The Treatment of the Permeability of Material Layers Using the Zeeman Topology Z_{pl}

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Abstract: - The proposed media analysis relates to the interaction of forces between sources in physical media defined with respect to connecting lines. The mathematical basis is a countable dense subspace A of R^3 whose elements have finite distances. A constructive induction proof of the non-regularity of the Zeeman topology Z_{pl} [1] by St. Popvassilev [2] provides the modeling for A. The autohomeomorphism group $H(Z_{pl})$ with respect to the Zeeman like topology Z_{pl} shows the immediate mobility of sources. The Superposition principle of Standard Analysis has to be specified.

Key-words: -invariant properties, Zeeman like topologies and applications, Permeability of membranes

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1. Introduction

Using a construction by Popvassilev, it is possible to inductively define a countable dense set A in Euclidean space R^3 with the following properties:

• Any arbitrary line in R^3 intersects the set A at most in two points.

• Any two points in A have a finite distance along their connecting line. (An arbitrary subset A of $R^3$with this property will be called a *-set A)

• The *-set A is dense in the Euclidean closure Cl(A).

• A point P in the Euclidean closure Cl(A) \ A cannot be disjointly separated by Z_{pl}-open sets from the Z_{pl}- closed set A. (Z_{pl} is not regular).

For the physical application, we consider a region A in R^3 consisting of three layers (Fig. 3). The central layer [$x1<x<x2$, $y1<y< y2$, $0\leq z\leq z1$] lies between the layers $z1<z< z2$ and $z3<z< 0$. The distribution of matter within these layers can be chosen differently (e.g., mass points, molecules, etc.). The *-property of A will be jointly constructed in the three layers.

The invariance of the Z_{pl}-topological structure, maintained by the autohomeomorphism group $H(Z_{pl})$, will be used to characterize the flexibility of the sources of the three layers to get a qualitativ notion of permeability.

Some useful facts are the following:

• The group of autohomeomorphisms $H(Z_{pl})$ is equal to the group of autobijections that leave invariant the family Σ_{pl} of piecewise linear arcs: $H(Z_{pl}) = Bij(\Sigma_{pl})$.

• The difference set Cl(A) \ A consists of accumulation points of "Zeno sequences"(see II) with residues in A.

• Since Z_{pl} is not first-countable, the convergence of seequuences cannot be used to characterize the Z_{pl}-continuity of the autohomeomorphisms $h \in H(Z_{pl})$. Instead, the Z_{pl}-continuity of $h \in H(Z)$ must be defined via the open sets of Z_{pl}.

• There exist Euclidean-discontinuous autotransformations $h \in H(Z_{pl})$ that leave the set Σ_{pl} invariant.

The general topological properties of Z_{pl} are transformed by $h\in H(Z_{pl})$

We prepare the mathematical modeling of permeability in the following sections:

II. Zeeman Topology: General definition
III. The Z_{pl} method of source distribution
IV. Remarks on autohomeomorphism groups, (invariance, symmetry, and Noether's theorem)
V. A special geometric transformation by Rubin
VI. Physical Application:Permeability of Membrane
VII. A short note on forces with large and small distances

2. The General Zeeman Topology

In a seminal paper[1], E.C. Zeeman introduced a new topology Z in Minkowski space, whose homeomorphism group is essentially the Lorentz group (including translations and homotheties). The specific relationship between timelike lines and the Euclidean topology in $\mathbb{R}^4$ was uniquely established by this "finest" topology Z, which coincides with the Euclidean topology of $\mathbb{R}^4$ on all timelike lines. Following this, numerous studies have explored more general topological domains (X, Σ) with geometrically defined subspaces, focusing on the generalized Zeeman topology Z(X, Σ) [see 6].

