International Journal of Mathematical and Computational Methods
http://www.iaras.org/iaras/journals/ijmcm

Laplace–Floquet Theory for Quantum–Classical Transduction: A Unified Framework for Energy–Flow Shaping and Harmonic Remapping

VICTORIA MELLOR
University of Portsmouth, Portsmouth,
UNITED KINGDOM

Abstract: - Emerging hybrid quantum-classical systems such as quantum base stations and quantum-enhanced sensors for future wireless networks, require a signal-processing stage that converts quantum-derived classical observables into waveforms compatible with existing communication infrastructure. This transduction stage must simultaneously shape energy flow through resonant structures and translate harmonic content between incommensurate timing domains, two operations that existing frameworks treat separately: Laplace-domain methods handle energy storage and stability but assume time-invariance, while Floquet and Fourier methods handle periodic modulation but discard transient dynamics. Neither alone captures the coupled physics of resonant transduction under periodic modulation. This paper addresses that gap by developing a unified Laplace-Floquet framework. The central result (Equation 7) is a harmonic transfer relation that retains both Laplace-domain pole structure (encoding causality, damping, and transient response) and Floquetdomain harmonic coupling (encoding deterministic frequency translation under periodic modulation). From this single relation, the paper derives closed-form conditions for causality, BIBO stability, parametric stability margins, and noise propagation across harmonic channels. A numerical illustration demonstrates the framework on a representative transduction scenario, quantifying how resonant and modulation parameters jointly determine stability margins and noise gain. The framework thereby provides the analytical machinery needed to co-design the resonant and modulation parameters of quantum-classical interfaces, directly supporting the engineering of quantum-enabled network infrastructure.

Key-words: - Laplace-Floquet theory, quantum-classical interfaces, harmonic remapping, parametric systems, resonant transduction.

Received: December 11, 2025. Revised: March 7, 2026. Accepted: April 8, 2026. Published: July 1, 2026.

1 Introduction

1.1 Motivation and Context

Hybrid quantum–classical systems are becoming central to proposals for next-generation wireless networks [6], [13], precision sensing [12], and distributed quantum computing. In each of these settings, a quantum process, such as an atomic clock, a spin ensemble, or a superconducting qubit, produces a measurable classical signal (an expectation value) that must be integrated with conventional communication or control infrastructure [7]. The interface between the quantum source and the classical network is a *transduction stage* that performs two coupled operations: it shapes the energy flow of the quantum-derived signal through a resonant intermediary, and it remaps the signal's harmonic content from the quantum timing reference (e.g., a beat frequency ω_b) to a classical one (e.g., a local oscillator ω_{LO}).

The engineering significance of this transduction stage is growing rapidly. Proposals for quantum base stations [14], entanglement-assisted radar and sensing [16], and hybrid opto-electro-mechanical converters [15] all require a rigorous analytical framework for the classical signal-processing layer that sits between the quantum measurement and the communications network.

1.2 Limitations of Existing Approaches

No existing single framework adequately captures both operations simultaneously:

- **Laplace-domain / LTI methods** [2], [5] provide rigorous tools for analysing energy storage, dissipation, pole-based stability, and transient response. However, they assume time-invariance and therefore cannot describe the controlled periodic modulation that enables frequency translation.
- **Fourier / Floquet methods** [1], [9], [10] naturally describe harmonic coupling under periodic modulation but operate in steady-state and discard the transient and pole-structure information needed for stability analysis, causal design, and noise propagation through resonant elements.
- **Ad hoc combinations** such as performing LTI analysis on a frozen snapshot of a time-varying system [3], can be actively misleading: a system that is stable at every frozen instant can still exhibit parametric instability due to coherent energy pumping between harmonics [8].

Table I summarizes the capabilities of each approach and highlights the specific gaps that the proposed Laplace-Floquet framework addresses.

This gap means that designers of quantum-classical interfaces currently lack a single analytical tool that can simultaneously answer questions about resonant energy shaping, deterministic frequency translation, stability under modulation, and noise performance, all of which must be co-optimized in practice.

