Discrete Model of Hydraulic Fracture Crack Propagation in Media with Filtration

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Abstract: - An infinite homogeneous isotropic elastic medium with a penny shape crack is considered. The crack is subjected to the pressure of fluid injected in the crack center with a positive injection rate. Description of the crack growth is based on the lubrication equation (balance of the injected fluid and the crack volume), equation for crack opening caused by fluid pressure on the crack surface, Poiseullie equation related local fluid flux with crack opening and pressure gradient, and the criterion of crack propagation of linear fracture mechanics. The crack growth is simulated by a discrete process consisting of three basic stages: increasing the crack volume by a constant crack size, crack jump to a new size defined by the fracture criterion, and filling the new appeared crack volume by the fluid. It is shown that the model results a reasonable dependence of the crack radius on the time as well as the pressure distribution on the crack surface. The model is applied to the case of media with filtration, and numerical examples of hydraulic fracture crack growth with the "leak-off" effect are presented.

Key-Words: - Fracture mechanics, hydraulic fracture, penny shape crack, filtration.

1 Introduction

For importance in gas and petroleum industry, the process of hydraulic fracture has been the object of intense theoretical and experimental studies for about sixty years. The number of publications dedicated to this problem is huge. Publications before 21-st century can be found in the books [1], [2]. More recent publications are mentioned, e.g., in [3], [4], [5]. Mathematically the problem can be reduced to a system of non-linear integro-differential equations in the region with moving boundary. Analytical solutions of this system do not exist even in simplest cases, and only numerical methods are efficient. By application of conventional numerical methods, the original integro-differential equations are to be discretized with respect to time and space variables, and then, hydraulic fracture crack geometry should be reconstructed at each discrete time step. The principal unknown of the problem is pressure distribution on the crack surface. It turns out that construction of this distribution is an ill-posed problem. Application of conventional numerical methods for solution of ill-posed problems can result substantial numerical errors, and only specific methods are efficient [6]. Because the ill-posed problem should be solved at each time step of the crack growth, the errors accumulate and a reliable solution can be lost. In addition to non-linearity and moving boundary, this is another principal difficulty in numerical solution of the hydraulic fracture problem.

In the present work, hydraulic fracture growth of a penny shape crack in homogeneous and isotropic elastic medium is considered. The fluid is injected in the crack center with an arbitrary positive injection rate. Description of the crack growth is based on the lubrication equation (balance of the injected fluid and the crack volume), equation for crack opening in elastic media caused by fluid pressure distributed on the crack surface, the Poiseullie equation related local fluid flux with the crack opening and pressure gradient, and the classical criterion of crack propagation of linear fracture mechanics. Time discretization of these equations is interpreted as an actual process that consists of three stages: growth of the crack volume by a constant crack radius, an instant crack jump to a new radius, and filling the new appeared volume by the fluid. For solution of the ill-posed problem of reconstruction of the pressure distribution at each time step of crack growth, a specific class of approximating functions is used. These positive, monotonously decreasing functions are appropriate for approximation of actual pressure distributions and allow solving the ill-posed problem with sufficient accuracy. The case of the medium with filtration is considered, and the “leak-off” effect is taken into account.

2 A Penny Shape Crack Subjected to Fluid Injection

Consider an infinite isotropic homogeneous elastic medium containing an isolated penny shape crack. The crack is subjected to internal pressure caused by the fluid injected in the crack center with given positive injection rate Q(t) (Fig.1).

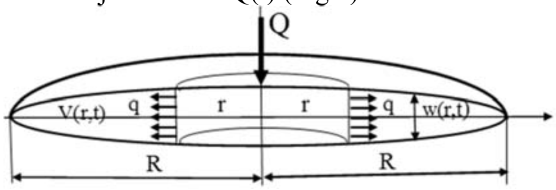

Fig.1

It follows from the symmetry of the problem that the growing crack remains circular with increasing radius $R(t)$. Crack opening $w(r,t)$ and pressure distribution $p(r,t)$ on the crack surface are functions of time *t* and the distance *r* from the crack center. Let us introduce the fractional crack volume $v(r,t)$ by the equation

$$v(r,t)=2\pi\int_r^{R(t)} w(x,t)x dx. \qquad (1)$$

Thus, $v(r,t)$ is the crack volume between the circle of radius *r* and the crack edge $r=R(t)$. Let $q(r,t)$ be the fluid flux in the radial direction through the crack cross-section with coordinate r. For non-compressible fluid, the equation of balance of fractional volume $v(r,t)$ and the injected fluid (lubrication equation) has the form

$$\frac{\partial v}{\partial t}=2\pi r q(r,t). \qquad (2)$$

The fluid flux $q(r,t)$, crack opening $w(r,t)$, and pressure $p(r,t)$ relate by the Poiseuille law [1]

$$q(r,t)=-\frac{w(r,t)^3}{12\eta}\frac{\partial p(r,t)}{\partial r}. \qquad (3)$$

