Tridiagonal Kernel Enhanced Multivariance Products Representation (TKEMPR), Recursive EMPR, Decomposition Method.

1 Introduction

Tridiagonal Kernel Enhanced Multivariance Products Representation (TKEMPR) is based on the Enhanced Multivariance Products Representation method. For this reason, we will give the basic information of these method here.

EMPR is an expansion for a given multivariate function $f\left(x_1,...,x_N\right)$ as

$$\begin{aligned} f\left(x_1,...,x_N\right) \;=\;& f_0\prod_{i=1}^{N}s_i\left(x_i\right)+\sum_{j=1}^{N}f_j\left(x_j\right)\prod_{\substack{i=1\\i\neq j}}^{N}s_i\left(x_i\right) \\ &+\sum_{\substack{j_1,j_2=1\\j_1<j_2}}^{N}f_{j_1,j_2}\left(x_{j_1},x_{j_2}\right)\prod_{\substack{i=1\\i\neq j_1,j_2}}^{N}s_i\left(x_i\right) \\ &+\sum_{\substack{j_1,j_2,j_3=1\\j_1<j_2<j_3}}^{N}f_{j_1,j_2,j_3}\left(x_{j_1},x_{j_2},x_{j_3}\right)\prod_{\substack{i=1\\i\neq j_1,j_2,j_3}}^{N}s_i\left(x_i\right) \\ &+\cdots+f_{12\ldots N}(x_1,...,x_N) \end{aligned} \quad (1)$$

where N is the number of the independent variables. Subindexed fs are called constant, univariate, bivariate EMPR components and so on respectively [1–4]. s_is are univariate support functions.x_js, are independent variables in some interval on the real line such that

$$x_i\in\left[\,a_i,b_i\,\right],\qquad i=1,...,N \quad (2)$$

The main purpose of this method is to determine the general structure of the EMPR components. In this work, we focused on bivariate functions on the unit interval $\left[\,0,1\,\right]$. Therefore, we can rewrite the EMPR expansion for a given bivariate function $f\left(x,y\right)$ as

$$\begin{aligned} f\left(x,y\right) \;=\;& f_0u(x)v(y)+f_1(x)v(y)+f_2(y)u(x) \\ &+f_{1,2}(x,y) \end{aligned} \quad (3)$$

where u and vs are the support functions. The weight functions and support functions, used in this method, are defined as the following form

$$W_1\left(x\right)\equiv 1,\qquad \int_0^1 dxu\left(x\right)^2=1, \quad (4)$$

$$W_2\left(y\right)\equiv 1,\qquad \int_0^1 dyv\left(y\right)^2=1 \quad (5)$$

For the determination of the components the vanishing conditions should be imposed as

$$\int_0^1 dxf_1(x)u(x)=0,\quad \int_0^1 dyf_2(y)v(y)=0 \quad (6)$$

$$\int_0^1 dxf_{1,2}(x,y)u(x)=0,\int_0^1 dyf_{1,2}(x,y)v(y)=0 \quad (7)$$

These equations allow us to get the EMPR components uniquely as the following form

$$f_0 = \int_0^1 dx \int_0^1 dy f\left(x,y\right) u\left(x\right) v\left(y\right) \tag{8}$$

$$f_1\left(x\right) = \int_0^1 dy f\left(x,y\right) v\left(y\right) - f_0 u\left(x\right) \tag{9}$$

$$f_2\left(y\right) = \int_0^1 dx f\left(x,y\right) u\left(x\right) - f_0 v\left(y\right) \tag{10}$$

$$\begin{aligned} f_{1,2}\left(x,y\right) &= f\left(x,y\right) - f_0 u\left(x\right) v\left(y\right) \\ &- f_1\left(x\right) v\left(y\right) - f_2\left(y\right) u\left(x\right) \end{aligned} \tag{11}$$

2 TKEMPR Decomposition Method

In the EMPR expansion, there are 2^N componenets for a given $N-$variate function. Computation of them are very expensive task. This issue urges us to truncate EMPR expansion to low the multivairance and convert the expansion to an approximation method. But, in this work , we are going to choose another way to low the multivariane by vanishing the bivariate EMPR component, shown at the right side of the expansion given in (3).

Let us begin by briefly analysing TKEMPR method when applied to any bivariate square integrable functions on the interval $[\,0,1\,]$ [5–28]. For this reason, we need to start with constructing a recursion formula by using EMPR method.

