Approximate solution for magnetohydrodynamics fluid flow between two circular discs

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Abstract: This study is related to the unsteady magnetohydrodynamic (MHD) squeezing flow of a viscous incompressible fluid between two parallel discs one of which is fixed and permeable. The flow is assumed 2-D and the effects of heat transfer are taken into account. The governing Navier Stokes and heat equation given in cylindrical coordinates with the help of the similarity transformations are reduced into a system of coupled non-linear ordinary differential equations. Then, the solution of the resulting system is obtained in the form of truncated Taylor series where the coefficients are evaluated by using differential transform method, namely DTM. This method is a generalized algebraic way for obtaining Taylor series coefficients of a smooth function and that reduces the computational cost. Here, we have investigated the flow properties in terms of suction/injection coefficient as well as the other flow parameters. The illustrative examples show that the present results are very consistent with existing ones.

Magnetohydrodynamic Fluid, Navier Stokes Equations, Heat Equation, Differential Transform Method: Typing manuscripts, LaTeX

1 Introduction

The flow of a fluid between parallel plates, which move symmetrically towards to each other, produces squeezing flow. Under the influence of Magneto-Hydro-Dynamics (MHD) fields, such squeezing flow problems are suitable for thrust bearing applications of liquid-metal lubrication. Since it prevents the unexpected change of viscosity of the lubricant using MHD fluid as a lubricant for many industrial applications is meaningful [1].

The laminar flow through channel with porous walls problem was first considered by [2],[3], [4] and [5] for low Reynolds number and higher Reynolds number. Additionally, effects of a magnetic field in lubrication through a channel were first investigated in [6],[7]. In [6], Hughes and Elco investigated the electrically conducting, incompressible viscous fluid flow between two parallel discs in the presence of magnetic field and one of the discs rotates at angular velocity. In [7], magnetohydrodynamic squeezed films have been both studied experimentally and theoretically and theory of MHD was applied to squeezed films between circular plates. This study was extended to include liquid-inertia effects and lift forces.

The flow between two rotating discs, considering the magnetic field effects, was studied in [8], where the magnetic field is applied perpendicular to the discs. In his study, Hamza also investigated the effects of the magnetic field and centrifugal inertial forces on the velocity and the load capacity and the torques. Later, S. Bhattacharyya ([9]) studied the motion of a conductive viscous fluid film between two parallel discs. He assumed that the lower disc rotates at arbitrary angular velocity and he searched uniform axial magnetic field effects on the flow.

Heat transfer in machines with fast moving motors and lubricants inside is an active research area because such hydrodynamic machines are used for loading of mechanical components, liquid metal lubrication bearings and squeezed films in power transmission. From all this point of view, in [10], two dimensional, viscous and unsteady MHD flow between two infinite parallel plates was considered. Two parallel plates which moves symmetrically towards to each other or away from each others cause squeezing flow. Domairry, G. Aziz examined the MHD squeezing flow between two parallel infinite discs. Here, one of the discs is impermeable and the other is porous by suction or injection. The combined effects of inertia, electromagnetic forces and suction or injection have been investigated in [11]. Joneidi, A.A. Domairry, G. Babaelahi, M. ([12]) examined the magnetohydrodynamic squeezing flow between two parallel discs by increasing the porosity of the discs. Moreover, they considered the suction or injection of the permeable wall, as well as the Reynolds and Hartmann numbers effects on velocity profiles. In [14], heat transfer effects on squeezing flow between parallel discs were analyzed and additional details which related to the subject can be obtained from [14]. The present work is devoted to seven sections. In section 2, description and mathematical formulations of the problem are given and by using the similarity transformation, partial differential equations are reduced to a couple of ordinary differential equations. In section 3, we give fundamental properties of DTM and in section 4, the solutions of the ordinary differential equations are obtained as Taylor series, in which the series coefficients are computed algebraically without differentiating the function. In section 5, we mention about the error estimes and Chapter 6 illustrates the figures in terms of flow parameters and the present solution technique matches with the results of [14]. The last section is the Conclusion.