In this paper we introduce in 3-dimensional Euclidean space R^3 the finest topology $Z(R^3,\Sigma_l):= Z_l$ using the family of lines Σ_l Z_l is equivalently defined by the family of all piecewise lines Σ_{pl}. Thus,

$$Z(R^3, \Sigma_l) = Z_l = Z(R^3, \Sigma_{pl}):= Z_{pl}$$

An example of an open set of the topology Z_{pl} is an open sphere from which the residue of a convergent sequence towards the center has been extracted, distributed along countably infinitely many straight lines through the center (called "Zeno sequence"). The evaluation of the properties of Z_{pl} for physical problems will be shown here. This paper highlights some physical applications of the Zeeman topology $Z(\mathbb{R}^3, \Sigma_{pl}):= Z_{pl}$.

In the proof of the non-regularity of Z_{pl}, Popvassilev constructs a subset A of $\mathbb{R}^3$ using the following construction: (see more detailed in Popvassilev [2]):
a. Consider the countable subset Q^3 of $\mathbb{R}^3$.
b. Choose balls $K(x_i, 1/k)$ centered on these countable points $x_i \in Q^3$ with radii 1/k, k = 1, 2, .. and obtain a countable covering (O_i) of $\mathbb{R}^3$(or covering a subset of $\mathbb{R}^3$).
c. Start inductively by choosing a point $P_1 \in O_1$ and draw a line g_1 through P_1 that intersects some point $P_2 \in O_2$.
d. Delete all points on the line g_1 except the points P_1 and P_2.
e. Start drawing a line g_2 through the point P_2 to a point $P_3 \in O_3$.
f. By induction, find the set $A = (P_1, P_2, ...)$ that is dense in $\mathbb{R}^3$ (or in an appropriate subspace of $\mathbb{R}^3$).
The set A defined inductively in this way has the following properties:

- Any line g in $\mathbb{R}^3$ intersects A in at most two points.
- A is a closed set with respect to the topology Z_{pl}.
- Any two points in A have a finite distance on their connecting line
- An arbitary point of the Euclidean closure Cl(A) \ A cannot be separated by disjoint Z_{pl}-open sets from the closed set A(Nonregularity).

3. The Z_{pl} Method for Source Distribution

It is evident that the topology Z_{pl} can be equivalently defined by the set Σ_l of all straight lines l in $\mathbb{R}^3$. The finest topology does not distinguish between straight lines and piecewise linear arcs. The following facts are central to this study:

It is easily shown that Z_{pl} lacks a countable neighborhood basis. Consequently, the Z_{pl}-continuity of mappings cannot be characterized by convergent sequences.

This paper inter alia utilizes the non-regularity of the Z_{pl} topology, as proven by St. Popvassilev [2], to develop a physical applications.

The specific inductive construction of the Z_{pl}-closed set A – which cannot be Z_{pl}-separated from a point P in Cl(A) \ A – has significant implications for our physical application in $\mathbb{R}^3$ (see Section VI). The key properties are:

Density: The resulting subset set A is dense in its Euclidean closure Cl(A).

Non-regularity: A point P in Cl(A) \ A cannot be separated from any Z_{pl}-neighborhood U(A) of A by Z_{pl}- open neighborhoods W(P) of P.

Linear Intersection: The inductive construction ensures that every straight line in $\mathbb{R}^3$ intersects the set A at most in two points.

Z_{pl}-Closure: Since A is a Z_{pl}-closed set, any two points in A maintain a finite distance along their existing straight-line connection.

Zeno Sequences and Homeomorphisms: The image of Zeno sequences (P_i, P) under Z_{pl}-homeomorphisms h ∈ H(Z_{pl}) – where (P_i) is lying in A and the Euclidean accumulationpoint P is in Cl(A) \ A – preserves a specific proximity between h(P) and h(A). This proximity relationship of a point P to the set A using a straight line connection (modified pl-arc connection) has a possible interpretation for the superposition of forces (Section VI).

A Zeno sequence (P_i, P) is defined as an Euclidean convergent sequence P_i --> P where the accumulation point P can be separated from the remainder of sequence (P_i) by a Z_{pl}-open neighborhood of P.

More detailed steps of the inductive construction of set A can be gathered from the work of Popvassilev[2].

For the application in Section V (permeability of membranes), we consider a subset of $\mathbb{R}^3$ to construct the set A with the above properties. In particular, all three layers can be constructed collectively with the *-property.