It is assumed that the fluid is Newtonian with constant viscosity η. From equations (2) and (3) follows that the lubrication equation can be written in the form

$$\frac{\partial v(\rho,t)}{\partial t}=-2\pi\rho\frac{w(\rho,t)^3}{12\eta}\frac{\partial p(\rho,t)}{\partial \rho}. \qquad (4)$$

Here dimensionless radial coordinate $\rho=r/R(t)$ is introduced. For an isotropic elastic medium and radially symmetric pressure distribution $p(\rho,t)$ on the crack surface, crack opening $w(\rho,t)$ and fractional volume $v(\rho,t)$ of a penny shape crack of radius R is defined by the equations [7,8]

$$w(\rho,t)=\frac{4(1-\nu)}{\pi\mu}R(t)\int_0^1 G(\rho,\varsigma)p(\varsigma,t)d\varsigma, \qquad (5)$$

$$v(\rho,t)=\frac{4(1-\nu)}{\pi\mu}R(t)^3\int_0^1 K(\rho,\varsigma)p(\varsigma,t)d\varsigma. \qquad (6)$$

Here μ and ν are shear modulus and Poisson ratio of the medium. The kernel $G(\xi,\varsigma)$ has the form

$$G(\xi,\varsigma)=\begin{cases}\frac{\varsigma}{\rho}F(\sin^{-1}(\kappa),\frac{\varsigma}{\xi}), & \varsigma<\xi\\ F(\sin^{-1}(\kappa^{-1}),\frac{\xi}{\varsigma}), & \varsigma>\xi\,,\end{cases} \qquad (7)$$

where

$$F(\phi,m)=\int_0^{\phi}\frac{d\theta}{\sqrt{1-(m\sin\theta)^2}},\ \kappa=\sqrt{\frac{1-\xi^2}{1-\varsigma^2}}. \qquad (8)$$

The kernel $K(\rho,\varsigma)$ is expressed in terms of $G(\xi,\varsigma)$

$$K(\rho,\varsigma)=2\pi\int_\rho^1 G(\xi,\varsigma)\xi\, d\xi \qquad (9)$$

and is a smooth integrable function of the variables (ρ,ς). The integral operators with the kernels $G(\rho,\varsigma)$ and $K(\rho,\varsigma)$ have the following remarkable properties. Actions of these operators on polynomial functions of ρ with even exponents

$$p(\rho)=a_0+a_1\rho^2+a_2\rho^4+\ldots+a_n\rho^{2n}, \qquad (10)$$

are polynomials of the same power 2n multiplied with $(1-\rho^2)^{\frac{1}{2}}$ (for the *G*-kernel) and $(1-\rho^2)^{\frac{3}{2}}$ (for the *K*-kernel). Coefficients of these polynomials are expressed in terms of the coefficients $a_0,a_1,..,a_n$ in equation (10) in explicit analytical forms [9].

Note that calculation of pressure distributions $p(\rho,t)$ from equation (6) with the given left hand side $v(\rho,t)$ is in fact solution of the Fredgholm integral equations of the first kind with integrable kernel $K(\rho,\varsigma)$. It is a well-known ill-posed problem [6]. For such problems, small deviations (errors) of the left hand sides $v(\rho,t)$ cause large errors in the pressure distribution $p(\rho,t)$.

For calculation of the crack radius in the hydraulic fracture process, the classical criterion of linear fracture mechanics is used. For radial pressure distribution $p(\rho)$, the stress intensity factor K_I for the fracture mode I at the crack edge is [7]

$$K_I(p,R)=\frac{\sqrt{2R}}{\pi}\int_0^1\frac{p(\rho)\rho}{\sqrt{1-\rho^2}}d\rho, \qquad (11)$$

and the fracture criterion takes the form

$$K_I(p,R)=K_{Ic}, \qquad (12)$$

where K_{Ic} is the so-called fracture toughness. This specific physical parameter defines resistance of the medium to crack propagation.

Lubrication equation (4), equations (5) and (6), and fracture criterion (12) compose a complete system of equations for penny shape crack growth in a homogeneous isotropic elastic medium by fluid injection. A natural principal unknown of the problem is the fluid pressure $p(\rho,t)$ on the crack surface. All other crack parameters (crack radius, crack opening, and fractional volume) are expressed in term of the pressure.