2 Problem statement and mathematical formulation

Here, we consider the unsteady MHD flow of an incompressible viscous fluid flows through two parallel infinite discs with the presence of the heat transfer effects. The distance between the discs as a function of time is $h(t) = H(1-at)^{1/2}$ (see Figure 1). Here, H denotes the distance between discs at $t = 0$ and a defines a constants. It is assumed that the magnetic field, $B_0(1-at)^{1/2}$, is employed normal to the discs. T_w and T_h indicate the constant temperatures of the lower and upper disc respectively. At $z = h(t)$, the upper impermeable disc moves toward or away from the lower permeable disc with the velocity $\frac{aH(1-at)^{-1/2}}{2}$. For the sake of simplicity, cylindrical coordinates, (r, ϕ, z) , can be used and due to the rotational symmetry of the flow $(\partial/\partial\phi = 0)$ the azimuthal velocity v of $V = (u, v, w)$ disappears. Hence, the governing

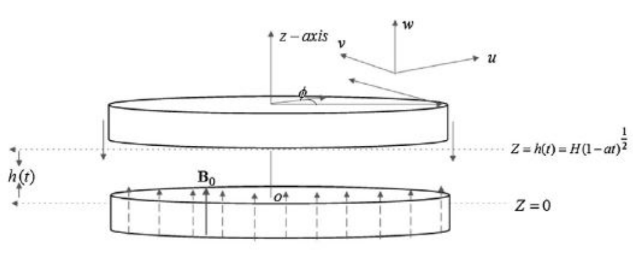


Figure 1: Flow configuration of the flow.

equations can be written as in [13],[14]as follows,

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial \omega}{\partial z} = 0, \tag{1}$$

$$\begin{aligned} \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \omega\frac{\partial u}{\partial z}) &= -\frac{\partial P}{\partial r} + \mu(\frac{\partial^2 u}{\partial r^2} \\ &+ \frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}) - \frac{\sigma}{\rho}B^2 u \quad (2) \\ \rho(\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial r} + \omega\frac{\partial \omega}{\partial z}) &= -\frac{\partial P}{\partial z} + \mu(\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} \\ &+ \frac{1}{r}\frac{\partial \omega}{\partial r}) \quad (3) \\ C_p(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \omega\frac{\partial T}{\partial z}) &= \frac{K_0}{\rho}(\frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{r}\frac{\partial T}{\partial r} - \frac{u}{r^2}) \\ &+ \vartheta(2\frac{u^2}{r^2} + (\frac{\partial u}{\partial z})^2 + 2(\frac{\partial \omega}{\partial z})^2 \quad (4) \\ &+ 2(\frac{\partial u}{\partial r})^2 + 2(\frac{\partial \omega}{\partial r})^2 + 2\frac{\partial u}{\partial z}\frac{\partial \omega}{\partial r}). \end{aligned}$$

Boundary conditions are:

$$\text{At} \qquad z = h(t), u = 0, w = \frac{dh}{dt}, T = T_h \tag{5}$$

$$\text{At} \qquad z = 0, u = 0, w = w_0, T = T_w \tag{6}$$

where u and w are the velocity components in the r and z direction respectively, μ is the kinematic viscosity, P is the pressure and ρ is the density of the fluid. Apart from these, T denotes temperature, T_0 is the thermal conductivity, ϑ is the kinematic viscosity and w_0 is the injection/suction velocity. If we use the following transformations

$$\begin{aligned} u &= \frac{ar}{2(1-at)}f'(\eta), \omega = -\frac{ah}{(1-at)^{\frac{1}{2}}}f(\eta), \\ B(t) &= \frac{B_0}{(1-at)^{\frac{1}{2}}}, \eta = \frac{z}{H(1-at)^{\frac{1}{2}}}, \quad (7) \\ \theta &= \frac{T - T_h}{T_\omega - T_h}. \end{aligned}$$

Then, Eqs.(1)-(5) are reduced to

$$\begin{aligned} & f'''' - S(\eta f''' + 3f'' - 2ff'''^2 f'' = 0, \quad (8) \\ & \theta'' + S.Pr(2f\theta' - \eta\theta') + Pr.Ec(f''^2 \\ + \quad & 12\chi^2 f'^2) = 0. \quad (9) \end{aligned}$$