TABLE I
COMPARISON OF ANALYTICAL FRAMEWORKS FOR QUANTUM-CLASSICAL TRANSDUCTION

Capability	**LTI / Laplace (e.g., [2], [5])**	**Floquet / Fourier (e.g., [1], [10])**	**Frozen-time (e.g., [3])**	**Laplace-Floquet (this work)**
Energy storage & dissipation modelling	✓	-	✓*	✓
Transient response & pole-based stability	✓	-	✓*	✓
Harmonic coupling under periodic modulation	-	✓	-	✓
Deterministic frequency translation	-	✓	-	✓
Parametric stability analysis	-	Partial†	-	✓
Closed-form BIBO gain for noise budgeting	✓	-	-	✓
Joint resonant–modulation co-design	-	-	-	✓
Reduction to standard LTI as special case	✓	-	✓	✓

*Valid only instantaneously; does not capture inter-harmonic coupling. †Floquet multipliers available but without Laplace transient structure.

1.3 Contribution of This Paper

This paper closes that gap by developing a **Laplace-Floquet framework** that explicitly unifies:

1) Laplace-domain descriptions of resonant energy flow, damping, and transient response, with
2) Floquet-style harmonic coupling induced by controlled periodic modulation.

The central result is a harmonic transfer relation (Equation 7) from which the paper derives:

- closed-form conditions for causality and BIBO stability,
- parametric stability margins that identify and avoid instability zones,
- a BIBO gain constant that quantifies noise amplification across harmonic channels,
- clear reduction to standard LTI theory when modulation is absent.

The novelty of this work lies not in the individual mathematical components-Laplace transforms and Floquet theory are well established-but in their systematic unification into a single harmonic transfer relation (Equation 7) that retains both pole structure and harmonic coupling simultaneously, together with the derivation of closed-form design conditions (Equations 8–15) that are not available from either framework alone. This combination provides analytical capabilities (see Table I) that no existing single framework offers.

A numerical illustration (Section 8) demonstrates the framework on a representative transduction scenario, showing how it enables quantitative co-design of resonant and modulation parameters.

The resulting theory is compatible with standard systems and control engineering, directly supports the design of quantum base stations and hybrid classical-quantum sensors, and provides a rigorous foundation that generalizes both LTI Laplace theory and Fourier/Floquet analysis to the coupled regime relevant to quantum-classical transduction.

2 Problem Formulation

Consider a quantum system whose measured observable produces a classical signal $x(t)$ exhibiting a beat-synchronous structure at frequency ω_b. The engineering objective is to transform this signal into a classical waveform compatible with a different timing reference (e.g., a communication local oscillator ω_{LO}) while preserving phase coherence and minimizing added noise.

Concretely, the transduction stage must:

1) **Shape the energy flow** through resonant storage and controlled decay, selecting the frequency component of interest and suppressing off-resonance noise.
2) **Remap the harmonic content** from the quantum timing domain (ω_b) to the classical timing domain (ω_{LO}) via deterministic frequency translation.
3) **Remain causal, stable, and physically realizable** throughout, including during transient turn-on, frequency switching, and in the presence of parameter uncertainty.

The need to satisfy all three requirements simultaneously through energy shaping, harmonic remapping, and guaranteed stability, is what necessitates a framework that retains both Laplace and Floquet structure, rather than treating them in isolation.

3 Resonant Energy–flow Model

The first stage of transduction is energy-flow shaping. I model the resonant intermediary as a linear resonant system with complex amplitude $a(t)$:

$$\dot{a}(t) = -(\gamma + j\omega_r)a(t) + \kappa x(t), \tag{1}$$

with output

$$y_r(t) = \mathrm{Re}\{a(t)\}. \tag{2}$$

The Laplace-domain transfer function is

$$G(s) = \frac{1}{s + \gamma + j\omega_r}. \tag{3}$$

This representation explicitly encodes the three physical quantities that govern energy-flow shaping:

- **Energy storage:** the resonant frequency ω_r determines the oscillation rate and hence which spectral component of $x(t)$ is selected.
- **Dissipation:** the damping rate γ controls how quickly stored energy decays, setting the bandwidth and quality factor $Q = \omega_r/\gamma$.
- **Causality and stability:** the pole at $s = -\gamma - j\omega_r$ lies in the left-half plane for any $\gamma > 0$, guaranteeing causal, stable response.