Another importent property of a subset A of R^3 with the *-property is useful for the computation of the superposition of Coulomb/Newton forces between sources in A and a test source in R^3 \ A. The infinite series of terms ($1/r_i^2$) or potentialterms ($1/r_i$) due to forces on connecting lines must converge. Therefore the *-property must be brought in line to get a realistic result.

Any two points in A have a finite distance.

This type of superposition differs from the classical computation using the integration of potentials and afterwards the gradient of the potential to get the resulting force. This method avoids unrealistic continuous charge distribution in the continuum $\mathbb{R}^3$.(see singularites of potential theory, Martensen [8]).

4. Remarks on Autohomeomorphism Groups

Similar to how E. Zeeman used the autohomeomorphism group to represent the physically important Lorentz group, we look in this paper for physical applications of the homeomorphism group H(Z_{pl}). Invariance and symmetry are fundamental concepts in mathematical physics (cf. Dragon [7]; Noether, Poincaré, Lorentz, and conformal diffeomorphism groups, etc.).

Although different topologies may share the same homeomorphism group (see Lee [3]), we focus on investigating the invariance of the aforementioned construction specifically within the unique topology Z_{pl}.

Since the autohomeomorphism group H(Z_{pl}) includes Euclidean-discontinuous transformations (Rubin [4]), the standard invariance of coordinate systems cannot be further assumed to represent the invariance of all physical representations with coordinates. Physical connection could also be invariant with respect to other groups of automorphisms. This will be referred further in Section VI.

The group of autobijections Bij(Σ_{pl}) in $\mathbb{R}^3$ is defined by transformations that preserve the property that elements of Σ_{pl} remain embedded after transformation remaining piecewise linear arcs. Since the group Bij(Σ_{pl}) contains discontinuous transformations, its scope is generally limited.

The usual topological properties of Z_{pl} are invariant under autohomeomorphisms h ∈ H(Z_{pl}). This invariance is central to the physical application in Section VI. It can be proven that every Z_{pl}-homeomorphism h preserves piecewise linear arcs. Conversely, every autobijection h that preserves piecewise linear arcs is a Z_{pl}-homeomorphism:

H(Z_{pl}) = Bij(Σ_{pl})

Proof Sketch:

Let h be an element of H(Z_{pl}) and g a line segment. This segment can be covered by finitely many Z_{pl}-open neighborhoods; therefore, h maps piecewise linear arcs to piecewise linear arcs. Conversely, let h be an element of Bij(Σ_{pl}) and U a Z_{pl}-open neighborhood of a point P in $\mathbb{R}^3$. Consider a Z_{pl}-open set withall line segments through P having interval in U. The image h(U) contains h(P) and first intervals on the transformed piecewise lines through h(P). Given the properties of the inverse h^{-1}, h(U) constitutes a Z_{pl}-neighborhood of h(P).

5. A Special Geometric Transformation by Rubin

For our purposes, we utilize an autobijection $h \in Bij(\Sigma_{pl})$ of $\mathbb{R}^3$ proposed by L. Rubin, which exhibits a Euclidean discontinuity. The construction is illustrated in Figures 1 and 2. In section IV we proved the geometrical version of the autohomeomorphism group $H(Z_{pl})$:

$H(Z_{pl}) = Bij(\Sigma_{pl})$

A discontinuous transformation of $Bij(\Sigma_{pl})$ in $\mathbb{R}^3$ can be constructed as described in Fig 1 and Fig 2:

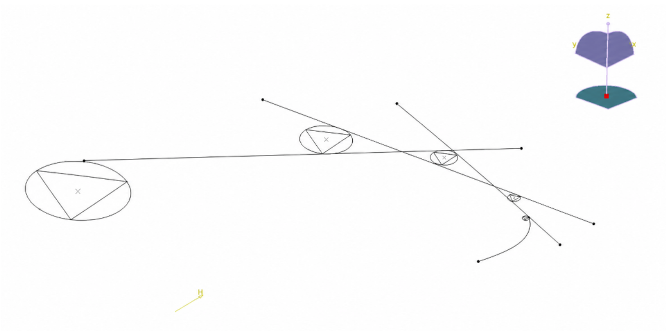

Fig. 1: Construction of a sequence of circumcircles of equilateral triangles in the xy-plane, such that any arbitrary line intersects only a finite number of circles.