In Eqs.(8)-(9), $S = \frac{aH^2}{2\vartheta}$, $M = \sqrt{\frac{\sigma B_0^2 H^2}{\vartheta}}$, $Pr = \frac{\mu C_p}{K_0}$, $Ec = \frac{1}{C_p(T_\omega - T_h)}(\frac{ar}{2(1-at)})^2$, denote the squeezing, Hartman, Prandtl and Eckert number respectively. Moreover, $\chi = \sqrt{\frac{H^2(1-at)}{r^2}}$ is the non-dimensionalized length. Hence, boundary conditions can be rewritten as follows,

$$f(0) = A, f'(0) = 0, \theta(0) = 1 \tag{10}$$

$$f(1) = \frac{1}{2}, f'(1) = 0, \theta(1) = 0, \tag{11}$$

where A is the permeability of the lower disc which is related to suction/injection cases.

3 Approximation by DTM

In this section, we will look for Taylor series solutions for Eqs.(8)-(9) in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{12}$$

where a_n 's are Taylor series coefficients and x_0 is taken as zero for simplicity. To compute the series coefficients where we will use differential transform method (DTM), which is an iterative way for obtaining the series coefficients without differentiating the function.

With this method, given differential equation or system of equations together with the initial conditions (including with boundary condition(s)) are transformed into an algebraic system of equations which gives the recurrence relation for the coefficients. As a result, the succesive coefficients are obtained in terms of the previous ones. This method is an useful tool for solving linear/non-linear, ordinary and partial differential equations as well as the system of such equations. It does not require any linearization or perturbation and usually large computational work is no needed.

The basic definitions of DTM can be found in the references [15, 16, 17] and some properties of DTM are given here as follows:

Definition 1 *If $f(\eta)$ is an analytic function in the domain $\Omega = [0, T]$ then, this function is continuously differentiable with respect to η :*

$$\frac{d^k f(\eta)}{d\eta^k} = \phi(\eta, k), \forall \eta \in \Omega. \tag{13}$$

For any point $\eta = \eta_i$ in $[0, T]$, the function ϕ is defined as $\phi(\eta, k) = \phi(\eta_i, k)$, where $k \in N$ and N_0, denotes non-negative integers. Therefore, Eq.(13) is written as

$$F(k) = \phi(\eta_i, k) = (\frac{d^k f(\eta)}{d\eta^k})_{\eta=\eta_i}, \forall k \in N_0, \tag{14}$$

where $F(k)$ is called the differential transform of $f(\eta)$ and $F(k) \in R^{nxn}$.

Definition 2 *If $f(\eta)$ can be expressed as a Taylor series about fixed point η_i, then,*

$$f(\eta) = \sum_{k=0}^{\infty} \frac{f^k(\eta_i)}{k!} (\eta - \eta_i)^k. \tag{15}$$

If $f_n(\eta)$ is the n-partial sums of Taylor series, Eq.(15), then

$$f_n(\eta) = \sum_{k=0}^{n} \frac{f^k(\eta_i)}{k!} (\eta - \eta_i), \tag{16}$$

where $f_n(\eta)$ is also called nth Taylor polynomial for $f(\eta)$ about η_i and $R_{n+1}(\eta)$ is the remainder term. By using Eq.(14), Eq.(15) is written as follows;

$$f(\eta) = \sum_{k=0}^{n} F(k)(\eta - \eta_i)^k + R_{n+1}(\eta). \tag{17}$$

Assuming that $\eta_i = 0$, and remainder term approaches zero for sufficiently large n, then Eq.(17) becomes,

$$f(\eta) \approx \sum_{k=0}^{n} F(k)\eta^k. \tag{18}$$

From the above definitions, it is clear that the concept of DTM is to compute Taylor series coefficients algebraically. By the help of Eqs.(14)-(15), the fundamental mathematical operations of one-dimensional differential transform can readily be obtained and listed below.