This model is consistent with the input-output formalism widely used in quantum optics [4], [7] when restricted to classical expectation values. The Laplace-domain representation provides the foundation onto which periodic modulation is layered in the next section.

4 Periodic Modulation and Floquet Structure

Energy-flow shaping alone does not accomplish frequency translation. To remap harmonic content from one timing domain to another, the resonant output is subjected to controlled time-varying modulation:

$$y_m(t) = p(t)y_r(t), \tag{4}$$

where $p(t)$ is periodic with fundamental frequency Ω.

For generality, the modulation waveform is expanded as a Fourier series:

$$p(t) = \sum_{k\in\mathbb{Z}} p_k e^{jk\Omega t}. \tag{5}$$

This step converts the system from a linear time-invariant (LTI) system into a linear periodically time-varying (LPTV) system. The key physical consequence is that spectral content at any frequency ω in the input is replicated at frequencies $\omega + k\Omega$ for all integers k weighted by the modulation coefficients p_k. It is this mechanism that enables deterministic frequency translation from ω_b to ω_{LO} when Ω is chosen appropriately.

The mathematical description of LPTV systems is provided by Floquet theory [1], [9]. The concept of harmonic transfer functions for LPTV systems was developed by Wereley and Hall [10] in the context of helicopter rotor dynamics. The next section extends this approach by combining it with the Laplace-domain resonant model to obtain the unified framework for transduction.

5 Laplace-floquet Harmonic Representation

5.1 Derivation of the Central Result

Define harmonic-shifted Laplace components by windowing the input and output signals against complex exponentials at each harmonic of the modulation frequency:

$$X_m(s) = \mathcal{L}\{x(t)e^{-jm\Omega t}\}, \qquad Y_k(s) = \mathcal{L}\{y_m(t)e^{-jk\Omega t}\}. \tag{6}$$

The system mapping is then given by the **Laplace-Floquet harmonic transfer relation**:

$$\boxed{Y_k(s) = G(s + jk\Omega) \sum_{m\in\mathbb{Z}} p_{k-m} X_m(s)} \tag{7}$$

This equation constitutes the core analytical result of the paper.

5.2 Interpretation and Significance

Equation (7) encodes, in a single relation, the two coupled operations that define quantum-classical transduction:

- **Laplace structure** (s variable): The transfer function $G(s + jk\Omega)$ retains the full pole structure of the resonant intermediary, evaluated at a harmonically shifted Laplace variable. This preserves information about energy storage rates, damping, transient dynamics, and stability-all of which are discarded by pure Floquet or Fourier approaches.
- **Floquet structure** (indices k, m): The summation over m with coefficients p_{k-m} describes the harmonic coupling induced by periodic modulation. Each modulation harmonic p_{k-m} mixes input spectral content from band m into output band k, enabling deterministic frequency translation.
- **Reduction to LTI theory:** When modulation is absent ($p_k = \delta_{k0}$), the framework reduces exactly to the standard Laplace-domain transfer function $Y_0(s) = G(s)X_0(s)$, confirming backward compatibility with classical systems theory.

The practical significance of retaining both structures simultaneously is that it enables co-design: the resonant parameters (ω_r, γ) and the modulation parameters (Ω, $\{p_k\}$) can be jointly optimized for stability, noise, and frequency-translation accuracy within a single analytical framework.

C. Distinction from Existing Harmonic Transfer Approaches

Harmonic transfer functions for LPTV systems were introduced by Wereley and Hall [10] and have been applied in rotorcraft dynamics and power electronics [11]. The present work differs from these treatments in three specific respects:

1) **Explicit resonant pole structure:** Equation (7) retains the physical resonant transfer function $G(s + jk\Omega)$ rather than working with abstract operator matrices. This makes the dependence on damping γ, resonant frequency ω_r, and coupling strength κ analytically transparent and directly amenable to co-design.
2) **Closed-form stability and noise conditions:** Sections 6.2-6.4 derive explicit expressions (Equations 10-15) for BIBO gain, parametric stability margins, and the damping-selectivity trade-off that are specific to the resonant transduction problem. These closed-form results are not available from the general harmonic transfer function formalism.
3) **Application to quantum-classical interfaces:** The framework is specialized (Section 7) to the transduction of post-measurement quantum observables, a domain not previously addressed by harmonic transfer function theory.

