Computation of Non-Linear Fourth-Order Partial Differential Equation Using a Modified Second Order Fully Implicit Difference Scheme

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Abstract: - In the work reported herein, a second order accurate difference scheme is formulated and analyzed for fourth order nonlinear differential equations. Encouraging conclusions are obtained for the error and accompanying error norms. Computed simulations are compared with those in the literature and are found to not only corroborate theory and accuracy, but also justify the applicability of the scheme in more rigorous settings involving nonlinearity and higher order differential equations.

Key-Words: - fourth order differential equation, nonlinearity, second order difference scheme, errors and error norms.

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1 Introduction

Fourth order differential equations are very significant in real-world engineering problems. A firm understanding of the processes and theories which they represent demands simulations based on numerical solutions especially when the governing equations are nonlinear. Some examples in this field include the Euler-Bernoulli beam theory which is a fourth-order ordinary differential equation that predicts transverse deflection of a cantilevered beam subjected to uniform transverse loading and appropriate boundary conditions. Nandini [1] applied a B-spline collocation technique to determine the deflection of a geometrically nonlinear cantilevered beam. Other examples of note include Kuramoto-Sivashinsky equation, which is a nonlinear differential equation used for the study of many physical phenomena in engineering and physics of the continuum such as pattern formation, reaction-diffusion systems, phase-turbulence in the Belousov-Zhabotinsky reaction, combustion etc. Some of the studies related to these areas can be found in [2], [3], [4], [5], [6], [7], [8], [9], [10], [21].

The extended Fisher-Kolmogorov (EFK) equation is a time-dependent, transient, partial differential equation used for the study of physical and biological systems involving phase transitions, bistability, tumor growth dynamics, disease spread, travelling waves, etc. The EFK has remained an active area for numerical work and can be found widely cited in scientific literature [11], [12], [13], [14], [15], [16], [17], [18], [19], [20].

In this work, a modified second order finite difference scheme is applied to arrive at the numerical solutions of the extended Fisher-Kolmogorov equation (EFK) represented as:

$$\frac{\partial u}{\partial t}+\gamma\frac{\partial^4 u}{\partial x^4}-\frac{\partial^2 u}{\partial x^2}+f\left(u\right)=0,$$

$$x\in\Omega,\ t\in(0,T) \qquad \ldots(1)$$

where

$$f\left(u\right)=u^3-u$$

with initial condition

$$u\left(x,0\right)=u_0\left(x\right),\quad x\in\left[a,b\right] \qquad \ldots(2)$$

and boundary conditions

$$u\left(a,t\right)=z_0,$$

$$\frac{\partial^2 u}{\partial x^2}\left(a,t\right)=0,\ \ u\left(b,t\right)=0,$$

$$\frac{\partial^2 u}{\partial x^2}\left(b,t\right)=0, \qquad \ldots(3)$$

We note that when $\gamma=0$ in (1), we arrive at the standard Fisher-Kolmogorov (FK) equation. A stabilizing fourth-order derivative term added by van Saaloos [22] and van Saaloos [23] converted the standard (FK) equation to equation (1).

2 Mathematical Formulation

We motivate the numerical discretization of equation (1)-(3) by converting the fourth order partial differential equation into a second order system of coupled differential equations.

Following this procedure, we define;

$$m(x,t) = u_{xx}(x,t)$$

Hence, equations (1)-(3) become:

$$m(x,t) = u_{xx}(x,t) \qquad ...(4)$$

$$u_t + \gamma m_{xx} - m + F(u) = 0 \qquad ...(5)$$

with initial condition

$$u(x,0) = u_0 \qquad x \in [a,b] \qquad ...(6)$$

and boundary conditions:

$$u(a,t) = g_0 \qquad u(b,t) = g_1$$

$$m(a,t) = 0 \qquad u(b,t) = 0 \qquad ...(7)$$

Without any loss of generality, we embark on the numerical solution by considering:

$$\frac{\partial u}{\partial t} = F_1(u)\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(F_2(u)\frac{\partial u}{\partial x}\right) + F_3(u) \qquad ...(8)$$

A Crank-Nicolson (CN) scheme-type application to equation (8), yields:

$$\left(\frac{\partial u}{\partial t}\right)_i^{k+\frac{1}{2}} = \frac{1}{2}\left[F_1(u)\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(F_2(u)\frac{\partial u}{\partial x}\right) + F_3(u)\right]_i^k +$$

$$\frac{1}{2}\left[F_1(u)\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(F_2(u)\frac{\partial u}{\partial x}\right) + F_3(u)\right]_i^{k+1} \qquad ...(9)$$