- $w(\eta) = u(\eta) \pm v(\eta)$, $W(k) = U(k) \pm V(k)$
- $w(\eta) = cu(\eta)$,$W(k) = cU(k)$
- $w(\eta) = \frac{d}{d\eta} u(\eta)$, $W(k) = (k+1)U(k+1)$
- $w(\eta) = \frac{d^n}{d\eta^n} u(\eta)$, $W(k) = (k+1)(k+2)\cdots(k+m)U(k+m)$
- $w(\eta) = u(\eta)v(\eta)$, $W(k) = \sum_{l=0}^{k} U(l)V(k-l)$
- $u(\eta) = \eta^m f(\eta)$, $U(k) = \delta(k-m)F(k-m)$

Here, lower case letters denote original functions and upper case letters are transformed functions.

4 Solution of the problem by DTM

In this part, by using the DTM, we will evaluate the Taylor series coefficients, $F[k], \Theta[k]$, of $f(\eta), \theta(\eta)$, which are velocity and temperature distribution of MHD squeezing flow between paralle discs with the presence of heat transfer. Then, Eqs.(8)-(9) are approximated by:

$$f(\eta) = \sum_{k=0}^{N} F[k]\eta^k, \tag{19}$$

$$\theta(\eta) = \sum_{k=0}^{N} \Theta[k]\eta^k, \tag{20}$$

and transformed into:

$$
\begin{aligned}
& (k+1)(k+2)(k+3)(k+4)F[k+4] \\
- & S(\sum_{s=0}^{k}\delta(s-1)(k-s+1)(k-s+2) \\
& \mathrm{x}(k-s+3)F[k-s+3] \\
+ & 3(k+1)(k+2)F[k+2] - 2\sum_{s=0}^{k}(k-s+1) \\
& \mathrm{x}(k-s+2)(k-s+3)F[k-s+3]F[s] \\
- & M^2(k+1)(k+2)F[k+2] = 0, \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
& (k+1)(k+2)\Theta[k+2] \\
+ & S.Pr(2\sum_{s=0}^{k}(k-s+1)F[s]\Theta[k-s+1] \\
- & \sum_{s=0}^{k}\delta(s-1)(k-s+1)\theta[k-s+1]) \\
- & Pr.Ec(\sum_{s=0}^{k}(s+1)(s+2)(k-s+1) \\
& \mathrm{x}(k-s+2)F[s+2]\Theta[k-s+2]) \\
+ & 12\alpha^2\sum_{s=0}^{k}(s+1)(k-s+1)F[s+1]F[k-s+1] \\
= & 0. \qquad (22)
\end{aligned}
$$

Hence, the successive series coefficients are obtained from Eqs.(21)-(22). From the first three conditons (Eq.(10)), we have:

$$F[0] = A, F[1] = 0,\ \theta[0] = 1. \qquad (23)$$

To include all the boundary conditions to the problem, we also define:

$$F[2] = a_1, F[3] = a_2, \text{and } \theta[2] = a_3, \qquad (24)$$

where $a_1, a_2,$ and a_3 are unknown parameters. From Eqs.(21)-(22), for $k = 0, 1, 2 \ldots, N$, by using the recurrence relations, the successive series coefficients can be obtained in terms of $a_1, a_2,$ and a_3 and some of them follow as,

$$
\begin{aligned}
F[4] &= -\frac{1}{2}A.S.a_2 + \frac{1}{12}M^2a_1 + \frac{1}{4}S.a_1, \qquad (25) \\
\Theta[2] &= -A.Pr.S.a_3 + 2Ec.Pr.a_1^2, \ldots \qquad (26)
\end{aligned}
$$

As a result, the computed coefficients are substituted into Eqs.(19)-(20) and the values of $a_1, a_2,$ and a_3 are solved by applying the last three boundary conditions (Eq.(11)) to the series solutions.

