Distribution of Maximum and Minimum of K-μ-g Random Variables

DRAGANA KRSTIĆ, MIHAJLO STEFANOVIC, VESAD DOLJAK, ZORAN POPOVIC*, RADMILA GEROV
Faculty of Electronic Engineering, University of Niš
Aleksandra Medvedeva 14, 18000 Niš
*Technical College of vocational studies
Zvecan, Serbia
SERBIA
dragana.krstic@elfak.ni.ac.rs

Abstract: - In this paper, the distributions of the maximum and the minimum of κ-μ-g random variables are derived. The κ-μ-g random variable was obtained from κ-μ random variable whose power has Gamma distribution. The expressions for probability density functions (PDF) of the maximum of two κ-μ-g random variables and minimum of two κ-μ-g random variables are performed and analyzed. Using these expressions, performance analysis of wireless communication systems in the presence of κ-μ-g multipath fading and κ-μ-g cochannel interference can be made.

Key-Words: - κ-μ-g distribution; random variable; probability density function; maximum; minimum; short term fading; shadowing

1 Introduction

In wireless communications systems fading causes received signal envelope fluctuations over time [1], [2]. There are many different statistical models which describe fading distribution in channels. The most often used are Rayleigh, Rician, Nakagami-m, Weibull, Hoyt, α−μ, κ−μ, η−μ, α−κ−μ , α−η−μ. We will analyze here the κ−μ−g distribution of fading. This distribution describs composite fading model with gamma shadowed κ−μ multipath fading. The distribution κ−μ/gamma corresponds to a physical fading model [3]. This composite distribution is based on κ−μ generalized shadowed multipath fading model.

The κ-μ-g random variable can describe signal envelope in Gamma shadowed κ-μ multipath fading channels. The κ-μ-g distribution has three parameters [4]. The parameter κ is Rician factor. It is defined as the ratio of the power of direct component of signal and the power of the scattered components. The κ-μ-g fading is less severe for higher values of factor κ. The parameter μ is tied to the number of clusters in propagation environment and the κ-μ-g multipath fading is less severe for bigger values of parameter μ. The parameter c is severity of Gamma shadowing and Gamma long term fading is less severe for higher values of parameter c. The κ-μ-g distribution is general distribution and some distributions can be derived from this distribution as special cases [4].

The statistics of two random variables are very important for performance analysis of wireless mobile communication systems. The ratio of two random variables can be used in performance analysis of wireless systems operating over multipath fading channel in the presence of cochannel interference. In cellular radio interference limited environment, the ratio of signal envelope and interference envelope is important system performance [5].

Also, the ratios of random variables can be applied in performance analysis of wireless mobile communication systems which work over multipath fading environment in the presence of cochannel interference suffered to multipath fading [6].

The statistics of maximum of two random variables is often used for calculating the bit error probability (BER) of wireless communication system with selection combining (SC) receiver with two inputs, working over short term fading channel. The SC receiver output signal is equal to the maximum of signals at its inputs [7].

The statistics of minimum of two random variables is applicable in analysis of performance of wireless relay communication systems with two sections. Under determined conditions, signal envelope at output of relay system can be expressed as product of signal envelope at each section. Cumulative distribution of minimum of two random variables is used for calculation the outage probability of relay system with two sections [8].

In this paper, the analysis of maximum and minimum of two κ-μ-g random variables is presented. The probability density functions (PDF) of the maximum of two κ-μ-g random variables and minimum of two κ-μ-g random variables are calculated. The results from this work can be applied in performance analysis of wireless mobile communication systems in the presence of κ-μ-g fading.

2 The Statistics of the κ-μ-g Random Variable

2.1 Probability Density Function of κ-μ-g Random Variable

The κ-μ-g distribution describes signal envelope in Gamma shadowed κ-μ multipath fading environment. The probability density function of κ-μ-g random variable is determined by integration of conditional κ-μ distribution [3]:

$$p_x(x)=\int_0^\infty d\Omega\, p_y(x/\Omega)\,d\Omega \qquad (1)$$

where:

$$p_y(y/\Omega)=\frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}\Omega^{\frac{\mu+1}{2}}}y^{\mu}\cdot e^{-\frac{\mu(k+1)}{\Omega}y^2}\cdot I_{\mu-1}\left(2\mu_1\frac{\sqrt{k(k+1)}}{\Omega}y\right)=$$

$$=\frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}\Omega^{\frac{\mu+1}{2}}}\cdot\sum_{i_1=0}^{\infty}\left(\mu\sqrt{\frac{k(k+1)}{\Omega}}\right)^{2i_1+\mu-1}\frac{1}{i_1!\Gamma(i_1+\mu)}\cdot$$

$$\cdot y^{2i_1+2\mu-1}e^{-\frac{\mu(k+1)}{\Omega}y^2},\ y\ge 0 \qquad (2)$$

and:

$$p_\Omega(\Omega)=\frac{1}{\Gamma(c)\beta^c}\Omega^{c-1}e^{-\frac{1}{\beta}\Omega},\ \Omega\ge 0, \qquad (3)$$

