New Exact Solutions of Wave and Heat Equations via Andualem and Khan Transform

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Abstract: Recently, Andualem and Khan introduced a new integral transform method known as AK transform (AKT) to solve partial differential equations. Therefore, in this article, we use AKT to solve some partial differential equations. More exactly, waves and heat equations are solved for new exact solutions and after transformation, the results are obtained and plotted. The two counter examples considered in this work, are quite important in terms of engineering and sciences applications.

Keyword: AK transform, Wave equation, Heat equation, Exact solution

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1. Introduction

An equation that consists of derivatives of unknown function is called a differential equation. Differential equations have applications in all areas of science and engineering. Mathematical formulation of most of the physical and engineering problems lead to differential equations. Partial differential equations are mathematical formulations of problems involving two or more independent variables. Most of the problems that arise in the real world are modelled by partial differential equations. Partial differential equations arise in all fields of sciences including Physics, Chemistry and Mathematics [1-4].

In order to solve the differential equations, the integral transform was extensively used and thus there are several works on the theory and application of integral transform such as the Laplace, Fourier, Mellin, and Hankel, Fourier Transform, Sumudu Transform, Elzaki Transform and ZZ transform Aboodh Transform[5,6, 7]. New integral transform, named as AK Transformation [8] introduce by Mulugeta Andualem and Ilyas Khan [2022], AK transform was successfully applied to fractional differential equations. In this section we will discuss the solution of we discuss the exact solution of some well-known partial differential equations with constant coefficient used in the fields of Engineering and Sciences with the help of AK transform.

AK Transform

A new transform called the AK Transform of the function $y(t)$ belonging to a class A, where:

$$A = \left\{ y(t) : \exists\, N, \eta_1, \eta_2 > 0, |y(t)| < N e^{\frac{|t|}{\eta_i}}, if\ t \in (-1)^i \times [0\ \infty) \right\}$$

The AK transform of $y(t)$ denoted by $M_i[y(t)] = \bar{y}(s,\beta)$ and is given by:

$$\bar{y}(s,\beta) = M_i\{y(t)\} = s \int_0^{\infty} y(t) e^{-\frac{s}{\beta}t}\, dt \qquad (1)$$

Table 1: AK Transform of Some Basic functions

$f(t)$	$M_i(f(t))$
C (constant)	$C\beta$
t^n	$\Gamma(n+1)\frac{\beta^{n+1}}{s^n}$
$e^{\lambda t}$	$\frac{s\beta}{s-\lambda\beta}$
$\cos t$	$\frac{s^2\beta}{\beta^2+s^2}$

To obtain AK transform of partial derivatives we use integration by parts as follows:

$$M_i\left[\frac{\partial f(x,t)}{\partial t}\right] = s\int_0^\infty \frac{\partial f(x,t)}{\partial t}e^{\frac{-s}{\beta}t}dt$$

$$= s\lim_{\eta\to\infty}\int_0^\eta e^{\frac{-s}{\beta}t}\frac{\partial f(x,t)}{\partial t}dt$$

$$= \lim_{\eta\to\infty}\left(s[e^{\frac{-s}{\beta}t}f(x,t]_0^\eta + \frac{s^2}{\beta}\int_0^\infty e^{\frac{-s}{\beta}t}f(x,t)dt\right)$$

$$= -sf(x,0) + \frac{s}{\beta}\bar{f}(x,s,\beta)$$

$$= \frac{s}{\beta}\bar{f}(x,s,\beta) - sf(x,0) \quad (2)$$

Next, to find $M_i\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]$, let $\frac{\partial f(x,t)}{\partial t} = l(x,t)$ then by using equation (2) we have

$$M_i\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = M_i\left[\frac{\partial l(x,t)}{\partial t}\right]$$

$$= M_i\left[\frac{\partial l(x,t)}{\partial t}\right] - sl(x,0)$$

$$= \frac{s}{\beta}\left(\frac{s}{\beta}\bar{f}(x,s,\beta) - sf(x,0)\right) - s\frac{\partial f(x,0)}{\partial t}$$

$$= \frac{s^2}{\beta^2}\bar{s}(x,s,\beta) - s\frac{\partial f(x,0)}{\partial t} - \frac{s^2}{\beta}f(x,0)$$

2. Application of AK Transform for Partial Differential Equations

AK transform is applicable to solve many real life problems. Especially it is useful to solve initial-value problems and fractional differential equations. Here we can see some applications of AK transform for the solution of some well-known partial differential equations with constant coefficient used in the fields of Engineering and Sciences.