6 Stability and Causality

The interplay between Laplace-domain dynamics and Floquet harmonic structure introduces considerations for stability and causality that extend beyond classical LTI theory. This section establishes rigorous conditions under which the Laplace-Floquet system maintains both causal response and bounded behaviour, demonstrating the analytical power of the unified framework.

6.1 Causality

A system is causal if its output at time t depends only on input values at times $\tau \leq t$. For the resonant energy-flow block characterized by $G(s)$, causality is guaranteed when all poles lie in the left-half of the complex plane, i.e., $\text{Re}\{s\} < 0$ for all poles of $G(s)$.

From Equation (3), the single pole of $G(s)$ is located at

$$s_{\text{pole}} = -\gamma - j\omega_r. \tag{8}$$

Since $\gamma > 0$ by physical necessity (representing energy dissipation in the resonant cavity or coupling medium), there exists $\text{Re}\{s_{\text{pole}}\} = -\gamma < 0$, ensuring the resonant subsystem is causal. The imaginary component $-j\omega_r$ determines the oscillation frequency but does not affect causality, as it contributes only to the position on the imaginary axis.

The periodic modulation $p(t)$ introduces additional considerations. If $p(t)$ is bounded and piecewise continuous, the time-varying multiplication does not introduce acausal behaviour. This follows from the fact that multiplication by a bounded function in the time domain corresponds to convolution in the frequency domain, preserving causality when both factors are causal.

More formally, causality of the overall Laplace-Floquet system follows from the structure of Equation (7). Each harmonic component $Y_k(s)$ depends on $X_m(s)$ through the shifted transfer function $G(s + jk\Omega)$. The key observation is that adding a purely imaginary term $jk\Omega$ to the Laplace variable shifts the pole vertically in the complex plane:

$$s^{(k)}_{\text{pole}} = -\gamma - j(\omega_r - k\Omega). \tag{9}$$

Since $\text{Re}\{s^{(k)}_{\text{pole}}\} = -\gamma < 0$ for all $k \in \mathbb{Z}$, the harmonic-shifted transfer functions remain causal regardless of the modulation frequency Ω. This vertical invariance of causalityis a key result enabled by retaining the Laplace pole structure within the Floquet framework. The frequency translation does not compromise causal response .

For physical realizability, $\sum_k |p_k| < \infty$ is required, which ensures $p(t)$ is bounded and that the Fourier series converges uniformly. This condition is naturally satisfied in practical systems where modulation is generated by finite-bandwidth electronics.

6.2 BIBO Stability

A system is bounded-input, bounded-output (BIBO) stable if every bounded input produces a bounded output. For the Laplace-Floquet system, I establish the bound:

$$\sup_t |y_m(t)| \leq C \sup_t |x(t)|, \tag{10}$$

where the constant C depends on the system parameters γ, κ, and the modulation coefficients $\{p_k\}$.

To derive this bound, observe that the impulse response $h(t)$ of the resonant system satisfies

$$h(t) = \kappa e^{-\gamma t} \cos(\omega_r t) \cdot \mathbb{1}_{t \geq 0}, \tag{11}$$

where $\mathbb{1}_{t\geq 0}$ is the Heaviside function. The L^1 norm of this impulse response is

$$\|h\|_1 = \int_0^\infty \kappa e^{-\gamma t} |\cos(\omega_r t)|\, dt \leq \frac{\kappa}{\gamma}, \tag{12}$$

which is finite for $\gamma > 0$. The periodic modulation introduces an additional multiplicative factor bounded by $\sum_k |p_k|$. Therefore, the overall BIBO gain constant is:

$$C = \frac{\kappa}{\gamma} \sum_{k \in \mathbb{Z}} |p_k|. \tag{13}$$

This result has direct design implications: the BIBO gain C quantifies the worst-case noise amplification through the transduction stage. Designers can use it to set requirements on the damping rate γ and modulation depth $\{p_k\}$ to meet noise-floor specifications. This closed-form expression for C is a new result that is not available from either LTI or Floquet analysis alone, since it requires knowledge of both the resonant pole structure (via κ/γ) and the modulation spectrum (via $\sum |p_k|$).

BIBO stability is maintained provided that:

1) $\gamma > 0$ (dissipation is present in the resonant element),
2) $\sum_k |p_k| < \infty$ (the modulation waveform has finite energy per period).