We incorporate Newton-linearization into equation (2) to handle nonlinearity

$$\left(\frac{\partial u}{\partial t}\right)_i^{k+\frac{1}{2}} = \frac{1}{2}\left[F_1(u)\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(F_2(u)\frac{\partial u}{\partial x}\right) + F_3(u)\right]_i^k +$$

$$\frac{1}{2}\left[F_1(u)\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(F_2(u)\frac{\partial u}{\partial x}\right) + F_3(u)\right]_i^{k+1} =$$

$$\frac{1}{2}\left[\left(F_1(u)\frac{\partial u}{\partial x}\right)_i^k + \left(F_1(u)\frac{\partial u}{\partial x}\right)_i^{k+1}\right] +$$

$$\frac{1}{2}\left[\frac{\partial}{\partial x}\left(F_2(u)\frac{\partial u}{\partial x}\right)_i^k + \frac{\partial}{\partial x}\left(F_2(u)\frac{\partial u}{\partial x}\right)_i^{k+1}\right] +$$

$$\frac{1}{2}\left[\left(F_3(u)\right)_i^k + \left(F_3(u)\right)_i^{k+1}\right] \qquad ...(10)$$

We apply a Taylor series expansion to each term:

$$\left(F_2(u)\frac{\partial u}{\partial x}\right)_i^{k+1} \approx \left(F_2(u)\frac{\partial u}{\partial x}\right)_i^k + \frac{\partial}{\partial t}\left(F_2(u)\frac{\partial u}{\partial x}\right)_i^k \Delta t =$$

$$\left(F_2(u)\frac{\partial u}{\partial x}\right)_i^k + \left[\left(F_2(u)\frac{\partial^2 u}{\partial x \partial t}\right)_i^k + \left(\frac{\partial F_2(u)}{\partial t}\frac{\partial u}{\partial x}\right)_i^k\right]\Delta t$$

$$...(11)$$

Let

$$\left(F_3(u)\right)_i^{k+1} \approx \left(F_3(u)\right)_i^k + \left(\frac{\partial F_3(u)}{\partial t}\right)_i^k \Delta t \qquad ...(12)$$

then

$$\left(\frac{\partial u}{\partial t}\right)_i^{k+\frac{1}{2}}=\left[\left(F_1(u)\frac{\partial u}{\partial x}\right)_i^k+\frac{\partial}{\partial x}\left(F_2(u)\frac{\partial u}{\partial x}\right)_i^k+\left(F_3(u)\right)_i^k\right]+$$

$$\frac{\Delta t}{2}\left[\left(F_1(u)\frac{\partial^2 u}{\partial x\partial t}\right)_i^k+\left(\frac{\partial F_1(u)}{\partial t}\frac{\partial u}{\partial x}\right)_i^k+\left(\frac{\partial}{\partial x}\left(F2(u)\frac{\partial^2 u}{\partial x\partial t}\right)+\frac{\partial F_2(u)}{\partial t}\frac{\partial u}{\partial x}\right)_i^k+\left(\frac{\partial F_3(u)}{\partial t}\right)_i^k\right]\quad ...(13)$$

$$\left(\frac{\partial F_1(u)}{\partial t}\right)_i^k=\left(\frac{\partial F_1(u)}{\partial u}\right)_i^k\left(\frac{\partial u}{\partial t}\right)_i^k=\left(\frac{\partial F_1(u)}{\partial u}\right)_i^k\left(\frac{\Delta u^{k+1}}{\Delta t}\right)\quad ...(14)$$

$$\left(\frac{\partial^2 u}{\partial x\partial t}\right)_i^k=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)_i^k=\frac{\partial}{\partial x}\left(\frac{\Delta u^{k+1}}{\Delta t}\right)\quad ...(15)$$