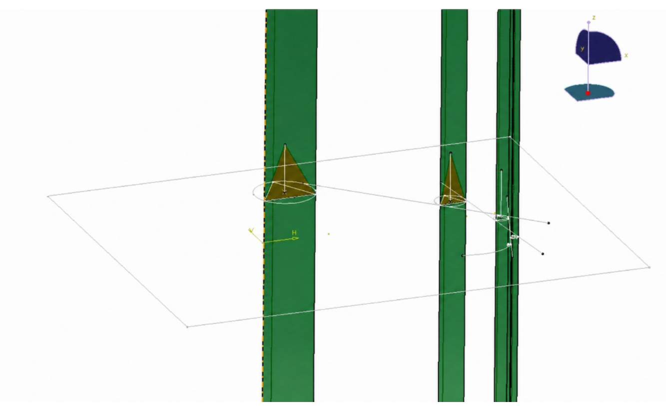

Fig. 2: Construction of "organ pipes" over the triangles, with centroids shifted along the z-axis at constant distances r to form pyramids reaching these shifted points. (More generally: the heights coult be different but not converging to zero)

The autobijection h of $\mathbb{R}^3$ adjusted to this deformation is a bijection that preserves piecewise linear arcs and is, therefore, a Z_{pl}-homeomorphism. The centers of the circles (P_i) form a sequence in the xy-plane converging to a point P, which remains fixed: h(P) = P. However, the points h(P_i) are elevated to a finite height r in the z-direction.

The study of invariance properties of the *-set A yields the following remarks:

The h(A) is a Z_{pl}-closed set.

Therefore the distances berween the transformated sources will remain finite.

The density of h(A) in its Euclidean closure Cl h(A) cannot be guaranteed.

The *-property of A loses this property, and have to be replaced by drawing shortest pl-arc connections.

The example of Nonregularity of Z_{pl} can be discussed after transformation.

Using the non-regularity of Z_{pl} and Zeno sequences, one can characterize the Euclidean closure Cl A of the Z_{pl}-closed set A.

The autotransformation h of $\mathbb{R}^3$ adjusted to this deformation h in Fig 2. is is a bijection that preserves piecewise linear arcs and is, therefore, a Z_{pl}-homeomorphism. The centers of the circles (P_i) form a sequence in the xy-plane converging to a point P, which remains fixed: h(P) = P. However, the points h(P_i) are elevated to a finite height r in the z-direction.

The invariance of Z_{pl}-properties under $H(Z_{pl})$ ensures that if A contains the centers P_i, the property of converging to points in the closure Cl(A) is preserved. Following Popvassilev's proof, an important line connection arises that can be transformed. While the density of h(A) may be altered, we utilize this specific invariance in our physical application (Section VI).

If one considers the Rubin transformation h in the reduced subspace A (section 3.), one finds in successive triangles two source points P_i and P_{i+1} , such that no further point from A lies on their connecting line. The connecting pl-arc transformed by h, which connects h(P_i) and h(P_{i+1}), contains no source from h(A). This property can be used for the permeability of the membrane.

6. Physical Application: Membrane Permeability

In this section, the permeability of a membrane is investigated using the Z_{pl} structure in $\mathbb{R}^3$. We consider a *-region A of $\mathbb{R}^3$ containing three distinct layers defined in Section I.

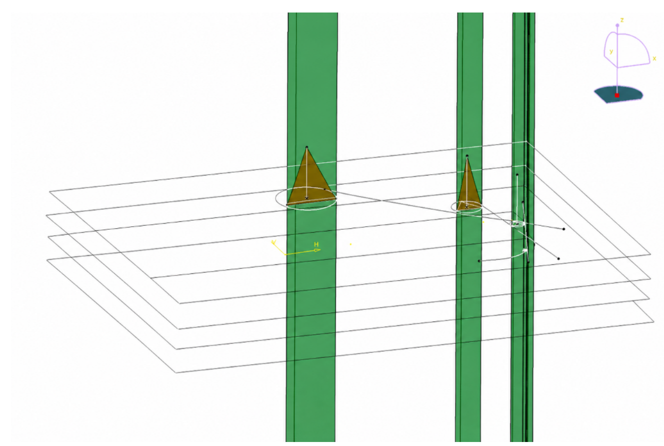

Fig .3

Each layer is covered with various physical sources, which are assumed to be stationary for the time being.(masses, charges) (see Fig. 3, as well as Fig. 1 and Fig. 2).