Example 1: Find the solution of the first - order initial value problem

$$\frac{\partial f(x,t)}{\partial x} - 2\frac{\partial f(x,t)}{\partial t} = f(x,t), \quad x > 0, \quad t > 0 \quad f(x,0) = e^{-3x} \quad (3)$$

Solution: Taking the AK transform on both sides of Equation of (3) with respect to t, leads to

$$M_i\left[\frac{\partial f(x,t)}{\partial x} - 2\frac{\partial f(x,t)}{\partial t} = f(x,t)\right]$$

Using the differentiation property of AK transform we get

$$\bar{f}'(x,s,\beta) - 2\left(\frac{s}{\beta}\bar{f}(x,s,\beta) - sf(x,0)\right) = \bar{f}(x,s,\beta)$$

Where $\bar{f}(x,s,\beta)$is the AK transform of $f(x,t)$

Substituting the given initial condition, we get

$$\bar{f}'(x,s,\beta) - 2\left(\frac{s}{\beta}\bar{f}(x,s,\beta) - se^{-3x}\right) = \bar{f}(x,s,\beta)$$

$$\Rightarrow \bar{f}'(x,s,\beta) - \left(2\frac{s}{\beta}+1\right)\bar{f}(x,s,\beta) = -2se^{-3x} \quad (4)$$

If you consider equation (4), which is linear first order differential equation of the form

$$y' + g(x)y = r(x)$$

This can be solved using the method of integrating factor

$$\mu(x) = e^{\int g(x)dx} = e^{-\int\left(2\frac{s}{\beta}+1\right)dx} = e^{-\left(2\frac{s}{\beta}+1\right)x}$$

Where, $\mu(x)$ is integrating factor

Now, multiplying equation (4) by integrating factor we obtain

$$e^{-\left(2\frac{s}{\beta}+1\right)x}\bar{f}'(x,s,\beta) - \left(2\frac{s}{\beta}+1\right)e^{-\left(2\frac{s}{\beta}+1\right)x}\bar{f}(x,s,\beta) = -2se^{-3x}e^{-\left(2\frac{s}{\beta}+1\right)x}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-\left(2\frac{s}{\beta}+1\right)x}\bar{f}(x,s,\beta)\right) = -e^{-\left(2\frac{s}{\beta}+1\right)x}.2se^{-3x}$$

If you integrate the above result with respect to x and after simple calculation, we get

$$\bar{f}(x,s,\beta) = \frac{s\beta}{s+2\beta}e^{-3x}$$

Since, the solution can be written in the form of $f(x,t)$ and also $\bar{f}(x,s,\beta)$ is the AK transform of $f(x,t)$.

Therefore, Taking the inverse AK transform of the above result to obtain the solution in the form of $f(x,t)$.

$$\Rightarrow f(x,t) = M_i^{-1}\left[\bar{f}(x,s,\beta)\right]$$

Where M_i^{-1} is inverse of AK transform

$$f(x,t) = M_i^{-1}\left(\frac{s\beta}{s+2\beta}e^{-3x}\right) = e^{-3x}M_i^{-1}\left(\frac{s\beta}{s+2\beta}\right)$$