The EMPR expansion for the target function $f(x,y)$ can be rewritten as the following form

$$\begin{aligned} f^{\,(0)}(x,y) &= \alpha_1 u_1(x) v_1(y) + \beta_1 u_2(x) v_1(y) \\ &\quad + \gamma_1 u_1(x) v_2(y) + f^{\,(1)}(x,y) \end{aligned} \tag{12}$$

such that

$$f^{\,(0)}(x,y) = f(x,y) \tag{13}$$

$$\alpha_1 = f_0, \qquad \beta_1 = \|f_1\|_x \qquad \gamma_1 = \|f_2\|_y \tag{14}$$

$$u_1(x) = u(x), \qquad v_1(y) = v(y) \tag{15}$$

$$u_2(x) = \frac{f_1(x)}{\|f_1\|_x}, \qquad v_2(y) = \frac{f_2(y)}{\|f_2\|_y} \tag{16}$$

The norm mentioned here is explicitly given below

$$\begin{aligned} \|f_1\|_x &\equiv \left(\int_0^1 dx f_1(x)^2\right)^{1/2} \\ \|f_2\|_y &\equiv \left(\int_0^1 dy f_2(y)^2\right)^{1/2} \end{aligned} \tag{17}$$

At the begining, there are only two $u_1(x)$ and $v_1(y)$ initial support functions. After using recursion, we obtain another generated two x and$y$ dependent $u_2(x)$ and $v_2(y)$ support functions. The left generated support functions, $u_1(x)$ and $u_2(x)$ are mutually orthogonal and have unit norms. Similary, the right generated support functions, $v_1(y)$ and $v_2(y)$ are also mutually orthogonal and have unit norms.

Now we can consider the following recursion which matches (12) for the j step,

$$\begin{aligned} f^{\,(j)}(x,y) &= \alpha_{j+1} u_{j+1}(x) v_{j+1}(y) \\ &\quad + \beta_{j+1} u_{j+2}(x) v_{j+1}(y) \\ &\quad + \gamma_{j+1} u_{j+1}(x) v_{j+2}(y) \\ &\quad + f^{\,(j+1)}(x,y), \quad j = 0, 1, ... \end{aligned} \tag{18}$$

where

$$f^{\,(j)}(x,y) = f_{1,2}^{(j)}(x,y), \tag{19}$$

$$\alpha_j = f_0^{(j)}, \tag{20}$$

and

$$\beta_j = \left\|f_1^{(j)}\right\|_x, \quad \gamma_j = \left\|f_2^{(j)}\right\|_y, \quad j = 1, 2, 3, ... \tag{21}$$

and generated support functions are defined as

$$u_{j+1}(x) = \frac{f_1^{(j)}(x)}{\left\|f_1^{(j)}\right\|_x}, \qquad v_{j+1}(y) = \frac{f_2^{(j)}(y)}{\left\|f_2^{(j)}\right\|_y} \tag{22}$$

This equality implies that $f^{\,(j)}(x,y)$ tends to vanish when j grows unboundedly up to infinity. This can happen only when the u_j functions and/or v_j functions form basis sets separately. If this happens then we can arrive at the following ultimate decomposition

$$\begin{aligned} f(x,y) &= \sum_{j=1}^{\infty} \alpha_j u_j(x) v_j(y) \\ &\quad + \sum_{j=1}^{\infty} \beta_j u_{j+1}(x) v_j(y) \\ &\quad + \sum_{j=1}^{\infty} \gamma_j u_j(x) v_{j+1}(y). \end{aligned} \tag{23}$$

where the univariate $u_j(x)$ functions form an infinite orthonormal function set so do the $v_j(y)$ functions separately. This equality can be rewritten in the following concise form [20–28]

$$f(x,y) = \mathbf{U}(x)^T \mathbf{K_T}\, \mathbf{V}(y) \tag{24}$$

Now, we obtain the concise matrix format like three factor matrix product whose kernel is in tridiagonal matrix form, symbolized with $\mathbf{K_T}$.

3 Numerical Implementations

This section includes discussion convergences, computational complexities of this procedure by presenting a number of numerical examples for given bivariate functions.

As a first example, $\cos(x+y)$ is chosen which is not multiplicative form.

$$\mathbf{K_T} \equiv \begin{bmatrix} 0.496751 & 0.227949 \\ 0.227949 & -0.041087 \end{bmatrix}$$

such that

$$\mathbf{U} \equiv \begin{bmatrix} 1 \\ -2.17922 + 3.69149 cos(x) - 2.01667 sin(x) \end{bmatrix},$$

$$\mathbf{V} \equiv \begin{bmatrix} 1 \\ -2.17922 + 3.69149 cos(y) - 2.01667 sin(y) \end{bmatrix},$$

In this first kind of example for TKEMPR method, the approximation is constructed by using $2x2$ matrices. In other words, recursion is used for two steps. When we also do the norm analysis here, we get the norm value as 0.000499992 in working precision 20. Red surface belong to the exact function, blue one belong to the approximated function.