The source positions for the three layers are determined by a common, layer-spanning *-set A according to Section III.

The central layer represents a membrane whose permeability is the subject of this analysis. The subset A, fulfills althe properties established in Section III.

We propose the following two hypotheses for the force system:

Hypothesis 1: Forces between sources in A are uniquely defined by laws that exclusively require the existence of a connecting line inn $\mathbb{R}^3$ to measure the distance.

Hypothesis 2: The invariance of the tooological structure, as determined by the *-set A with respect to the autohomeomorphism group $H(Z_{pl})$ represents the flexibility of the physical system.

Regarding Hypothesis 1,

We try to avoid the physically unrealistic continuous source distributions. Classical potential theory [8] uses the detour via potentials for force calculation. The gradient calculation yields singularities of the force field, especially at the edges of source regions.

The superposition of forces with respect to a testsource lying outeside of A must be computed by the countable series of forces with respect to connecting lines. For a testsource lying in Cl(A) \ A we can use the nonregularity of Z_{pl} to get an almostconnection of forces.

After reducing the space to the countable, densely packed set A the forcecalculation can be made by forming a converging series with countably many ordered terms (proportional $1/r_i^2$). The contributions arise on countably many straight lines through the testing point.

Regarding Hypothesis 2, The study of Zeno-sequences of A, which are transformed by non-continuous transformations h of Rubin [4] can be used for different changements of positions in the material layers.

The transformation h discussed in Section V is applied to the three layers parallel to the xy-plane (Fig. 3). We use a commonly determined *-property of A to get the Z_{pl}-closed set h(A).

The permeability of the membrane between layer 1 and layer 3 can be simulated by a Zeno sequence consisting of sources in side the triangles in Fig 1 which can be transformed by an appropriate h, elevating height of different pyramides.

Rubin's transformation h shows how a special Zeno sequence can be transformed while preserving its properties by piecewise straight-line connection.

The transformed straight line connecting two sources in O_i and O_{i+1} runs in image h(A) along a pl-arc. The shortest pl-arc, which is a straight line segment, is used for force calculation.

If one considers the Rubin transformation h in the reduced subspace A (section V.), one finds in successive triangles two source points P_i and P_i+1 each, such that no further point from A lies on their connecting line. The H_{pl} transformed line by h, being a pl-arc connecting $h(P_i)$ and $h(P_{i+1})$, contains no source from h(A). The computation of force uses a direct connecting line.

The permeability of the membranelayer can be characterized by special Rubintransformations.

7. A Short Note on Forces at Large and Small Distances

For Newton's law of mass force, which also applies to forces at a far distance regardless of time, the following remark can be useful regarding its mathematical explanation using the topology Z_{pl}**:**

The force measurement between two mass points in a laboratory frame can be transformed discontinuously by $h \in H(Z_{pl})$ to any two other points in R^3. If the shortest ol-connection is defined for distance measurement, then Newton's law of force can be transferred invariantly without the usual coordinate transformation using isometry and without time delay.

In Section V, we use, in particular, L. Rubin's autotransformation for the mobility of sources in matter. This autotransformation is defined as the autobijection of R^3, which leaves pl-arcs invariant and is discontinuous with respect to the Euclidean topology. Figure 2 shows the transformation of a Zeno sequence in the xy-plane and its pl-connections.

Remark on Quantum mechanical properties: The construction of A cannot exactly determine the positions of the two possible souces on a intersecting line. If the velocity of sources is zero (at -273 °C), the uncertainty is obvious.

A dynamical version of the Z_{pl}-method permeability model presented here will be treated in a subsequent paper.

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