$$f(x,t) = e^{-3x}.e^{-2t} = e^{-3x-2t}$$

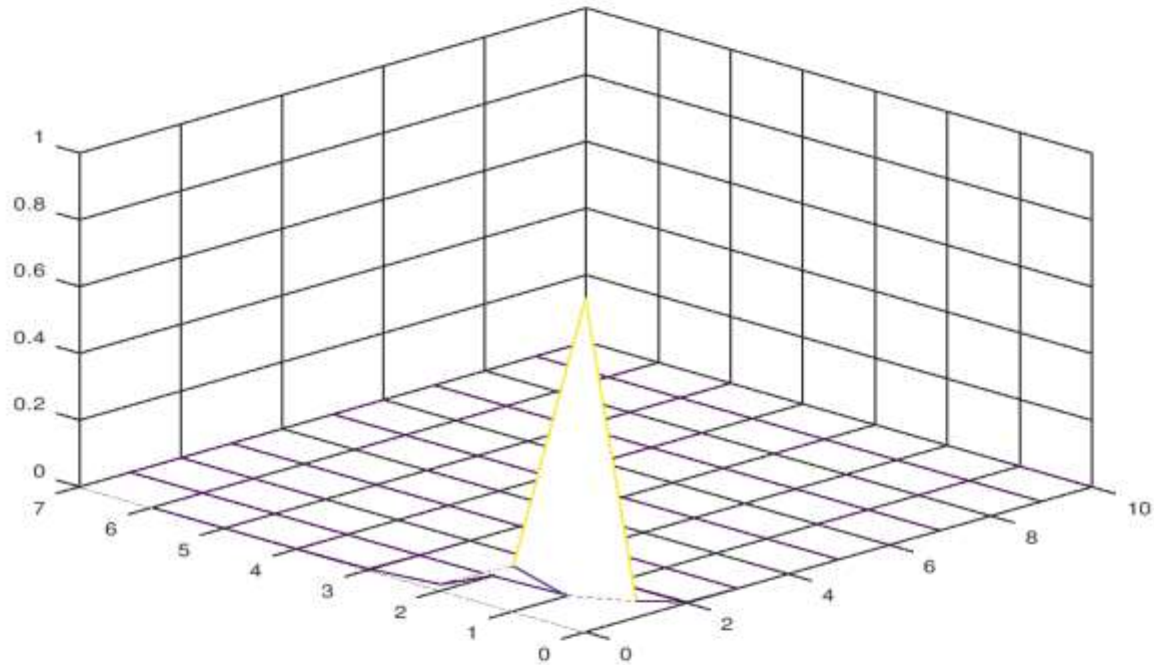
One may readily check that this is indeed the solution to the initial value problem.

$$\Rightarrow f(x,0) = e^{-3x-0} = e^{-3x}$$

Therefore, the solution of equation (3) is

$$f(x,t) = e^{-3x-2t}, \quad x > 0, \ t > 0$$

Figure 1: 3D analytic solution of equation 1.4 in the interval 0<x<10, and 0<t<7



Example 2: Consider initial/boundary value problem of heat equation

$$u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 2, \quad t > 0 \tag{5}$$

$$u(0,t) = 0 = u(2,t)$$

$$u(x,0) = 3\sin(2\pi x)$$

Solution: Applying the AK transform on both sides of Equation of (5) with respect to t, we get

$$\frac{s}{\beta}\bar{u}(x,s,\beta) - su(x,0) = \bar{u}''(x,s,\beta)$$

Now, apply the given initial condition

$$\frac{s}{\beta}\bar{u}(x,s,\beta) - 3s\sin(2\pi x) = \bar{u}''(x,s,\beta) \tag{6}$$

Equation (6) can be also rewritten as non-homogeneous, second order linear constant coefficient equation.

$$\bar{u}''(x,s,\beta) - \frac{s}{\beta}\bar{u}(x,s,\beta) = -3\frac{s}{\beta}\sin(2\pi x) \tag{7}$$

The general solution of the non-homogeneous equation is given by the sum of the homogeneous solution and the particular solution.

Now, first let us find the general solution of the homogeneous part, which means the general solution of

$$\bar{u}''(x,s,\beta) - \frac{s}{\beta}\bar{u}(x,s,\beta) = 0$$

Suppose the solution is $\bar{u}_h(x,s,\beta) = e^{rx} \Longrightarrow \bar{u}_h{}''(x,s,\beta) = r^2e^{rx}$

If we substitute this result to the homogeneous equation, we have

$$r^2e^{rx} - \frac{s}{\beta}e^{rx} = 0 \Longrightarrow e^{rx}\left(r^2 - \frac{s}{\beta}\right) = 0$$