Here, κ is Rician factor, μ is the number of clusters in propagation environment, $I_n(x)$ is modified Bessel function of the first kind and order n, $\Gamma(n)$ is (complete) gamma function, Ω is the signal envelope average power with Gamma distribution, $\beta=\overline{\Omega^2}$, c is Gamma fading severity parameter.

PDF of the κ-μ-g random variable may be written in the form [4]:

$$p_x(x)=\frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}\Omega^{\frac{\mu+1}{2}}}\cdot$$

$$\cdot\sum_{i_1=0}^{\infty}\left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1}\frac{1}{i_1!\Gamma(i_1+\mu)}\cdot x^{2i_1+2\mu-1}\frac{1}{\Gamma(c)\beta^c}\cdot$$

$$\cdot\left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_1}{2}-\frac{\mu}{2}}\cdot K_{c-i_1-\mu}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right). \qquad (4)$$

$K_n(x)$ is modified Bessel function of the second kind [9, eq. (3.471.9)].

2.2 Cumulative distribution function of κ-μ-g Random Variable

The cumulative distribution function (CDF) of x is [4]:

$$F_x(x)=\int_0^x dt\; p_x(t)=$$

$$=\frac{\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}}\cdot\sum_{i_1=0}^{\infty}\left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1}\frac{1}{i_1!\Gamma(i_1+\mu)}\cdot\frac{1}{\Gamma(c)\beta^c}\cdot$$

$$\cdot\frac{1}{i_1+\mu}x^{2i_1+2\mu}\cdot\sum_{j_1=0}^{\infty}\frac{1}{(i_1+\mu+1)(j_1)}\cdot\left(\mu(k+1)x^2\right)^{j_1}\cdot$$

$$\cdot\left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_1}{2}-\frac{\mu}{2}-\frac{j_1}{2}}\cdot K_{c-i_1-\mu-j_1}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right) \qquad (5)$$

2.3 Moment of n-th Order of the κ-μ-g Random Variable

The moment of n-th order of κ-μ-g random variable is [4]:

$$m_n=\overline{x^n}=\int_0^\infty dx\, x^n p_x(x)=$$

$$=\frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}}e^{k\mu}}\cdot\sum_{i_1=0}^{\infty}\left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1}\frac{1}{i_1!\Gamma(i_1+\mu)}\cdot\frac{1}{\Gamma(c)\beta^c}\cdot$$

$$\cdot\frac{1}{2}\frac{1}{\left(\mu(k+1)\right)^{i_1+\mu+\frac{n}{2}}}\Gamma\left(i_1+\mu+\frac{n}{2}\right)\cdot\beta^{c+\frac{n}{2}}\Gamma\left(c+\frac{n}{2}\right)\cdot \qquad (6)$$

2.4 Maximum of Two κ-μ-g Random Variables

The maximum of two κ-μ-g random variables x_1 and x_2 is:

$$x=\max(x_1,x_2) \qquad (7)$$

The probability density function of x is:

$$p_x(x) = p_{x_1}(x)F_{x_2}(x) + p_{x_2}(x)F_{x_1}(x) = \frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{k\mu}} \cdot$$

$$\cdot \sum_{i_1=0}^{\infty} \left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1} \frac{1}{i_1!\Gamma(i_1+\mu)} \cdot x^{2i_1+2\mu-1} \frac{1}{\Gamma(c)\beta^c} \cdot$$

$$\cdot \left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_1}{2}-\frac{\mu}{2}} \cdot K_{c-i_1-\mu}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right)$$

$$\cdot \frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{k\mu}} \cdot \sum_{i_2=0}^{\infty} \left(\mu\sqrt{k(k+1)}\right)^{2i_2+\mu-1} \frac{1}{i_2!\Gamma(i_2+\mu)} \cdot \frac{1}{\Gamma(c)\beta^c} \cdot$$

$$\cdot \frac{1}{i_2+\mu} x^{2i_2+2\mu} \sum_{j_1=0}^{\infty} \frac{1}{(i_2+\mu+1)(j_1)} \left(\mu(k+1)x^2\right)^{j_1} \cdot$$

$$\cdot \left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_2}{2}-\frac{\mu}{2}-\frac{j_1}{2}} \cdot K_{c-i_2-\mu-j_1}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right) \quad (8)$$