The stability margin increases with γ, but this comes at the cost of reduced quality factor $Q = \omega_r/\gamma$. This trade-off is fundamental in resonant transducer design: high-Q systems provide better frequency selectivity and noise suppression but exhibit slower transient responses and reduced stability margins.

6.3 Parametric Stability and Floquet Multipliers

While BIBO stability guarantees bounded response to bounded inputs, periodically time-varying systems can exhibit parametric instabilities not captured by frozen-time analysis [8]. These instabilities arise when the modulation frequency Ω is commensurate with the natural frequency ω_r, allowing coherent energy transfer between harmonics that can drive unbounded growth. This phenomenon is well known in the parametric amplifier and oscillator literature [18] but has not previously been analysed in the context of quantum–classical transduction using a combined Laplace-Floquet formalism.

The stability of LPTV systems is characterized by Floquet multipliers, which are the eigenvalues of the monodromy matrix over one modulation period $T = 2\pi/\Omega$ [1], [3]. For the Laplace–Floquet system, stability requires all Floquet multipliers μ_i to satisfy $|\mu_i| < 1$.

Classical parametric resonance theory identifies instability zones near

$$\Omega \approx \frac{2\omega_r}{n}, \quad n \in \mathbb{Z}^+, \tag{14}$$

where energy can be coherently pumped into the resonant mode. The strongest instabilities occur at $n = 1$ (principal parametric resonance) when $\Omega \approx 2\omega_r$.

For quantum-classical transduction, the Laplace-Floquet framework provides explicit guidance for avoiding these instabilities by design. The modulation frequency Ω should satisfy

$$\left|\Omega - \frac{2\omega_r}{n}\right| > \Delta_{\text{safe}}, \quad \forall n \in \{1, 2, \ldots, N_{\text{max}}\}, \tag{15}$$

where Δ_{safe} is a safety margin (typically $\Delta_{\text{safe}} \approx 10\gamma$ to ensure robust stability) and N_{max} is determined by the bandwidth of $p(t)$.

In practice, the dominant modulation coefficients p_k are concentrated near $k = \pm 1$ for simple mixing operations, which limits the parametric coupling strength. The effective parametric gain scales as $|p_k|^2$, so systems with small modulation depths are inherently less susceptible to parametric instabilities.

6.4 Relationship Between Damping and Stability Robustness

The damping coefficient γ serves a dual role in the Laplace-Floquet framework: it determines both the decay rate of transients (Laplace-domain property) and the robustness of stability against parametric coupling (Floquet-domain property). This dual role illustrates why a unified framework is necessary-optimizing γ requires understanding both aspects simultaneously.

Increasing γ provides:

1) **Larger stability margins:** Poles move farther into the left-half plane, increasing robustness to uncertainties in ω_r or parasitic couplings.
2) **Reduced parametric gain:** The susceptibility to parametric resonance decreases as γ increases, since the resonance becomes broader and less sharp.
3) **Faster transient decay:** The system reaches steady-state more quickly, improving tracking performance for time-varying inputs.

However, increasing γ reduces the quality factor $Q = \omega_r/\gamma$, which degrades:

- Frequency selectivity (broader resonance peak),
- Noise suppression (less rejection of off-resonance noise),
- Energy storage efficiency (faster energy leakage).

For quantum-classical transduction, the Laplace-Floquet framework enables principled navigation of this trade-off. A typical design rule is to choose Q in the range 10-100, depending on the application requirements and the coherence time of the quantum source.

6.5 Comparison with Standard Approaches

Traditional stability analysis for LTI systems relies on pole locations in the s-plane and Nyquist/Bode criteria [5]. For time-varying systems, frozen-time analysis, where time-varying parameters are temporarily treated as constant, can be misleading. A system that appears stable at every frozen instant can still be unstable due to parametric coupling between harmonics, as discussed in Section 6.3.

The Laplace–Floquet framework resolves this limitation by retaining both structures simultaneously:

- **Laplace domain:** Captures transient dynamics, pole locations, and exponential decay rates.
- **Floquet domain:** Captures harmonic coupling, parametric resonance, and periodic steady-state behaviour.