The following derivatives can be expressed as:

$$\frac{\partial}{\partial x}\left(F_2(u)\frac{\partial u}{\partial x}\right)_i^k=\frac{\left(F_2(u)\frac{\partial u}{\partial x}\right)_{i+\frac{1}{2}}^k-\left(F_2(u)\frac{\partial u}{\partial x}\right)_{i-\frac{1}{2}}^k}{\Delta x}\quad ...(16)$$

$$\left(F_2(u)\frac{\partial \Delta u_i^{k+1}}{\partial x}\right)_i^k=\frac{\left(F_2(u)\frac{\partial \Delta u^{k+1}}{\partial x}\right)_{i+\frac{1}{2}}^k-\left(F_2(u)\frac{\partial \Delta u^{k+1}}{\partial x}\right)_{i-\frac{1}{2}}^k}{\Delta x}\quad ...(17)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F_2(u)}{\partial u}\frac{\partial u}{\partial x}\Delta u_i^{k+1}\right)_i^k=\frac{\left(\frac{\partial F_2}{\partial u}\frac{\partial u}{\partial x}\Delta u_i^{k+1}\right)_{i+\frac{1}{2}}^k-\left(\frac{\partial F_2}{\partial u}\frac{\partial u}{\partial x}\Delta u_i^{k+1}\right)_{i-\frac{1}{2}}^k}{\Delta x}\quad ...(18)$$

This is accompanied by the following definitions:

$$(\Delta u)_i^{k+1}=u(i,k+1)-u(i,k)\quad ...(19)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i+\frac{1}{2}}^k=\frac{u(i+1,k)-u(i,k)}{\Delta x}\quad ...(20)$$

$$\left(\frac{\partial u}{\partial t}\right)_i^{k+\frac{1}{2}}=\frac{u(i,k+1)-u(i,k)}{\Delta x}=\frac{(\Delta u)_i^{k+1}}{\Delta t}\quad ...(21)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i-\frac{1}{2}}^k=\frac{u(i,k)-u(i-1,k)}{\Delta x}\quad ...(22)$$

$$(\Delta u)_{i+\frac{1}{2}}^k=\frac{\Delta u(i+1,k+1)+\Delta u(i,k+1)}{2}\quad ...(23)$$

$$(\Delta u)_{i-\frac{1}{2}}^{k+1}=\frac{\Delta u(i,k+1)+\Delta u(i-1,k+1)}{2}\quad ...(24)$$

$$\frac{\Delta u_i^{k+1}}{\Delta t}=\left(F_1(u)\frac{\partial u}{\partial x}\right)_i^k+\left(\frac{\partial}{\partial x}\left(F_2(u)\frac{\partial u}{\partial x}\right)\right)_i^k+\left(F_3(u)\right)_i^k+$$
$$\frac{1}{2}\left[F_1\frac{\partial}{\partial x}\left(\Delta u_i^{k+1}\right)+\frac{\partial F_1}{\partial u}\frac{\partial u}{\partial x}\Delta u^{k+1}+\frac{\partial}{\partial x}\left(F_2\frac{\partial}{\partial x}\left(\Delta u_i^{k+1}\right)\right)\right]_i^k+$$
$$\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial u}\frac{\partial u}{\partial x}\Delta u_i^{k+1}\right)+\frac{\partial F_3}{\partial u}\Delta u_i^{k+1}\right]_i^k\quad ...(25)$$

$$\frac{\Delta u_i^{k+1}}{\Delta t} = \left[\frac{u_{i+1}^k - u_{i-1}^k}{2\Delta x}\right]\left(F_1(u)\right)_i^k + \left(F_3 + \frac{1}{2}\frac{\partial F_3}{\partial u}\Delta u_i^{k+1}\right)_i^k +$$

$$\frac{1}{2}\left[\left(\frac{\Delta u_{i+1}^{k+1} - \Delta u_{i-1}^{k+1}}{2\Delta x}\right)\left(F_1(u)\right)_i^k + \left[\frac{u_{i+1-}^k u_{i-1}^k}{2\Delta}\right]\frac{\partial F_1}{\partial u}\Delta u_i^{k+1}\right] +$$

$$\frac{F_2(u)_{i+\frac{1}{2}}^k\left(u_{i+1}^k - u_i^k\right) - F_2(u)_{i-\frac{1}{2}}^k\left(u_i^k - u_{i-1}^k\right)}{\Delta x^2} +$$

$$\frac{F_2(u)_{i+\frac{1}{2}}^k\left(\Delta u_{i+1}^{k+1} - \Delta u_i^{k+1}\right) - F_2(u)_{i-\frac{1}{2}}^k\left(\Delta u_i^{k+1} - \Delta u_{i-1}^{k+1}\right)}{\Delta x^2} +$$

$$\frac{1}{4}\frac{\left(\frac{\partial F_2}{\partial u}\right)_{i+\frac{1}{2}}^k\left(u_{i+1}^k - u_i^k\right)\left(\Delta u_{i+1}^{k+1} + \Delta u_i^{k+1}\right)}{\Delta x^2} -$$

$$\frac{\left(\frac{\partial F_2}{\partial u}\right)_{i-\frac{1}{2}}^k\left(u_i^k - u_{i-1}^k\right)\left(\Delta u_i^{k+1} + \Delta u_{i-1}^{k+1}\right)}{\Delta x^2} \qquad \ldots(26)$$