Cumulative distribution function (CDF) of the maximum of two κ-μ-g random variables x_1 and x_2 is:

$$F_x(x) = F_{x_1}(x) = F_{x_2}(x) =$$

$$= \frac{2\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{k\mu}} \cdot \sum_{i_1=0}^{\infty} \left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1} \frac{1}{i_1!\Gamma(i_1+\mu)} \cdot \frac{1}{\Gamma(c)\beta^c} \cdot$$

$$\cdot \frac{1}{i_1+\mu} x^{2i_1+2\mu} \sum_{j_1=0}^{\infty} \frac{1}{(i_1+\mu+1)(j_1)} \left(\mu(k+1)x^2\right)^{j_1} \cdot$$

$$\cdot \left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_2}{2}-\frac{\mu}{2}-\frac{j_1}{2}} \cdot K_{c-i_1-\mu-j_1}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right). \quad (9)$$

2.5 Minimum of Two κ-μ-g Random Variables

The minimum of two κ-μ-g random variables x_1 and x_2 is:

$$x = \min(x_1, x_2) \quad 10)$$

The probability density function of x is:

$$F_x(x) = 1 - \left(1 - F_{x_1}(x)\right)\left(1 - F_{x2}(x)\right) =$$

$$= 1 - \left(1 - F_x(x)\right)\left(1 - F_x(x)\right) = 1 - \left(1 - F_x(x)\right)^2 =$$

$$= 1 - (1 - \frac{\mu(k+1)^{\frac{\mu+1}{2}}}{k^{\frac{\mu-1}{2}} e^{k\mu}} \cdot$$

$$\cdot \sum_{i_1=0}^{\infty} \left(\mu\sqrt{k(k+1)}\right)^{2i_1+\mu-1} \frac{1}{i_1!\Gamma(i_1+\mu)} \cdot \frac{1}{\Gamma(c)\beta^c} \cdot$$

$$\cdot \frac{1}{i_1+\mu} x^{2i_1+2\mu} \cdot \sum_{j_1=0}^{\infty} \frac{1}{(i_1+\mu+1)(j_1)} \cdot \left(\mu(k+1)x^2\right)^{j_1} \cdot$$

$$\cdot \left(\mu(k+1)x^2\beta\right)^{\frac{c}{2}-\frac{i_1}{2}-\frac{\mu}{2}-\frac{j_1}{2}} \cdot K_{c-i_1-\mu-j_1}\left(2\sqrt{\frac{\mu(k+1)x^2}{\beta}}\right)\Bigg)^2 \quad (11)$$

3 Numerical Results

The maximum of two κ-μ-g random variables is shown in Figs. 1 and 2. The maximum of two κ-μ-g random variables depending on Rician factor κ_1 for $c_1=c_2=\mu_1=\mu_2=2$ and changeable parameters κ_2, β_1 and β_2 is presented in Fig. 1.

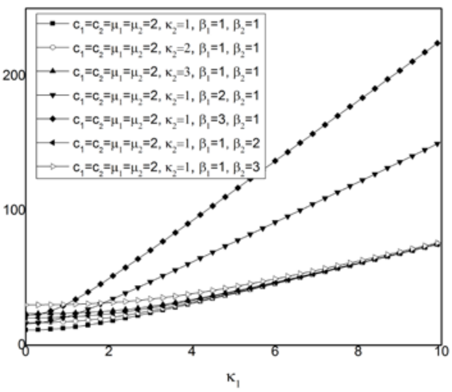

Fig.1. The maximum of two κ-μ-g random variables versus Rician factor κ_1 for $c_1= c_2=\mu_1= \mu_2=2$ and changeable parameters κ_2, β_1 and β_2.

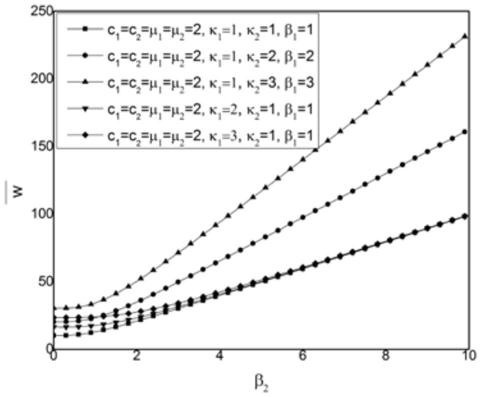

Fig. 2. The maximum of two κ-μ-g random variables depending on parameter β_2, for $c_1= c_2=\mu_1=$ $=\mu_2=2$ and variable parameters κ_1, κ_2 and β_1.