This unified treatment enables:

1) Rigorous stability analysis that accounts for time-variation without resorting to frozen-time approximations,
2) Systematic design of modulation waveforms $p(t)$ that achieve the desired frequency translation while avoiding instability zones,
3) Prediction of transient response during turn-on or frequency switching,
4) Quantitative noise analysis across all harmonic channels via the BIBO gain constant C.

These capabilities are essential for quantum base stations and hybrid sensors, where both quantum coherence timescales (captured by Floquet structure) and classical control bandwidths (captured by Laplace dynamics) must be simultaneously managed.

7 Specialization to Quantum-classical Transduction

The preceding sections developed the Laplace–Floquet framework in generality. This section clarifies how the framework specializes to the quantum-classical interface and what physical constraints this imposes.

For quantum-origin signals:

- **Classical signal only:** $x(t)$ represents expectation values obtained after quantum measurement, not quantum states or operators. The framework operates entirely in the classical domain, downstream of the quantum-to-classical boundary. This is consistent with the standard quantum-optical input-output formalism [4], [7] when restricted to mean-field quantities.
- **Timing synchronization:** The periodic modulation $p(t)$ serves to align the timing of quantum beats (at ω_b) with classical communication clocks (at ω_{LO}), achieving deterministic frequency translation $\omega_b \to \omega_{LO}$ when $\Omega = \omega_{LO} - \omega_b$.
- **No quantum coherence transmitted:** Since the framework acts on post-measurement classical signals, no quantum coherence is transmitted or required to be preserved beyond the measurement stage. This ensures compatibility with standard quantum measurement theory [17].

The Laplace-Floquet framework therefore respects quantum measurement principles while providing the analytical tools needed for deterministic, stable, and low-noise classical signal processing of quantum-derived observables.

8 Numerical Illustration

To demonstrate the framework's analytical utility, this section presents a numerical evaluation for a representative transduction scenario. The purpose is not to simulate a specific physical device, but to illustrate how the closed-form results of Sections 5-6 enable quantitative co-design of resonant and modulation parameters.

8.1 Scenario and Parameters

Consider a transduction stage with the parameters shown in Table II. These values are representative of a microwave-frequency quantum-classical interface, such as a superconducting cavity coupled to a classical receiver [14].

8.2 Pole Locations and Causality

From Equation (9), the harmonic-shifted poles are located at:

$$s_{\text{pole}}^{(k)} = -2\pi(50\times10^6) - j\,2\pi(5\times10^9 - k\times100\times10^6). \quad (16)$$

For all harmonic orders k, the real part remains $\text{Re}\{s_{\text{pole}}^{(k)}\} = -2\pi \times 50$ MHz, confirming causality across all harmonic channels as established in Section 6.1. The poles are vertically displaced in the complex plane by $k\times2\pi\times100$ MHz but never cross into the right-half plane.

TABLE II
PARAMETERS FOR THE NUMERICAL ILLUSTRATION

Parameter	Symbol	Value
Resonant frequency	$\omega_r/2\pi$	5 GHz
Damping rate	$\gamma/2\pi$	50 MHz
Quality factor	$Q = \omega_r/\gamma$	100
Coupling strength	κ	1 (normalized)
Quantum beat frequency	$\omega_b/2\pi$	5.1 GHz
Local oscillator frequency	$\omega_{LO}/2\pi$	5.0 GHz
Modulation frequency	$\Omega/2\pi = (\omega_{LO} - \omega_b)/2\pi$	-100 MHz
Modulation coefficients	$p_{\pm 1}$	0.4
	p_0	1.0
	$\lvert k\rvert \geq 2$	0 (truncated)

8.3 BIBO Gain Evaluation

With the specified modulation coefficients ($p_0 = 1.0$, $p_{\pm 1} = 0.4$, all others zero), the modulation norm is:

$$\sum_k |p_k| = |p_{-1}| + |p_0| + |p_1| = 0.4 + 1.0 + 0.4 = 1.8. \quad (17)$$

From Equation (13), the BIBO gain constant is:

$$C = \frac{\kappa}{\gamma} \sum_k |p_k| = \frac{1}{2\pi \times 50 \times 10^6} \times 1.8 \approx 5.73 \times 10^{-9} \text{ s.} \quad (18)$$

This means any bounded input signal $x(t)$ produces a bounded output with $\sup |y_m(t)| \leq 5.73 \times 10^{-9} \sup |x(t)|$. The gain constant provides a direct noise budget: if the input noise floor is known, C gives the worst-case output noise level.