We define

$$r = \frac{\Delta t}{2\Delta x}; \quad r1 = \frac{\Delta t}{\Delta x^2}$$

Factorization of equation (14) yields:

$$\left[\frac{r}{2}\left(F_1(u)\right)_i^k - \frac{r1}{2}F_2(u)_{i-\frac{1}{2}}^k + \frac{r1}{4}\left(\frac{\partial F_2(u)}{\partial u}\right)_{i-\frac{1}{2}}^k\left(u_i^k - u_{i-1}^k\right)\right]\Delta u_{i-1}^{k+1} +$$

$$\left[1 - \frac{r}{2}\left(\frac{\partial F_1(u)}{\partial u}\right)_i^k\left(u_{i+1}^k - u_{i-1}^k\right) + \frac{r1}{2}\left(F_2(u)_{i+\frac{1}{2}} + F_2(u)_{i-\frac{1}{2}}^k\right) - \frac{1}{2}\left(\frac{\partial F_2}{\partial u}\right)_{i+\frac{1}{2}}^k\left(u_{i+1}^k - u_i^k\right)\right]\Delta u_i^{k+1} +$$

$$\left[\frac{1}{2}\left(\frac{\partial F_2(u)}{\partial u}\right)_{i-\frac{1}{2}}^k\left(u_i^k - u_{i-1}^k\right) - \frac{\Delta t}{2}\left(\frac{\partial F_3(u)}{\partial u}\right)_i^k\right]\Delta u_i^{k+1} -$$

$$\left[\frac{r}{2}\left(F_1(u)\right)_i^k + \frac{r_1}{2}F_2(u)_{i+\frac{1}{2}}^k + \frac{r1}{4}\left(\frac{\partial F_2(u)}{\partial u}\right)_{i+\frac{1}{2}}^k\left(u_{i+1}^k - u_i^k\right)\right]\Delta u_{i+1}^{k+1} =$$

$$r\left(F_1\right)_i^k\left(u_{i+1}^k - u_{i-1}^k\right) + r1\left[\left(F_2\right)_{i+\frac{1}{2}}^k\left(u_{i+1}^k - u_i\right) - r1\left[\left(F_2\right)_{i-\frac{1}{2}}^k\left(u_i^k - u_{i-1}^k\right)\right]\right] +$$

$$\Delta t\left(F_3\right)_i^k \qquad \ldots(27)$$

Equation (15) can now be put in a typical tri-diagonal matrix form to read:

$$A_i^k \Delta u_{i-1}^{k+1} + B_i^k \Delta u_i^{k+1} + C_i^k \Delta u_{i+1}^{k+1} \qquad ...(28)$$

where

$$A_i^k = \frac{1}{2}\left(F_1(u)\right)_i^k - \frac{r1}{2} F_2(u)_{i-\frac{1}{2}}^k + \frac{r1}{4}\left(\frac{\partial F_2}{\partial u}\right)_{i-\frac{1}{2}}^k \left(u_i^k - u_{i-1}^k\right) \qquad ...(29)$$

$$B_i^k = 1 - \frac{1}{2}\left(\frac{\partial F_1}{\partial u}\right)_i^k \left(u_{i+1}^k - u_{i-1}^k\right) + \frac{r1}{2}\left(F_2(u)_{i+1}^k - F_2(u)_{i-1}^k\right) - \frac{1}{2}\left(\frac{\partial F_2}{\partial u}\right)_{i+\frac{1}{2}}^k \left(u_{i+1}^k - u_i^k\right) + \frac{1}{2}\left(\frac{\partial F_2}{\partial u}\right)_{i-\frac{1}{2}}^k \left(u_i^k - u_{i-1}^k\right) - \frac{\Delta t}{2}\left(\frac{\partial F_3}{\partial u}\right)_i^k \qquad ...(30)$$