Since, for any x and $r, e^{rx} \neq 0$. So, we have $r = \pm\sqrt{\frac{s}{\beta}}$

Therefore, the general solution of the homogeneous problem is

$$\bar{u}_h(x,s,\beta) = c_1 e^{x\sqrt{\frac{s}{\beta}}} + c_2 e^{-x\sqrt{\frac{s}{\beta}}}$$

With the associated conditions

$$\bar{u}(0,s,\beta) = 0 = \bar{u}(2,s,\beta)$$

In order to find the constants c_1 and c_2 we use the boundary conditions

$$0 = \bar{u}_h(x,s,\beta) = c_1 + c_2 \Rightarrow c_1 = -c_2 \qquad (*)$$

$$\bar{u}_h(2,s,\beta) = 0 = c_1 e^{2\sqrt{\frac{s}{\beta}}} + c_2 e^{-2\sqrt{\frac{s}{\beta}}} \qquad (**)$$

Now substitute the value of c_1 in to (**) that we obtain from (*)

$$0 = -c_2 e^{2\sqrt{\frac{s}{\beta}}} + c_2 e^{-2\sqrt{\frac{s}{\beta}}}$$

$$\Rightarrow c_2\left[e^{-2\sqrt{\frac{s}{\beta}}} - e^{2\sqrt{\frac{s}{\beta}}}\right] = 0$$

$$\Rightarrow e^{-2\sqrt{\frac{s}{\beta}}} - e^{2\sqrt{\frac{s}{\beta}}} = 0 \text{ or } c_2$$

Since $e^{-2\sqrt{\frac{s}{\beta}}} - e^{2\sqrt{\frac{s}{\beta}}} \neq 0$. Therefore the value of $c_2 = 0$, consequently the value of $c_1 = 0$

Next, we find the particular solution of non-homogeneous problem, the particular solution of such non-homogeneous problem is given by $A\cos(bx) + B\sin(bx)$. When we come to our problem, we have

$$\bar{u}_p(x,s,\beta) = A\cos(2\pi x) + B\sin(2\pi x)$$

In order to find the constant A and B, we use the method of undetermined coefficients

$$\Rightarrow \frac{d}{dx}\bar{u}_p(x,s,\beta) = -2\pi A\sin(2\pi x) + 2\pi B\cos(2\pi x)$$

$$\frac{d^2}{dx^2}\bar{u}_p(x,s,\beta) = -4\pi^2 A\cos(2\pi x) - 4\pi^2 B\sin(2\pi x)$$

Therefore

$$\bar{u}''_p(x,s,\beta) - \frac{s}{\beta}\bar{u}_p(x,s,\beta) = -4\pi^2 A\cos(2\pi x) - 4\pi^2 B\sin(2\pi x) - \frac{s}{\beta}[A\cos(2\pi x) + B\sin(2\pi x)]$$

$$= \left(-4\pi^2 - \frac{s}{\beta}\right)[A\cos(2\pi x) + B\sin(2\pi x)] = -3\frac{s}{\beta}\sin(2\pi x)$$

From the above result we conclude that

$$\left(-4\pi^2 - \frac{s}{\beta}\right)A = 0 \Rightarrow A = 0 \text{ and } \left(-4\pi^2 - \frac{s}{\beta}\right)\mathrm{B} = -3\frac{s}{\beta}$$

$$\Rightarrow B = 3\frac{s}{\beta}\left(\frac{\beta}{(4\beta\pi^2 + s)}\right) \Rightarrow B = \frac{3s}{s + 4\pi^2\beta}$$