5 Error Estimate For the Solution

Eqs. (19),(20), the truncated Taylor series, are the approximate solutions to Eqs.(8),(9), with given initial and boundary conditions (Eqs.(10),(11)). Since these solutions should approximately satisfy the governing equations hence, we define the residuals by the notation D_1, D_2 as:

$$\left|D_1(f(\eta),\theta(\eta)), D_2(f(\eta),\theta(\eta))\right|. \qquad (27)$$

These residuals correspond to error at particular points $\eta = \eta_i$, $i = 0, 1, 2, \ldots, N$ for Eqs.(8) and (9) and denoted by $E_1(\eta_i)$, $E_2(\eta_i)$ respectively. Since the global error functions should approach zero or $E(\eta_i) \leq \varepsilon$, where ε is positive small number for desired accuracy, we define $\max(\varepsilon) = 10^{-\alpha}$ of $E_1(\eta_i)$. Thus, the truncation limit of N is increased up to $E_j(\eta_i) < 10^{-\alpha}, j = 1, 2$. When N is sufficently large then $E_j(x) \to 0$ and the global error decreases.

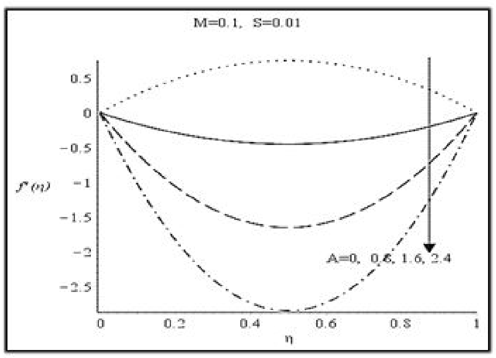
6 Results and Discussion

The present solutions examine the effects of some flow parameters on the velocity and temperature distrubitions of the squeezing flow between parallel discs. Here,especially, the cases where the permeability parameter, A, either is positive or negative denoting the suction and injection of the permeable wall respectively have been investigated. Moreover, velocity and temperature profiles for given A have been computed in terms of the following flow parameters; squeezing number, magnetic parameter, Prandtl number, Eckert number and the dimensionless length. When the illustrative figures are compared by the work of [14] and it is seen as follow below that all the results match very well indeed.

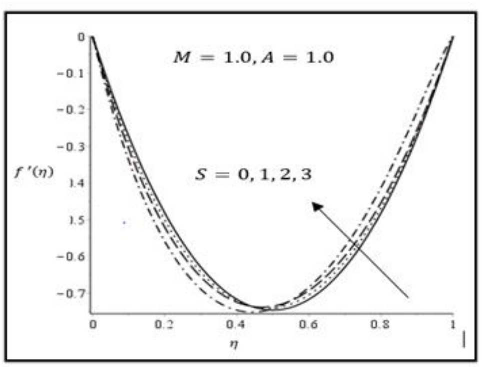
Suction Case ($A > 0$): Figure 2(a) indicates the effects of permeability parameter A on the radial velocity. With increasing A, the radial velocity of the fluid decreases. Moreover, permeable structure of the upper disc allows the fluid particles to move closer to the boundary and the boundary layer becomes thinner. In Figure 2(b), for $A = 1$, in the interval $0.0 < \eta < 0.4$, with increasing S the radial velocity increases , on the other hand, decreases with increasing S in the interval $0.4 < \eta < 1.0$. This decrease in speed is related to the fineness of the boundary layer.

Figure 3(a) shows that $\theta(\eta)$ is an increasing function of A. In other words, the thermal boundary layer is thinned by the increase in A. Figure 3(b) shows that $\theta(\eta)$ increases for the increasing values of S. On the other hand, the thermal boundary layer is inversely proportional to Pr in Figure 4(a). The low values of Pr are related to high viscous fluids with high thermal conductivity and high viscosity Pr, which corresponds to low viscous liquid. Figure 4(b) indicates that inverse relation holds for Ec number as in Figure 3(b). In Figure 5, it has been shown that $\theta(\eta)$ increases with increasing χ.

Injection Case ($A < 0$): Figures 6(a),6(b) to Figure 9 show the effects of physical parameters on the radial velocity and temperature distribution when $A < 0$. An inverse relationship is observed in physical properties on axial velocity for suction and injection cases. On the other hand, the temperature distribution figures remain the same as in the suction case.