3 Discretization of the Equations of Hydraulic Fracture process

The system of equations of crack growth can be solved only numerically. Conventional numerical methods of solution of partial differential equations are based on (time and space) discretization procedure. Taking a discrete time step Δt and changing the partial time derivative with the finite difference equation (4) can be presented in the form

$$v(\rho,t+\Delta t)\approx v(\rho,t)-\frac{2\pi w^3(\rho,t)}{12\eta}\rho\frac{\partial p(\rho,t)}{\partial\rho}\Delta t. \quad (13)$$

If the right hand side of this equation is known at the moment *t*, one can calculate the function $v(\rho,t+\Delta t)$ at the moment $t=t+\Delta t$. The difficulty in carrying out this scheme is that the crack radius at $t=t+\Delta t$ is unknown in advance (the crack has a moving boundary). As the result, the variables ρ on the left hand side of (13) $\rho=r/R(t+\Delta t)$ differs from the similar variable on the right hand side $\rho=r/R(t)$. If the time step Δt is sufficiently small, one can accept that $R(t)\approx R(t+\Delta t)$, calculate the fractional volume $v(\rho,t+\Delta t)$ from equation (13), obtain the new pressure distribution from equation (6), and then, find the new crack radius from the fracture criterion (12). As it was mentioned above, calculation of the pressure distribution $p(\rho,t+\Delta t)$ from equation (6) with known left-hand side $v(\rho,t+\Delta t)$ is an ill-posed problem, and small errors in $v(\rho,t+\Delta t)$ result large errors in calculation of $p(\rho,t+\Delta t)$. Nevertheless, if the pressure $p(\rho,t+\Delta t)$ is constructed, one can obtain new crack radius $R(t+\Delta t)$ from the fracture criterion (12)

$$K_I\big(p(t+\Delta t),R(t+\Delta t)\big)=K_{Ic}, \quad (14)$$

and then, go to the next time interval.

Formal discretization of equation (4) permits the following physical interpretation. Let at the moment t, the crack radius be $R(t)$, crack volume $V(t)=v(0,t)$, and the pressure distribution on the crack surface is $p(\rho,t)$. For such radius and pressure distribution, the stress intensity factor at the crack edge is $K_I(p,R)=K_{Ic}$. Suppose that the process of crack radius growth from $R(t)$ to $R(t+\Delta t)$ consists of three stages (Fig.2). First, during the time interval Δt, the fluid is injected inside the crack but the crack radius does not change. For incompressible fluid, balance of the injected fluid and increment of the crack volume (the lubrication equation (4)) should be satisfied, meanwhile fracture condition (12) is neglected. At the end of this stage, the crack volume increases (dashed line in Fig.2) from $V(t)$ to $V(t+\Delta t)$, and the pressure distribution is $p^+(\rho,t+\Delta t)$. For this distribution, the stress intensity factor at the crack edge is more than K_{Ic}. At this moment, the crack jumps instantly to the new radius $R(t+\Delta t)$ (second stage). Pressure on the crack surface changes, and we assume that it is define by the equation

$$p\left(\frac{r}{R(t+\Delta t)},t+\Delta t\right)=\alpha p^+\left(\frac{r}{R(t+\Delta t)},t+\Delta t\right), \quad (15)$$

where coefficient α $(\alpha<1)$ is to be defined from fracture criterion (12). Because of an instant jump, the fluid inside the crack fills the new region near the crack edge (third stage). The new crack radius $R(t+\Delta t)$ and the coefficient α in equation (15) are to be found from the equations

$$v(r/R(t+\Delta t),t+\Delta t)|_{r=0}=V(t+\Delta t); \quad (16)$$

$$K_I[p(r/R(t+\Delta t),t+\Delta t),R(t+\Delta t)]=K_{Ic}. \quad (17)$$

where left hand sides of equations (16) and (17) are defined in (6) and (11).