Hence the particular solution is

$$\bar{u}_p(x,s,\beta) = \frac{3s}{s + 4\pi^2\beta}\sin(2\pi x)$$

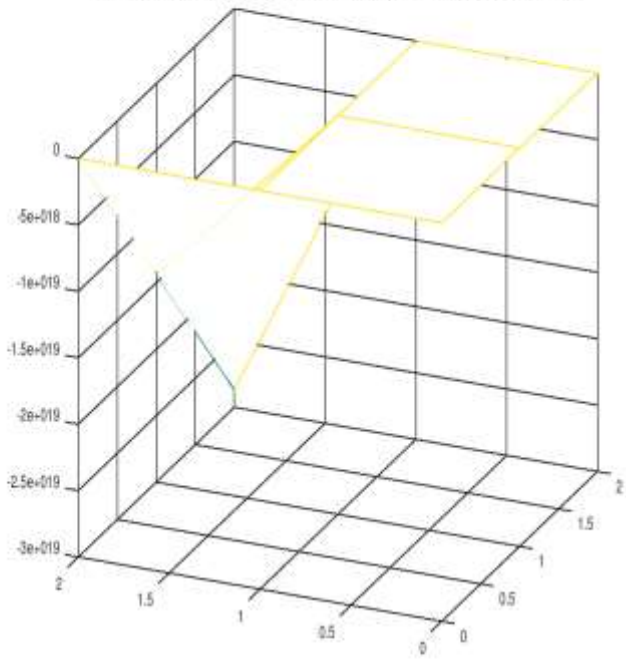
Apply inverse of AK transform of the above result to obtain the solution

$$M_i^{-1}[\bar{u}_p(x,s,\beta)] = M_i^{-1}\left(\frac{3s}{s + 4\pi^2\beta}\sin(2\pi x)\right)$$

$$u(x,t) = \sin(2\pi x)\, 3M_i^{-1}\left(\frac{s}{s - (-4\pi^2\beta)}\right)$$

$$= \sin(2\pi x).\, 3e^{-4\pi^2 t}$$

Figure 2: 3D analytical solution of equation 1.6 in the interval 0<x<2, and 0<t<2



Example 3: Find the solution of initial/boundary value problem

$$u_{tt}(x,t) = u_{xx}(x,t), 0 < x < \pi, \quad t > 0 \qquad 1.9$$

With boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

And initial conditions

$$u(x,0) = \sin x, \;\; u_t(x,0) = 0,$$

Solution: First applying the AK transform on both sides of Equation of (1.9), we get

$$\frac{s^2}{\beta^2}\bar{u}(x,s,\beta) - s\frac{\partial u(x,0)}{\partial t} - \frac{s^2}{\beta}u(x,0) = \bar{u}''(x,s,\beta)$$

Using initial conditions, we get

$$\frac{s^2}{\beta^2}\bar{u}(x,s,\beta) - \frac{s^2}{\beta}\sin x = \bar{u}''(x,s,\beta)$$

$$\Rightarrow \bar{u}''(x,s,\beta) - \frac{s^2}{\beta^2}\bar{u}(x,s,\beta) = -\frac{s^2}{\beta}\sin x \qquad 2.0$$

If we look equation 2.0, which is second order linear non-homogeneous differential equation. A general solution of the non-homogeneous equation is given by in the form

$$\bar{u}(x,s,\beta) = \bar{u}_h(x,s,\beta) + \bar{u}_p(x,s,\beta)$$

Now, first let us find the general solution of the homogeneous part

$$\bar{u}''(x,s,\beta) - \frac{s^2}{\beta^2}u(x,s,\beta) = 0$$

Suppose the solution is $\bar{u}_h(x,s,\beta) = e^{rx} \Rightarrow \bar{u}_h''(x,s,\beta) = r^2 e^{rx}$

If we substitute the above result to the homogeneous equation, we have

$$r^2 e^{rx} - \frac{s^2}{\beta^2}e^{rx} = 0$$

Since for any $x, e^{rx} \neq 0$. We have $\left(r - \frac{s}{\beta}\right)\left(r + \frac{s}{\beta}\right) = 0 \Rightarrow r = -\frac{s}{\beta},\; r = \frac{s}{\beta}$

Therefore, the general solution of the homogeneous problem is

$$\bar{u}_h(x,s,\beta) = c_1 e^{\frac{s}{\beta}x} + c_2 e^{-\frac{s}{\beta}x}$$

In order to find the constants c_1 and c_2 we use the boundary conditions.