8.4 Parametric Stability Check

The principal parametric resonance condition (Equation 14) gives $\Omega_{\text{danger}} = 2\omega_r = 2\pi \times 10$ GHz. The chosen modulation frequency is $|\Omega| = 2\pi \times 100$ MHz, which is far from any instability zone:

$$\left| \Omega - \frac{2\omega_r}{n} \right| \gg \Delta_{\text{safe}} = 10\gamma = 2\pi \times 500 \text{ MHz} \quad (19)$$

for all relevant n. The system is therefore well within the parametrically stable regime.

8.5 Quality Factor Trade-off

Table III illustrates how the BIBO gain and stability margin vary with quality factor Q for the same resonant frequency $\omega_r/2\pi = 5$ GHz and modulation norm $\sum |p_k| = 1.8$, demonstrating the damping-selectivity trade-off discussed in Section 6.4.

TABLE III
BIBO GAIN AND STABILITY MARGIN VERSUS QUALITY FACTOR

Q	$\gamma/2\pi$ (MHz)	C (ns)	Stability margin $\gamma/\gamma_{\min}$
10	500	0.573	Very high
50	100	2.86	High
100	50	5.73	Moderate
500	10	28.6	Low
1000	5	57.3	Very low

The table shows that increasing Q from 10 to 1000 increases the BIBO gain by a factor of 100 (i.e., the system amplifies noise 100 times more) while simultaneously reducing the stability margin. This quantitative trade-off, made explicit by the Laplace-Floquet framework, is precisely the type of co-design information that is unavailable from either LTI or Floquet analysis alone.

9 Relation to Existing Theories

To situate the contribution precisely, the proposed framework:

- **Generalizes LTI Laplace theory** [2], [5] to periodically modulated systems by introducing harmonic-shifted transfer functions $G(s + jk\Omega)$ while retaining full pole-structure information.
- **Extends Fourier/Floquet analysis** [1], [9] by explicitly retaining the transient dynamics, damping, and stability information encoded in the Laplace variable s, which pure Floquet approaches discard.
- **Builds on harmonic transfer function theory** [10], [11] by specializing to the resonant transduction problem and deriving closed-form design conditions not available from the general formalism.
- **Bridges quantum measurement models and classical systems engineering** by providing a rigorous signal-processing framework that operates on post-measurement classical observables [4], [7], [17] and is directly compatible with standard control and communications theory.

The framework does not replace quantum-optical or quantum-information treatments of the measurement process itself; rather, it provides the missing analytical layer for the classical transduction stage that follows measurement.

10 Future Research Directions

Several directions extend naturally from the framework presented here:

1) **Multi-mode and cascaded resonant systems:** The single-resonator model of Section 3 can be extended to coupled multi-mode systems, where Equation (7) generalizes to a matrix-valued harmonic transfer relation. This extension is relevant to opto-electro-mechanical transducers [15] with multiple intermediary modes.
2) **Noise spectral density analysis:** While the BIBO gain constant C provides a worst-case bound, a more refined treatment would compute the output noise spectral density by propagating input noise through the harmonic transfer relation. This would yield frequency-dependent signal-to-noise ratios for each harmonic channel.
3) **Nonlinear and weakly nonlinear extensions:** Practical transduction stages may include weak nonlinearities (e.g., saturation, Kerr effects). Describing-function or Volterra-series extensions of the Laplace-Floquet framework could capture harmonic generation and intermodulation.
4) **Experimental validation:** The closed-form predictions of Sections 6 and 8, (particularly the BIBO gain, pole locations, and parametric stability boundaries) are directly

testable in superconducting microwave cavities, electro-optic modulators, or piezoelectric transducers. An experimental programme to validate these predictions against measured transfer functions would strengthen the practical foundation of the theory.

5) **Optimization and control co-design:** The explicit dependence of stability margins and noise gain on the design parameters (γ, ω_r, Ω, $\{p_k\}$) makes the framework amenable to formal optimization. Convex or semidefinite programming techniques could be applied to find optimal parameter sets subject to stability, noise, and bandwidth constraints.