The maximum of two κ-μ-g random variables versus parameter β_2, for $c_1= c_2 =\mu_1= \mu_2=2$ and variable Rician parameters κ_1, κ_2 and parameter β_1 is plotted in Fig. 2.

It can be seen from these two figures that maximum of two κ-μ-g random variables rises with growth of Rician factor κ_1. The maximum is greater for bigger parameter β_2. Also, the maximum is higher for larger parameters β_1 and κ_2.

The minimum of two κ-μ-g random variables is presented in Figs. 3 to 6. The mean value of minimum of two κ-μ-g random variables versuse Rician factor κ_1 for $c_1=c_2 =\mu_1= \mu_2=2$, $\kappa_2=1$ and variable parameters β_1 and β_2 is shown in Fig. 3. The minimum of two κ-μ-g random variables versus Rician factor κ_2 for $c_1=c_2=\mu_1=\mu_2=2$ and changeable parameters κ_1, β_1 and β_2 is given in Fig. 4.

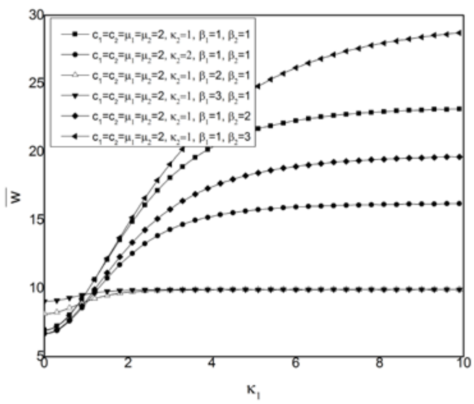

Fig. 3. Mean value of minimum of two κ-μ-g random variables depending on the Rician factor κ_1 for $c_1= c_2 =\mu_1= \mu_2=2$, $\kappa_2=1$ and variable parameters β_1 and β_2.

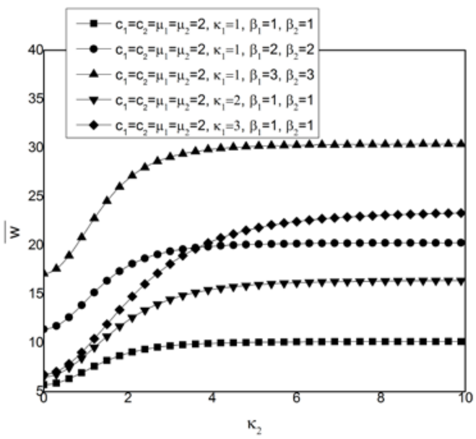

Fig. 4. Mean value of the minimum of two κ-μ-g random variables versus Rician factor κ_2 for $c_1= c_2 =\mu_1= \mu_2=2$ and changeable parameter κ_1, β_1 and β_2.

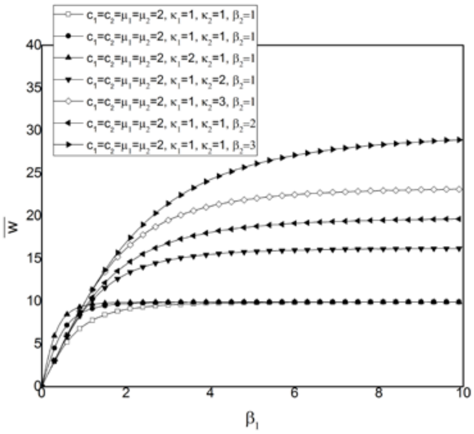

Fig. 5. Mean value of the minimum of two κ-μ-g random variables depending on the parameter β_1 for $c_1=c_2=\mu_1=\mu_2=2$, and variable parameters κ_1, κ_2, β_1 and β_2.

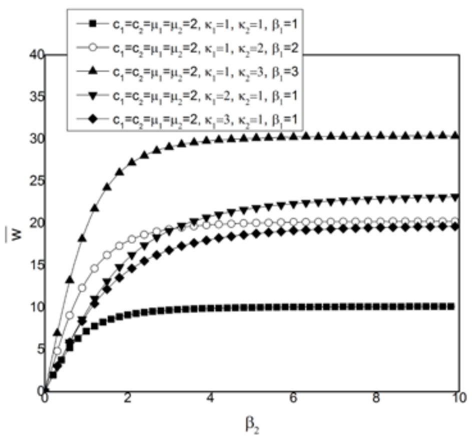

Fig. 6. Mean value of the minimum of two κ-μ-g random variables depending on the parameter β_2 for $c_1=c_2 =\mu_1=\mu_2=2$, and variable β_1 and Rician factors κ_1 and κ_2.