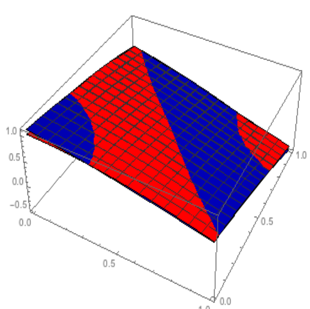

Figure 1: $cos(x+y)$, in the interval $[0,1]$, $n=2$

$$\mathbf{K_T} \equiv \begin{bmatrix} 0.496751 & 0.227949 & 0. \\ 0.227949 & -0.041087 & 0.012163 \\ 0. & 0.0121633 & -0.001016 \end{bmatrix}$$

such that

$$\mathbf{U} \equiv \begin{bmatrix} 1 \\ -2.17922 + 3.69149 cos(x) - 2.01667 sin(x) \\ -26.10199 + 23.74800 cos(x) + 13.31033 sin(x) \end{bmatrix}$$

$$\mathbf{V} \equiv \begin{bmatrix} 1 \\ -2.17922 + 3.69149 cos(y) - 2.01667 sin(y) \\ -26.10199 + 23.74800 cos(y) + 13.31033 sin(y) \end{bmatrix}$$

In the second kind of example, the approximation is constructed by using $3x3$ matrices. This means, the recursion for vanishing the residual component in the method is used for three steps. The norm value is determined as $2,90969.10^{-31}$ in working precision 20 [29–32].

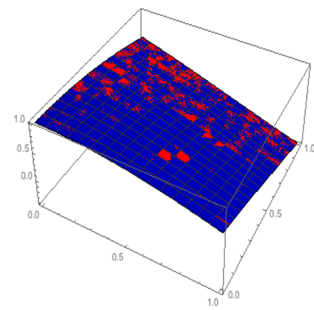

Figure 2: $cos(x+y)$, in the interval $[0,1]$, $n=2$

As a second example $(e^x + e^y)/(1+x+y)$ is chosen. Similarly, we apply the method into $2x2$ matrix product form. But in comparsion with the previous example, this function has more computational complexities which depend on its structure. So the decomposed left and right support vectors will be nonapparent when they are written. However, the kernel matrix $\mathbf{K_T}$ can be written as the following form

$$\mathbf{K_T} \equiv \begin{bmatrix} 1.72098 & 0.03487 \\ 0.03487 & 0.00477 \end{bmatrix}$$

In the figure given below, it is shown the exact function and approximated function. The norm value is determined as 0.0000378336.

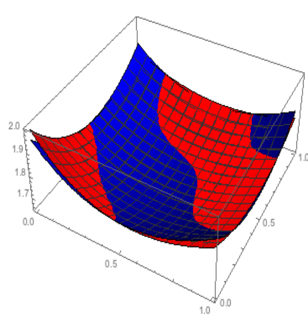

Figure 3: $(e^x + e^y)/(1+x+y)$, in the interval $[0,1]$, $n=2$

4 Concluding Remarks

The TKEMPR method can be considered as the concise matrix format like three factor matrix product whose kernel is in tridiagonal matrix form as $\mathbf{U}(x)^T\mathbf{K_T}\mathbf{V}(y)$. The results obtained here and in previous works support that the methodcan easily incorporate into any bivariate functions to approximate well. This is also can be expand to the multivariate functions. It will be discussed in the next future works. From the results shown in figures and error analysis, we can see that increasing the dimension, or namely increasing the recursion for vanishing the residual component, reduces the error of the approximation method. It is observed that there is no need for us to enhance the recursion steps to great values. However, there can be computational difficulties because of the target functions. We obtain approximation in sufficiently good quality through the method. After constructing this method into a general algorithm for all multivariate functions, the performance of the TKEMPR method can be seen in more detail.

References:

[1] B. Tunga and M. Demiralp : *The Influence of the Support Functions on the Quality of Enhanced Multivariance Product Representation* Journal of Mathematical Chemistry, Volume 48, Issue 3, pp 827-840, 2010.

[2] A. Okan, N.A. Baykara and M. Demiralp : *Weight Optimization in Enhanced Multivariance Product Representation (EMPR) Method* Int. Conf. on Numer. Anal. and Appl. Math., AIP Conference Proceedings, Volume 1281, pp. 1935-1938,Rhodes, Greece, 2010.