At the end of the third stage, the crack volume $V(t+\Delta t)$ is filled with fluid, and the radius $R(t+\Delta t)$ and pressure $p(r,t+\Delta t)$ are such as the fracture criterion (17) is satisfied. The total time ΔT of this three stages can be calculated from the equation

$$\Delta T=\frac{V(t+\Delta t)-V(t)}{Q(t)}. \quad (18)$$

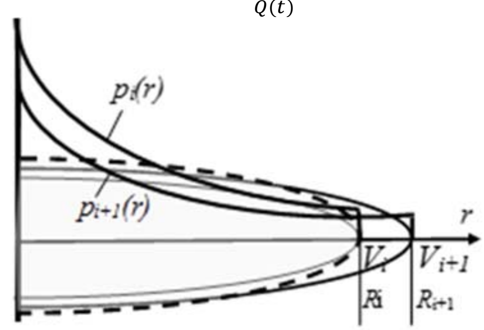

Fig.2

4 Approximation of the pressure distribution and solution of the ill-posed problem (6)

By positive injection rate $Q(t)$, the pressure $p(\rho,t)$ is a continuous function of variable ρ monotonously decreasing from the injection point to the crack edge. In addition, at the crack center, the following equation holds

$$\frac{\partial v}{\partial t}\Big|_{r=0}=-2\pi\frac{w^3(0,t)}{12\eta}\lim_{\rho\to 0}\rho\frac{\partial p}{\partial\rho}=Q(t). \quad (19)$$

Because crack opening at the center $w(0,t)$ is finite, the limit in this equation should be also finite. It means that the function $p(\rho, t)$ has logarithmic asymptotics at the crack center. Therefore, the function $p(\rho, t)$ can be approximated by the following series

$$p(r,t) = -p_0(t) ln \frac{r}{R(t)} + \sum_{n=1}^{N} p_n(t) \varphi_n \left(\frac{r}{R(t)}\right), \quad (20)$$

where $p_n(t) \geq 0$ $(n = 0,1,2, \dots, N)$, and $\varphi_n(\rho)$ are monotonously decreasing functions with the derivatives equal to zero at the crack center. For instance, the following ten functions $\varphi_n(\rho)$ can be used for approximation of the pressure distribution

$$\varphi_1 = 1, \varphi_2 = 1-\rho^{10}, \varphi_3 = 1-\rho^4,$$
$$\varphi_4 = 1-\rho^2, \varphi_5 = (1-\rho^2)^2, \varphi_6 = (1-\rho^2)^3,$$
$$\varphi_7 = (1-\rho^2)^4, \varphi_8 = (1-\rho^2)^8, \varphi_9 = (1-\rho^2)^{15},$$
$$\varphi_{10} = (1-\rho^2)^{40}. \quad (21)$$

The graphs of these functions are presented in Fig.3.

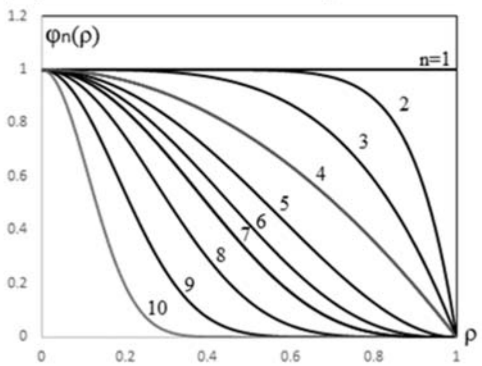

Fig.3

Because all these functions are polynomials similar to (10), action of the operators G and K in equations (5) and (6) on these functions can be presented in explicit analytical forms. The crack opening $w(\rho, t)$ and fractional volume $v(\rho, t)$ take forms

$$w(\rho,t) = \frac{4(1-\nu)}{\pi\mu} R(t) \sum_{n=0}^{N} p_n(t) w_n(\rho), \quad (22)$$

$$v(\rho,t) = \frac{4(1-\nu)}{\pi\mu} R(t)^3 \sum_{n=0}^{N} p_n(t) v_n(\rho), \quad (23)$$

where

$$w_n(\rho) = \int_0^1 G(\rho,\varsigma)\varphi_n(\varsigma) d\varsigma, \quad (24)$$

$$v_n(\rho) = \int_0^1 K(\rho,\varsigma)\varphi_n(\varsigma) d\varsigma, \quad (25)$$

and $\varphi_0(\varsigma) = -\ln\varsigma$. For $\varphi_n(\rho)$ in (21), the functions $w_n(\rho)$ and $v_n(\rho)$ can be found in explicit analytical forms [8].