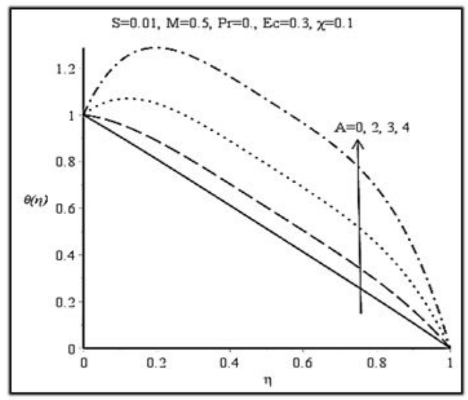


(a)

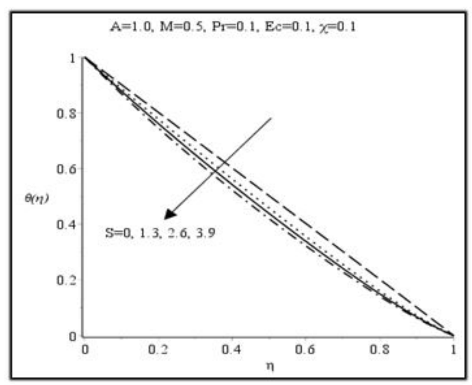


(b)

Figure 2: (a)Effects of A on radial velocity profiles (b)Effects of S on radial velocity

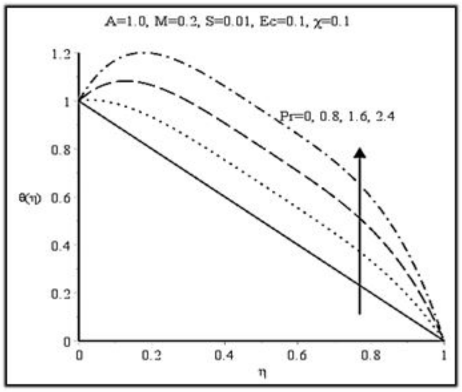


(a)

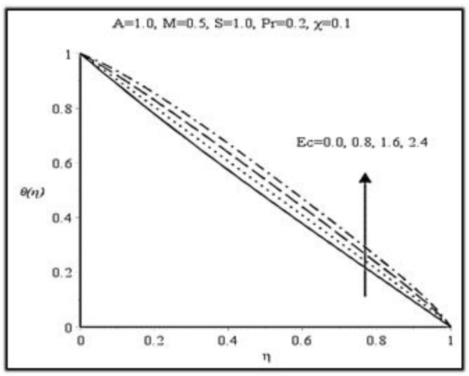


(b)

Figure 3: (a)Effects of A on temperature profile, (b) Effects of S on temperature profile for $A = 1.0$



(a)



(b)

Figure 4: (a)Effects of Pr on temperature profile for $A = 1.0$, (b)Effects of Ec on temperature profile for $A = 1.0$.

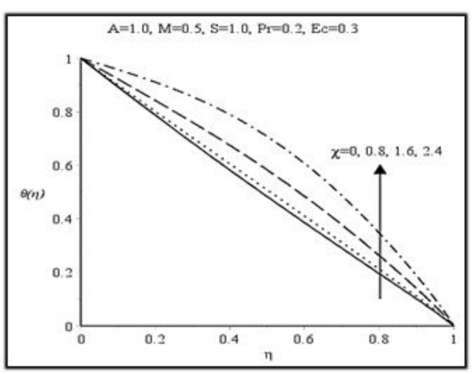
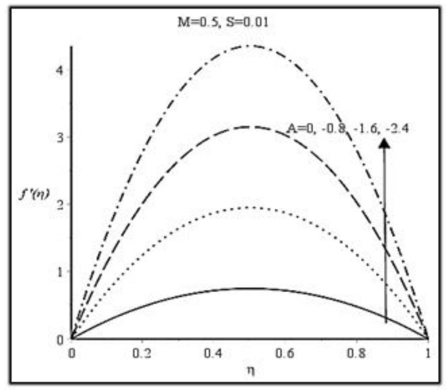
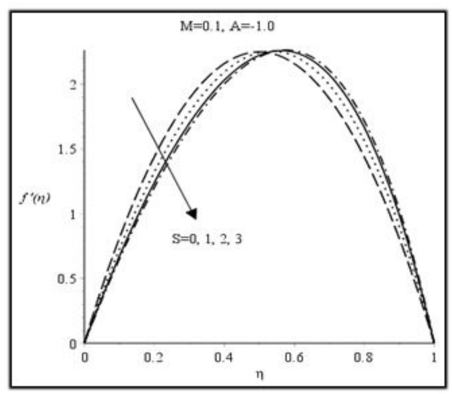