[3] M. A. Tunga : *Multivariate Data Modelling Through EMPR in Orthogonal Geometry* In 13th International Conference on Computational and Mathematical Methods in Science and Engineering (CMMSE 2013), pages 13421349, 24-27 June, Almeria, Spain, 2013.

[4] S. Tuna and B. Tunga : *Node Optimization Through Enhanced Multivariance Product Representation (EMPR)* In 13th International Conference on Computational and Mathematical Methods in Science and Engineering (CMMSE 2013), pages 13221330, Almeria, Spain, 24-27 June, 2013.

[5] M. Demiralp and E. Demiralp : *An Orthonormal Decomposition Method for Multidimensional Matrices* in AIP Proceedings for the International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2009), vol. 1168, Rethymno, Crete, Greece, 18-22 September 2009, pp. 424427, doi:http://dx.doi.org/10.1063/1.3241487, 2009.

[6] M. Demiralp and E. Demiralp : *Dimensionality Reduction and Approximation via Space Extension and Multilinear Array Decomposition* in AIP Proceedings for the International Conference of Computational Methods in Science and Engineering (ICCMSE 2009), Mini Symposium on Recent Developments in Numerical Schemes for Hilbert Space Related Issues in Science and Engineering, Rhodes, Greece, 29 September-4 October 2009, pp. 837840, doi:http://dx.doi.org/10.1063/1.4771824, 2009.

[7] M. Demiralp and E. Demiralp : *A New Straightforward Decomposition Method without Iteration to Approximate Matrices via Dominant Basis Matrices* in The International Conference on Scientific Computing - WorldComp09 (CSC09), Las Vegas, Nevada, USA, 13-16 July 2009, pp. 7983, 2009.

[8] B. Tunga and M. Demiralp : *An Iterative Scheme for Enhanced Multivariance Product Representation Method* in Proceedings for the 1st IEEEAM Conference on Applied Computer Science (ACS), Malta, pp. 247255, 2010.

[9] M. Demiralp : *New Generation HDMR Based Multiway Array Decomposers: Enhanced Multivariance Products Representation (EMPR)* in Proceedings for the 1st IEEEAM Conference on Applied Computer Science (ACS), ser. ISBN: 978-960-474-225-7, Malta, pp. 1616, keynote Speech, 2010.

[10] E. Demiralp and M. Demiralp : *Reductive Multilinear Array Decomposition Based Support Functions in Enhanced Multivariance Products Representation (EMPR)* in Proceedings for the 1st IEEEAM Conference on Applied Computer Science (ACS), Malta, pp. 448454, 2010.

[11] C. Gözükırmızı and M. Demiralp : *Numerical Studies on the Use of Enhanced Multivariance Product Representation as a Multiway Array Decomposer* in AIP Proceedings for the International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2010), Symposium 112, Recent Developments in Hilbert Space Tools and Methodology for Scientific Computing,

vol. 1281, Rhodes, Greece, pp. 19221925, doi:http://dx.doi.org/10.1063/1.3498300, 2010.

[12] A. Okan, ”Weight Optimization in Enhanced Multivariance Product Representation” (in Turkish), Marmara University, Mathematics Department, Master Thesis, December 2011.

[13] L. Divanyan and M. Demiralp : *Weighted Reductive Multilinear Array Decomposition* in AIP Proceedings for the 9th International Conference on Numerical Analysis and Applied Mathematics (ICNAAM2011), vol. 1389, Halkidiki, Greece, pp. 11561159, doi:http://dx.doi.org/10.1063/1.3637820, 2011.

[14] S. Tuna, N. A. Baykara, and M. Demiralp : *Weighted Singular Value Decomposition for Folded Matrices* in Proceedings of the 2nd International Conference on Applied Informatics and Computing Theory (AICT11), IEEEAM, ser. ISBN: 978-1-61804-034-3, N. Mastorakis, M. Demiralp, and N. A. Baykara, Eds., Prague, Czech Republic, pp. 7075, 2011.

[15] M. Demiralp : *Decomposing Functions, Arrays, Function Arrays* in Lecture Talk based on the symposium 48s preface, AIP Proceedings for the 9th International Conference on Numerical Analysis and Applied Mathematics (ICNAAM2011), vol. 1389, Halkidiki, Greece, pp. 11381138, doi:http://dx.doi.org/10.1063/1.3637815, 2011.