The stress intensity factor $K_I(t)$ at the crack edge is

$$K_I(t) = \frac{\sqrt{2R(t)}}{\pi} \sum_{n=0}^{N} p_n(t) k_n, \quad (26)$$

$$k_n = \int_0^1 \frac{\varphi_n(\rho)}{\sqrt{1-\rho^2}} d\rho. \quad (27)$$

If we denote the right hand side of equation (13) as $rhs(\rho, t)$

$$rhs(\rho,t) = v(\rho,t) - \frac{2\pi w^3(\rho,t)}{12\eta} \rho \frac{\partial p(\rho,t)}{\partial \rho} \Delta t, \quad (28)$$

and take into account equation (6), the lubrication equation (13) can be rewritten in the form

$$\frac{4(1-\nu)}{\pi\mu} R(t)^3 \int_0^1 K(\rho,\varsigma) p^+(\varsigma, t+\Delta t) d\varsigma = rhs(\rho,t). \quad (29)$$

The function $p^+(\rho, t + \Delta t)$ is approximated by the series similar to (20)

$$p^+(\rho, t+\Delta t) =$$
$$-p_0^+(t+\Delta t)\ln(\rho) + \sum_{n=1}^{N} p_n^+(t+\Delta t)\varphi_n(\rho). \quad (30)$$

Substituting this approximation in (29) and satisfying the resulting equation at M points ρ_k (nodes) homogeneously distributed along the crack radius

$$\rho_k = (k/M) \quad (k = 0,1,2, \dots, M)$$

we obtain the following system of linear algebraic equations for the coefficients $p_n^+(t+\Delta t)$

$$\sum_{n=0}^{N} S_{(k,n)} p_n^+(t+\Delta t) = rhs(\rho_k, t), \quad (31)$$

$$S_{(k,n)} = \frac{4(1-\nu)}{\pi\mu} R(t)^3 \int_0^1 K(\rho_k,\varsigma)\, \varphi_n(\varsigma) d\varsigma, \quad (32)$$

$$k = 0,1,2, \dots, M.$$

This system can be presented in the matrix form as follows

$$\boldsymbol{S} \cdot \boldsymbol{X} = \boldsymbol{RHS}, \quad \boldsymbol{X} = \{p_0{}^+, p_1{}^+, \dots, p_N{}^+\}^T. \quad (33)$$

Here T is the transposition operator. According to the method of solution of ill-posed problems [6], the vector **X** can be found from the equation

$$\min_Y \|\boldsymbol{S} \cdot \boldsymbol{Y} - \boldsymbol{RHS}\| = \|\boldsymbol{S} \cdot \boldsymbol{X} - \boldsymbol{RHS}\|, \quad (34)$$

$$\|\boldsymbol{Y}\| = \sum_{n=1}^{N} Y_n^2. \quad (35)$$

Here minimum is to be found on vectors **Y** with positive components

$$Y_1 \geq 0, Y_2 \geq 0, \dots, Y_{N+1} \geq 0. \quad (36)$$

The matrix **S** in equation (32) can be not square: the numbers of the approximating functions and the nodes on the crack radius can be different. For seeking minimum in equation (34) with restrictions (36), standard methods of linear programming can be used.

According to the discrete model, at the first stage of the *(i+1)*th step of crack growth, the crack radius remains fixed $R = R(t_i)$, and pressure distribution $p^+(\rho, t_i + \Delta t)$ is to be constructed from equations (30), (34) and (36). The appropriate value of the time interval Δt in equations (28)-(31) should be taken such as the relative error δ of the solution of equation (33)

$$\delta = \frac{\|\boldsymbol{S} \cdot \boldsymbol{X} - \boldsymbol{RHS}\|}{\|\boldsymbol{RHS}\|} \quad (37)$$

does not exceed a prescribed tolerance (in the calculations, δ<0.01 was taken). Then, the new crack radius $R(t_i + \Delta t)$ and pressure distribution are calculated from equations (16) and (17).

Example of evolution of the pressure distributions on the crack surface in the process of hydro fracture is shown in Fig.4-6 for the fluid with viscosity $\eta = 0.01 Pa \cdot sec$, $0.1 Pa \cdot sec$, $1 Pa \cdot sec$, the material fracture toughness $K_{Ic} = 1.6 MPa\sqrt{m}$.