Figure 5: Effects of χ on temperature profile for $A = 1.0$

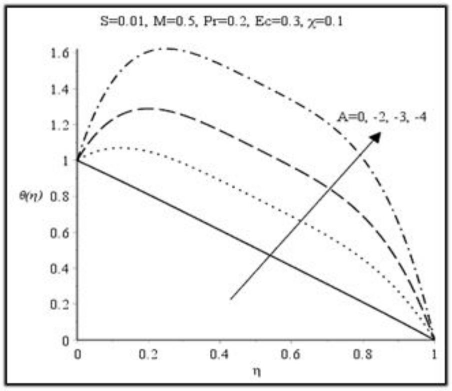


(a)

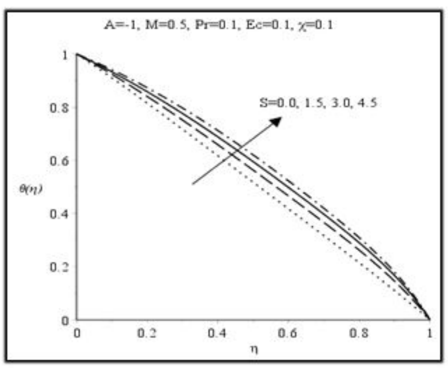


(b)

Figure 6: (a)Effects of negative values of A on the radial velocity profiles, (b)Effects of varying S on the radial velocity for $A = -1.0$.

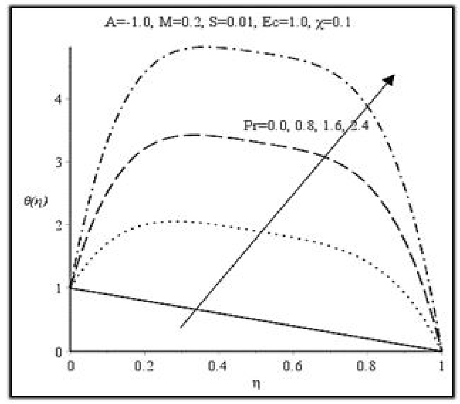


(a)

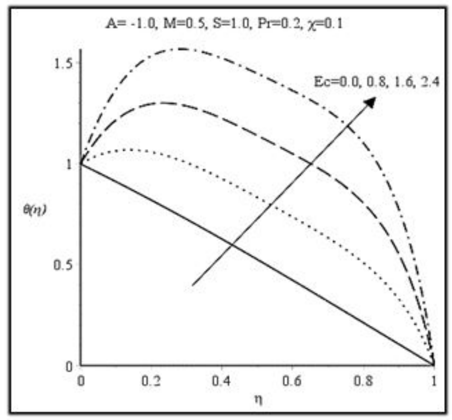


(b)

Figure 7: (a) Effects of the negative values of A on the temperature profile, (b)Effects of varying S on the temperature profile for $A = -1.0$.