One can see from these two that minimum increases with enlarging of Rician factors $\kappa_1$and κ_2, but when it reaches a maximum, it stays with it for all values of Rician factors.

Mean value of the minimum of two κ-μ-g random variables depending on the parameter β_1 for $c_1=c_2=\mu_1=\mu_2=2$ and variable parameters κ_1, κ_2, β_1 and β_2 is drawn in Fig. 5. In the Fig. 6, the minimum of two κ-μ-g random variables versus parameter β_2 is presented for $c_1=c_2 =\mu_1=\mu_2=2$, and variable β_1 and Rician factors κ_1 and κ_2.

The dependence of minimum from parameters β_1 and β_2 is bigger for smaller values of these parameters. The curves achieve maximums and stay with these values for all rest values of parameters β_1 and β_2.

4 Conclusion

The κ-μ-g random variable is discussed in this paper. The κ-μ-g random variable arises from the κ-μ random variable with Gamma distributed power of κ-μ random process.

The closed form expression for probability density function and cumulative distribution function of κ-μ-g random variable are determined. The obtained expressions can be used in performance analysis of wireless communication systems operating over κ-μ multipath fading channels undergo Gamma shadowing.

In this paper, the maximum of two κ-μ-g random variables is processed. PDF and CDF of the maximum of two κ-μ-g random variables are derived. Statistics of the maximum of two κ-μ-g random variables can be used in performance analysis of wireless communication systems with SC combiner with two branches in the presence of Gamma shadowed κ-μ multipath fading.

The statistics of minimum of two random variables is also analyzed. It is necessary for performance evaluation of wireless relay communication systems with two sections. Under determined conditions, signal envelope at output of relay system can be expressed as product of signal envelope at each section. Cumulative distribution of minimum of two random variables is used for calculation the outage probability of relay system with two sections

At the last section, the influence of Rician factors and Gamma long term fading severity parameters on the maximum and minimum of two κ-μ-g random variables is analyzed. This analysis is motivated by the fact that this distribution evinces an excellent agreement to experimental fading channel conditions. An application of these results for the wireless communications community is very important.

Acknowledgment

This paper has been supported by the Ministry of Education, Science and Technological Development of Republic of Serbia under projects TR-33035 and III-44006.

References:

[1] P. M. Shankar, *Fading and Shadowing in Wireless Systems*, Springer, Dec 7, 2011. DOI 10.1007/978-1-4614-0367-8

[2] M. K. Simon, M. S. Alouini, *Digital Communication over Fading Channels*, USA: John Wiley & Sons. 2000.

[3] P. C. Sofotasios, S. Freear, On the κ-μ/Gamma Generalized Multipath/ Shadowing Fading Distribution, https://arxiv.org/pdf/1505.04186.pdf

[4] D. Krstic, M. Stefanovic, V. Doljak, Z. Popovic, E. Leitgeb, The Κ-μ-g Random Variable and the Κ-μ-g Random Process, in press

[5] D. H. Pavlović, N. M. Sekulović, G. V. Milovanović, A. S. Panajotović, M. C. Stefanović, Z. J. Popović, Statistics for Ratios of Rayleigh, Rician, Nakagami-*m*, and Weibull Distributed Random Variables, Mathematical Problems in Engineering, Vol. 2013, Article ID 252804, http://dx.doi.org/10.1155/2013/252804

[6] E. Mekić, M. Stefanović, P. Spalević, N. Sekulović, A. Stanković, Statistical analysis of ratio of random variables and its application in performance analysis of multi-hop wireless transmissions, Mathematical Problems in Engineering, Vol. 2012, Article ID 841092

[7] S. Suljović, D. Krstić, S. Maričić, S. Zdravković, V. Milenković, M. Stefanović, Level Crossing Rate of SC Receiver Over Gamma Shadowed Weibull Multipath Fading Channel, to be published towards the end of December 2016. in Tehnički vjesnik/ Technical Gazette, Vol. 23, No. 6.

[8] A. Stanković, Č. Stefanović, N. Sekulović, Z. Popović, M. Stefanović, The Distribution of Minimum of Ratios of Two Random Variables and Its Application in Analysis of Multi-hop Systems, Radioengineering, Vol. 21, No. 4, Dec. 2012.

[9] Modified Bessel Function of the Second Kind, http://mathworld.wolfram.com/ModifiedBesselFunctionoftheSecondKind.html

[10] Gauss hypergeometric function, http://mathworld.wolfram.com/HypergeometricFunction.html

[11] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 7th edition, New York: Academic, 2007.