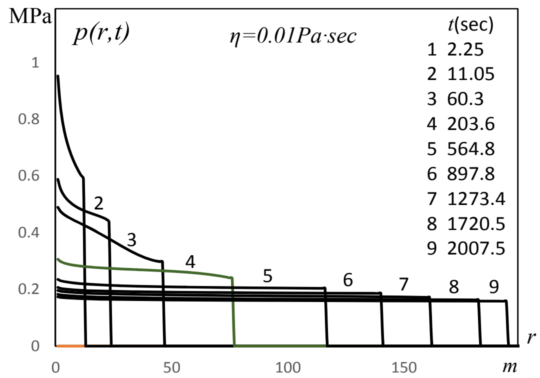

Fig.4

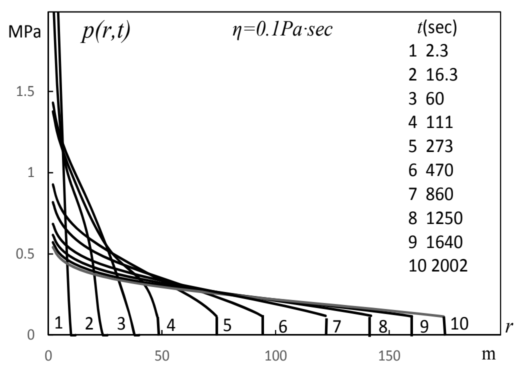

Fig.5

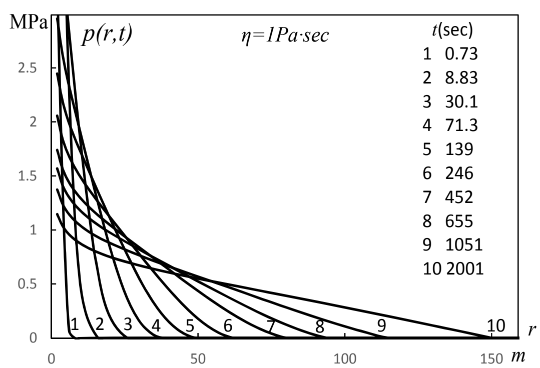

Fig.6

Dependences of the crack radius R on time and fracture toughness K_{Ic} are shown in Fig.7 ($\eta = 0.1 Pa \cdot sec$), and on time and fluid viscosity η in Fig.8 ($K_{Ic} = 1 MPa\sqrt{m}$).

Dependence of crack opening *w(r,t)* on time is shown in Fig.9 for $\eta = 0.01 Pa \cdot sec$, and in Fig.10 for $\eta = 1 Pa \cdot sec$, $\left(\mu = 6.25 MPa,\ \nu = 0.2,\ K_{Ic} = 1 MPa\sqrt{m}\right)$.

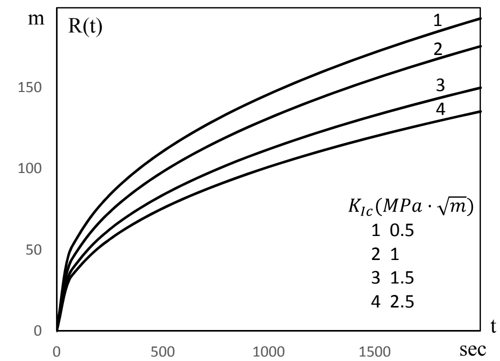

Fig.7

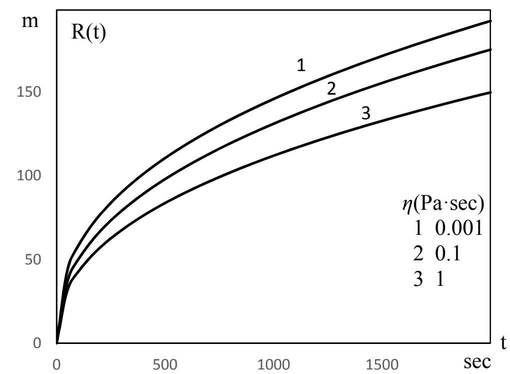

Fig.8

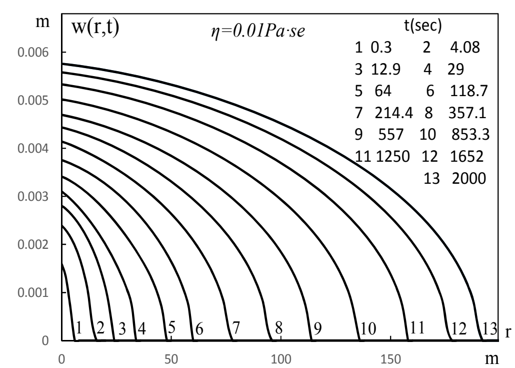

Fig.9

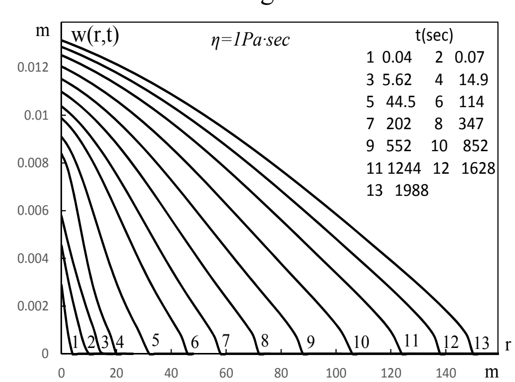

Fig.10

5 Hydraulic crack propagation in the medium with filtration

In the case of a medium with filtration, a portion of the fluid is leaked into the medium, and lubrication equation (2) should be changed as follows

$$\frac{\partial v(r,t)}{\partial t} = 2\pi r q(r,t) + 4\pi \int_r^{R(t)} l(z,t) z dz. \quad (38)$$