(a)



(b)

Figure 8: (a)Effects of varying Pr on the temperature profile for $A = -1.0$,(b)Effects of varying Ec on the temperature profile for $A = -1.0$

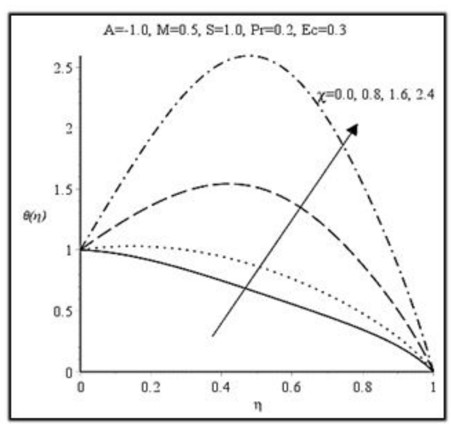


Figure 9: Effects of χ on temperature profile for $A = -1.0$

7 Conclusion

The basic goal of this work is to obtain solutions to the governing equations, which model the movement of MHD fluid in a channel with one permeable wall under the effects of the heat transfer, by DTM. This method evaluates the coefficients of Taylor series without differentiating the function and reduces the cost of computation. In $[0, 1]$ interval, reliable results are achieved by few terms in the series. The computations of the velocity and the temperature distribution are obtained by using the symbolic computation software, Maple. It is worth to note here that we also easily solve similar flow problems by using DTM.

References:

[1] M.A. Siddiqui,S. Irum, A. Ansari, Unsteady squeezing flow of a viscous MHD fluid between parallel plates, a solution using the homotopy perturbation, Mathematical Modelling and Analysis, 13(4)(2008), 565-576.

[2] A.S. Berman, Laminar flow in channel with porous wall. Journal of Applied Physics , 24 (1953), 1232-1235.

[3] J.R. Sellers, Laminar flow in channels with porous flow at high suction Reynolds number, Journal of Applied Physics, 26 (1955), 489-.

[4] S.W, Yuan, Further Investigation of Laminar Flow in Channels with Porous Walls, Journal of Applied Physics, 27(3) (1956), 267-269.

[5] R.M. Terill, G. M. Shrestha, Laminar flow through parallel and uniformly porous walls of different permeability, ZAMP 16 (4) (1965), 470-482.

[6] W.F. Hughes, R.A. Elco, Magnetohydrodynamic lubrication flow between parallel rotating disks, Journal of Fluid mechanics, 13 (1962), 21-32.

[7] D.J. Kuzma, E.R. Maki, R.J. Donelly, The magnetohydrodynamic squeeze film, Journal of Tribology, 110 (1988), 375-377.

[8] E.A. Hamza, The magnetohydrodynamic effects on a fluid film squeezed between two rotating surfaces, Journal of Physics, D: Applied Physics, 24 (1991), 547-554.

[9] S.Bhattacharyya, A.Pal, Unsteady mhd squeezing flow between two parallel rotating discs, Mechanics Research Communications, 24(6) (1997), 615-623.

[10] A.M. Siddiqui, S. Irum, A.R. Ansari, Unsteady squeezing flow of a viscous MHD fluid between parallel plates, Mathenmatical Modeling and Analysis, 13 (2008) 565576.

[11] G. Domairy, A. Aziz, Approximate analysis of MHD squeeze flow between two parallel discs with suction or injection by homotopy perturbation method, Mathematical Problems in Engineering, (article ID/ 2009/603916).

[12] A.A. Joneidi, G. Domairry, M. Babaelahi, Effect of mass transfer on the flow in the magnetohydrodynamic squeeze film between two parallel disks with one porous disk, Chemical Engineering and Communications, 198 (2011) 299311.

[13] M. Sajid, I. Ahmad, T. Hayat, M.Ayub, Series solution for unsteady axisym-metric flow and heat transfer over a radially stretching sheet. Communications in Non-Linear Science and Numerical simululations, 13 (2008), 2193-2200.

[14] S.I. Khan, N. Ahmed, U. Khan, S.U. Jan, S.T. Mohyud-Din, Heat Transfer analysis for squeezing flow between parallel disks, Journal of the Egyptian Mathematical Society,23 (2015), 445-450.

[15] Ayaz, F., On the two-dimensional differential transform method, Appl. Math. Comput., 143(2-3)(2003),361-374.

[16] Ayaz, F., Solutions of the system of differential equations by differential transform method, Appl. Math. Comput., 147(2)(2004),547-567.

[17] V. S. Erturk, S. Momani, Differential Transform Technique for Solving Fifth-Order Boundary Value Problems, Math. Comput. Appl. (2008), 13(2), 113-121