[16] M. Ayvaz and M. Demiralp : *Towards a New Multiway Array Decomposition Algorithm: Elementwise Multiway Array High Dimensional Model Representation (EMAHDMR)* in Proceedings of the 2nd International Conference on Applied Informatics and Computing Theory (AICT11), IEEEAM, ser. ISBN: 978-1-61804-034-3, N. Mastorakis, M. Demiralp, and N. A. Baykara, Eds., Prague, Czech Republic, pp. 7681, 2011.

[17] E. K. Özay : *A New Multi-way Array Decomposition via Enhanced Multivariance Product Representation*, AIP Proceedings for the 10th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM), Kos, Greece, pp. 2015-2018, Volume 1479, 2012.

[18] E. K. Özay and M. Demiralp : *A New Multi-way Array Decomposition*, 2012 SIAM Conference on Applied Linear Algebra, Valencia, Spain, pp. 78–78, 2012

[19] A.Okan, N.A. Baykara and M. Demiralp, *Parameter Optimization to Get Maximum Constancy in Constant Mobius Transformational High Dimensional Model Representation*, Proceedings of the 12th WSEAS International Conference on Systems Theory and Scientific Computation, Istanbul, Turkiye, 21-23 August 2012.

[20] E. K. Özay and M. Demiralp : *Tridiagonal Matrix Enhanced Multivariance Products Representation (TMEMPR) Studies: Decomposing the Planarly Unfolded Three-way Arrays*, Proceedings of the 14th International Conference on Computational and Mathematical Methods in Science and Engineering (CMMSE), Cadiz, Spain, 2014.

[21] A. Okan and M. Demiralp, *Tridiagonal Kernel Enhanced Multivariance Products Representation (TKEMPR) for Univariate Integral Operator Kernels*,The 2014 International Conference Mathematics and Computers in Sciences and Industry (MCSI 2014), 2014 International Conference,13-15 Sept., Varna, Bulgaria, pp. 195-200, doi: 10.1109/MCSI.2014.26, print ISBN: 978-1-4799-4744-7

[22] A. Okan and M. Demiralp, *Tridiagonal Kernel Enhanced Multivariance Products Representation (TKEMPR) for Outer Product Sums: Arrowheading EMPR for Kernel (AEMPRK)*, The 12th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2014), 22-28 September 2014, Rhodes, Greece.

[23] E. Demiralp, and M. Demiralp, *Tridiagonal Matrix Enhanced Multivariance Products Representation(TMEMPR) for Matrix Decomposition*, in Proceedings of 14th International Conference Computational and Mathematical Methods in Science and Engineering, 2014, vol. 2, pp. 446455.

[24] C. Gözükırmızı, and M. Demiralp, *The Influence of Initial Vector Selection on Tridiagonal Matrix Enhanced Multivariance Products Representation (TMEMPR)*, in The Proceedings of International Conference on Mathematics and Computers in Science and Industry, 2014.

[25] G. Özdemir,*Arrowheaded Enhanced Multivariance Products Representations for Matrices in Denumerably Infinite Matrices*, stanbul Technical University, Informatics nstitute, Master Thesis, December 2014.

[26] G. Özdemir and M. Demiralp, *Arrowheaded Enhanced Multivariance Products Representation for Matrices (AEMPRM): Specifically Focusing on Infinite Matrices and Converting Arrowheadedness to Tridiagonality*, 11th International Conference of Computational Methods in Sciences and Engineering, ICCMSE 2015, 20-23 March 2015, Athens, Greece, doi: 10.1063/1.4938939

[27] Z. Gündoğar, M. Demiralp, *Formulation of Tridiagonal Folmat Enhanced Multivariance Products Representation (TFEMPR)*, AIP Conference Proceedings, Vol. 1702, 2015,

[28] A. Okan and M. Demiralp, *Arrowheading Enhanced Multivariance Products Representation for a Kernel (AEMPRK) in a Taylor Series Expansion*, 11th International Conference of Computational Methods in Sciences and Engineering, ICCMSE 2015, 20-23 March 2015, Athens, Greece, doi: 10.1063/1.4912452

[29] M. D. Buhmann and D. Mache : *Advanced Problems in Constructive Approximation*, Springer, pages 209216 Germany, 2003.

[30] A. T. Bharucha-Reid : *Random Integral Equations*, Academic Press Inc., Volume 96, New York, 1973.

[31] F.G. Tricomi : *Integral Equations*, Courier Dover Publications, New York, 1985.

[32] M. Masujima : *Applied Mathematical Methods in Theoritical Physics*, John Wiley and Sons, Germany, 2006.