Here the integral term is the rate of the fluid flux filtrated into the medium from the crack surfaces between the circle of radius r and the crack edge $r=R(t)$. The fluid filtration rate $l(r,t)$ depends on the pressure distribution on the crack surface and is defined by the equation

$$l(r,t) = -\frac{k_f}{\eta}\frac{\partial P(r,z,t)}{\partial z}|_{z=0}. \quad (39)$$

Here k_f is the filtration coefficient of the medium, η is the fluid viscosity, $P(r,z,t)$ is the fluid pressure in the medium, (r,z) are cylindrical coordinates with the origin at the crack center and the z-axis orthogonal to the crack plane. If $z=0$, the function $P(r,z,t)$ coincides with the pressure distribution $p(r,t)$ on the crack surface.

The function $P(r,z,t)$ satisfies the filtration equation [10]

$$\frac{\partial P(r,z,t)}{\partial t} = D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) P(r,z,t), \quad (40)$$

The boundary conditions on the crack plane $z=0$

$$P(r,0,t) = p(r,t),\ r \le R(t);$$
$$\frac{\partial P(r,z,t)}{\partial z}|_{z=0} = 0,\ r > R(t), \quad (41)$$

conditions at infinity ($P(r,z,t) \to 0, r, z \to \infty$), and the initial condition at $t=0$

$$P(r,z,0) = 0. \quad (42)$$

Coefficient D in equation (40) is the hydraulic diffusivity coefficient [m^2/sec] defined by the equation

$$D = \frac{k_f}{\eta \phi c_t}, \quad (43)$$

where ϕc_t is the so-called the storativity [Pa^{-1}] of the medium (ϕ is rock porosity, c_t is compressibility of the rock material with fluid) [10].

Because filtration in the z-direction dominates, partial derivatives with respect to r in equation (40) can be neglected in comparison with the z-derivative, and this equation takes the form

$$\frac{\partial P(r,z,t)}{\partial t} = D\frac{\partial^2 P(r,z,t)}{\partial z^2}. \quad (44)$$

The boundary and initial conditions keep forms (41) and (42). The solution of this problem is presented in the form of the following integral [11]

$$P(r,z,t) = \frac{z}{2\sqrt{\pi D}}\int_0^t \frac{exp\left(-\frac{z^2}{4D(t-\tau)}\right)}{(t-\tau)^{\frac{3}{2}}} p(r,\tau) d\tau. \quad (45)$$

The equation for filtration flux $l(r,t)$ from the crack surface follows from equations (45) and (39) in the form

$$l(r,t) = -\frac{k_f}{4\eta D\sqrt{\pi D}} \times$$
$$\lim_{z\to 0}\int_0^t \frac{2D(t-\tau)-z^2}{(t-\tau)^{5/2}} exp\left(-\frac{z^2}{4D(t-\tau)}\right) p(r,\tau) d\tau. \quad (46)$$

Let the function $p(r,t)$ be pice-wise constant with respect to time

$$p(r,t) = p(r,t_j) = const(t), \quad (47)$$
$$t_{j-1} < t < t_j, \qquad j = 1,2,\ldots..$$

In this case, equation (46) for the flux rate $l(r,t)$ takes the form

$$l(r,t_i) = \frac{k_f}{\eta\sqrt{\pi D}} \times$$
$$\left[\sum_{j=1}^{i-1}\left(\frac{1}{\sqrt{t_i - t_{j-1}}} - \frac{1}{\sqrt{t_i - t_j}}\right) p(r,t_j) + \frac{p(r,t_i)}{\sqrt{t_i - t_{i-1}}}\right]. \quad (48)$$