So, we get

$$\bar{u}(0,s,\beta) = 0 \Rightarrow c_1 + c_2 = 0 \qquad 2.1$$

And also

$$0 = \bar{u}(\pi,s,\beta) \Rightarrow c_1 e^{\frac{s}{\beta}\pi} + c_2 e^{-\frac{s}{\beta}\pi} = 0 \qquad 2.2$$

From the two result (2.1 and 2.2), we have $c_1 = c_2 = 0$

Assume the particular solution is given by:

$$\bar{u}_p(x,s,\beta) = A\cos(x) + B\sin(x)$$

In order to find the constant A and B, we use the method of undetermined coefficients

$$\Rightarrow \frac{d}{dx}\bar{u}_p(x,s,\beta) = -A\sin(x) + B\cos(x)$$

$$\frac{d^2}{dx^2}\bar{u}_p(x,s,\beta) = -A\cos(x) - B\sin(x)$$

$$\Rightarrow \bar{u}_p{}''(x,s,\beta) - \frac{s^2}{\beta^2}\bar{u}_p(x,s,\beta) = -A\cos(x) - B\sin(x) - \frac{s^2}{\beta^2}(A\cos(x) + B\sin(x))$$

$$= \left[-1 - \frac{s^2}{\beta^2}\right](A\cos(x) + B\sin(x)$$

$$= -\frac{s^2}{\beta}\sin x$$

From the above result we conclude that

$$\left[-1 - \frac{s^2}{\beta^2}\right]A = 0 \Rightarrow A = 0 \text{ and, } \left[-1 - \frac{s^2}{\beta^2}\right]B = -\frac{s^2}{\beta} \Rightarrow B = \frac{s^2\beta}{s^2+\beta^2}$$

Therefore, the general solution $\bar{u}(x,s,\beta) = \bar{u}_h(x,s,\beta) + \bar{u}_p(x,s,\beta)$ is equal to

$$\Rightarrow \bar{u}(x,s,\beta) = \frac{s^2\beta}{s^2+\beta^2}\sin(x) \quad (2.3)$$

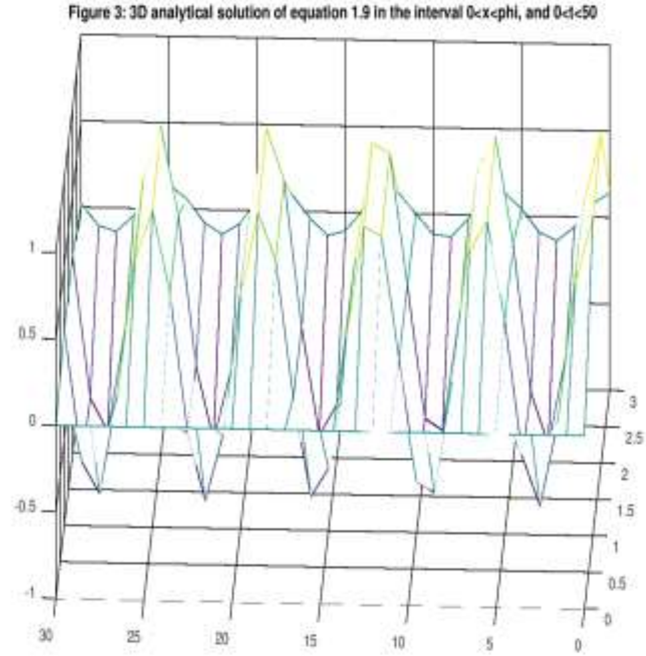
Now in order to find the solution of equation (1.9), applying the inverse AK transform on both sides of equation (2.3)

$$\Rightarrow M_i{}^{-1}[\bar{u}(x,s,\beta)] = M_i{}^{-1}\left(\frac{s^2\beta}{s^2+\beta^2}\sin(x)\right)$$

$$u(x,t) = \sin(x)\, M_i{}^{-1}\left(\frac{s^2\beta}{s^2+\beta^2}\right)$$

$$= \sin(x)\cos(t)$$

Figure 3: 3D analytical solution of equation 1.9 in the interval 0<x<phi, and 0<t<50



3. Conclusion

Authors successfully discussed the use of AK Transform for the solution of partial differential equations. We successfully found an exact solution in all the examples and the result gives a guaranty for AK transform, which plays a great role in finding exact solution of initial and boundary value problems of partial differential equations. AK can also be applied to similar nonlinear problems in future.

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