11 Conclusion

This paper has presented a unified Laplace-Floquet theory for quantum-classical transduction, motivated by the need for a single analytical framework that simultaneously captures resonant energy-flow shaping and deterministic harmonic remapping under periodic modulation. The central contribution is the Laplace-Floquet harmonic transfer relation (Equation 7), which encodes both Laplace-domain pole structure and Floquet-domain harmonic coupling in a single expression.

From this relation, the paper has derived closed-form conditions for causality (Section 6.1), BIBO stability with an explicit gain constant for noise budgeting (Section 6.2), parametric stability margins with design rules for avoiding instability zones (Section 6.3), and the damping-selectivity trade-off that governs resonant transducer design (Section 6.4). A numerical illustration (Section 8) demonstrated these results on a representative microwave-frequency scenario, showing quantitatively how the framework enables co-design of resonant and modulation parameters-a capability not available from existing approaches applied in isolation.

The framework reduces exactly to standard LTI theory when modulation is absent, confirming backward compatibility, and the comparative analysis of Table I shows precisely which analytical capabilities are uniquely enabled by the unified treatment.

The purpose of this work is to provide the analytical machinery that designers of quantum-classical interfaces currently lack: a rigorous, unified tool for co-optimizing resonant and modulation parameters with respect to stability, noise, and frequency-translation accuracy. This theory directly supports the engineering of quantum base stations, hybrid classical-quantum sensors, and other quantum-enabled network infrastructure.

References

[1] G. Floquet, “Sur les équations différentielles linéaires à coefficients périodiques,” *Annales scientifiques de l’École Normale Supérieure*, vol. 12, pp. 47–88, 1883.
[2] W. J. Rugh, *Linear System Theory*, 2nd ed. Upper Saddle River, NJ: Prentice Hall, 1996.
[3] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002.
[4] C. W. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics*, 3rd ed. Berlin: Springer, 2004.
[5] K. Ogata, *Modern Control Engineering*, 5th ed. Upper Saddle River, NJ: Prentice Hall, 2010.
[6] H. J. Kimble, “The quantum internet,” *Nature*, vol. 453, pp. 1023–1030, Jun. 2008.
[7] A. A. Clerk, M. H. Devoret, S. M. Girvin, F. Marquardt, and R. J. Schoelkopf, “Introduction to quantum noise, measurement, and amplification,” *Rev. Mod. Phys.*, vol. 82, pp. 1155–1208, Apr. 2010.
[8] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*. New York: Wiley, 1979.
[9] J. A. Richards, *Analysis of Periodically Time-Varying Systems*. Berlin: Springer, 1983.
[10] N. M. Wereley and S. R. Hall, “Frequency response of linear time periodic systems,” in *Proc. 29th IEEE Conf. Decision and Control*, Honolulu, HI, Dec. 1990, pp. 3650–3655.
[11] E. Möllerstedt and B. Bernhardsson, “Out of control because of harmonics—an analysis of the harmonic response of an inverter locomotive,” *IEEE Control Syst. Mag.*, vol. 20, no. 4, pp. 70–81, Aug. 2000.
[12] C. L. Degen, F. Reinhard, and P. Cappellaro, “Quantum sensing,” *Rev. Mod. Phys.*, vol. 89, p. 035002, Jul. 2017.
[13] S. Wehner, D. Elkouss, and R. Hanson, “Quantum internet: A vision for the road ahead,” *Science*, vol. 362, p. eaam9288, Oct. 2018.
[14] N. Lauk *et al.*, “Perspectives on quantum transduction,” *Quantum Sci. Technol.*, vol. 5, p. 020501, Feb. 2020.
[15] R. W. Andrews *et al.*, “Bidirectional and efficient conversion between microwave and optical light,” *Nature Phys.*, vol. 10, pp. 321–326, Apr. 2014.
[16] S. Barzanjeh *et al.*, “Microwave quantum illumination using a digital receiver,” *Sci. Adv.*, vol. 6, p. eabb0451, May 2020.
[17] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control*. Cambridge: Cambridge Univ. Press, 2009.
[18] W. H. Louisell, *Coupled Mode and Parametric Electronics*. New York: Wiley, 1960.