The discretized form of the lubrication equation (38) is

$$v(\rho, t+\Delta t) = v(\rho,t) -$$
$$-2\pi\frac{w^3(\rho,t)}{12\eta}\rho\frac{\partial p(\rho,t)}{\partial \rho}\Delta t + L(r,t)\Delta t. \quad (49)$$

$$L(r,t) = 4\pi\int_r^{R(t)} l(\varsigma,t)\varsigma d\varsigma. \quad (50)$$

For calculation the integral $L(r,t_i)$, we assume that in each time interval (t_j, t_{j+1}) the pressure is constant with respect to time $p(r,t) = p(r,t_j)$, and one can use equation (48) for calculation of $l(r,t_j)$. For approximation (20) of $p(r,t)$, the integral in equation (50) takes the form of the following sum

$$L(r,t_i) =$$
$$2\sum_{j=1}^{i}\lambda(j,i) R_j^2 \sum_{n=0}^{N} s\left(n, \frac{r}{R_j}\right) p_n(t_j), \quad (51)$$
$$R_j = R(t_j),$$
$$\lambda(j,i) = k_c\left[\frac{1}{\sqrt{t_i - t_j}} - \frac{1}{\sqrt{t_i - t_{j-1}}}\right],\ \ i > j,$$
$$\lambda(i,i) = \frac{k_c}{\sqrt{t_i - t_{j-1}}},\ \ k_c = \frac{k_f}{\eta\sqrt{\pi D}}. \quad (52)$$

$$s(n,\rho) = 4\pi\int_\rho^1 \varphi_n(\varsigma)\varsigma d\varsigma,\ \ \rho \le 1;$$
$$s(n,\rho) = 0,\ \ \rho > 1. \quad (53)$$

For functions $\varphi_n(\rho)$ in equation (21), the integrals $s(n,\rho)$ are calculated in explicit analytical forms, and the first six such integrals are

$$s(0,\rho) = \pi(1-\rho^2 + 2\rho^2 ln\rho),$$
$$s(1,\rho) = 2\pi(1-\rho^2),$$
$$s(2,\rho) = (\pi/3)(5 - 6\rho^2 + \rho^{18}),$$
$$s(3,\rho) = (2\pi/3)(2 - 3\rho^2 + \rho^6),$$
$$s(4,\rho) = \pi(1-\rho^2)^2,$$
$$s(5,\rho) = (2\pi/3)(1-\rho^2)^3. \quad (54)$$

Thus in the case of filtration, for calculation of pressure distribution $p(r,t_{j+1})$ at the $(i+1)$th time step, equation (38) should be used. The total time ΔT_{i+1} of this step is calculated from the balance equation

$$V(t_{i+1)}) - V(t_i) + L(0,t_i)\Delta T_i = Q(t_i)\Delta T_i, \quad (55)$$

$$\Delta T_i = \frac{1}{Q(t_i)} \frac{(V(t_{i+1}) - V(t_i))}{[1 - L(0,t_i)/Q(t_i)]}. \quad (56)$$

where $L(0, t_i)\Delta T_i$ is the portion of fluid filtrated into the medium in the time interval ΔT_i.

Dependences of the crack radius on time for various values of the parameter k_c in equation (51) are presented in Fig.11. The medium has shear modulus $\mu = 6.25GPa$, Poisson ratio $\nu = 0.2$, $K_{Ic} = 1MPa\sqrt{m}$, $\eta = 0.01Pa \cdot sec$, and the fluid injection rate is $Q = 0.2m^3/sec$.

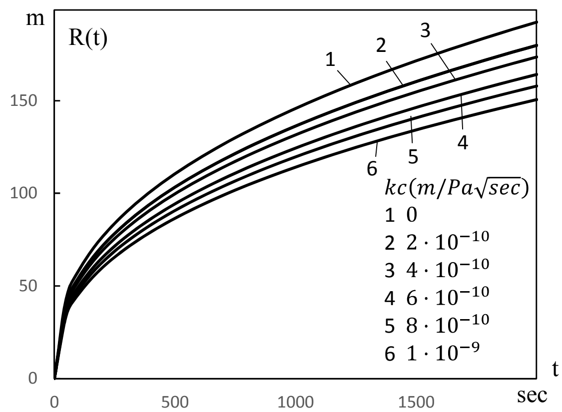

Fig.11

6 Conclusion

An efficient numerical method for solution of the hydraulic fracture problem for a penny shape crack in homogeneous isotropic elastic media with filtration is proposed. The numerical algorithm is based on a specific class of approximating functions that on the one hand, allow excluding numerical integration and differentiation that can be sources of numerical errors. On the other hand, these functions help to solve efficiently the ill-posed problem of reconstruction of pressure distribution on the crack surface in each time step of the crack growth.

The proposed model of discrete crack growth can be considered as a physical interpretation of the formal procedure of discretization of the lubrication equation (4). This model results a specific numerical algorithm different from existing in the literature. For the medium without filtration, the method predictions are close to the results of other authors [8]. In the present work, extension of the model to the case of media with filtration is performed.

Note that external stresses (lithographic pressure) that usually act in actual rocks were neglected here for simplicity. Accounting these stresses in the framework of the proposed numerical algorithm is straightforward.

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