$$C_i^k = -\left[\frac{r1}{2}\left(F_1(u)\right)_i^k + \frac{r_1}{2} F_2(u)_{i+\frac{1}{2}}^k + \frac{r1}{4}\left(\frac{\partial F_2}{\partial u}\right)_{i+\frac{1}{2}}^k \left(u_{i+1}^k - u_i^k\right)\right] \qquad ...(31)$$

$$D_i^k = r\left(F_1\right)_i^k \left(u_{i+1}^k - u_{i-1}^k\right) + r1\left[\left(F_2\right)_{i+\frac{1}{2}}^k \left(u_{i+\frac{1}{2}}^k - u_i^k\right) - \left(F_2\right)_{i-\frac{1}{2}}^k \left(u_i^k - u_{i-1}^k\right)\right] + \Delta t\left(F_3\right)_i^k \qquad ...(32)$$

For clarity, we outline the fundamental features of this scheme as follows:

A typical second-order Crank-Nicolson scheme is represented as:

$$\frac{u^{k+1} - u^k}{\Delta t} = 0.5\left[F\left(u^{k+1}\right) + F\left(u^k\right)\right] \qquad (33)$$

This can be put in an easily computable form to read:

$$u^{k+1} = \frac{\Delta t}{2} F\left(u^{k+1}\right) = u^k + \frac{\Delta t}{2} F\left(u^k\right) \qquad (34)$$

To prepare way for Newton linearization, we define:

$$\Gamma\left(u^{k+1}\right) = u^{k+1} - \frac{\Delta t}{2} F\left(u^{k+1}\right) - \left(u^k + \frac{\Delta t}{2} F\left(u^k\right)\right) \qquad (35)$$

We now need to solve a nonlinear algebraic system:

$$\Gamma\left(u^{k+1}\right) = 0 \qquad (36)$$

Following the Newton method and starting with an initial guess: $u^{k+1,(0)}$ we commence iteration:

$$u^{k+1(n+1)} = u^{k+1(n)} + \rho u^{(n)} \qquad (37)$$

where ρu^n is a perturbation obtained from the solution of

$$\frac{\partial \Gamma}{\partial u}\left(u^{(n)}\right)\rho u^{(n)} = -\Gamma\left(u^{(n)}\right) \qquad (38)$$

Central to the application of the Newton method is the precise resolution of the Jacobian :

$$\frac{\partial \Gamma}{\partial u}(u) = J = I - \frac{\Delta t}{2} F'(u) \qquad (39)$$

We finally compute the nonlinear equations to give us a tri-diagonal system, which after each iteration is expressed as:

$$\left(1 - \frac{\Delta t}{2} F'\left(u^n\right)\right)\rho u^{(n)} = -\left[u^n - \frac{\Delta t}{2} F\left(u^n\right) - \left(u^k + \frac{\Delta t}{2} F\left(u^n\right)\right)\right] \qquad (40)$$

The Taylor series technique applied to the time, space and nonlinear term provides the error inherent in this scheme

$$u_i^{k+1} = u_i^k + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + O\left(\Delta t^3\right) \qquad (41)$$

$$\frac{u_{i-1}^k - 2u_i^k + u_{i+1}^k}{h^2} = u_{xx} + O\left(h^2\right) \qquad (42)$$

$$f\left(u_i^{k+1}\right) = f\left(u_i^k\right) + f'\left(u_i^k\right)\left(u_i^{k+1} - u_i^k\right) + O\left(\Delta t^2\right) \qquad (43)$$

To highlight the impact, equations (41),(42) and (43) are substituted into :

$$u_t = u_{xx} + f(u)$$

After cancellation of the leading terms, we obtain:

$$\xi_i^k = O\left(\Delta t^2 + h^2\right) \qquad (44)$$

This shows that the scheme is locally second-order accurate. Numerical and exact solutions offer clue about the global error. We define

$$e_i^k = U_i^k - u_i^k \qquad (45)$$

which is further defined by

$$e^{k+1} = \Omega e^n + \Delta t \xi^n \qquad (46)$$

where Ω is the scheme operator.

From discrete energy estimates and discrete Gronwall inequality

$$\left\| e^{k+1} \right\| \le (1 + C\Delta t)\left\| e^n \right\| + C\Delta t\left(\Delta t^2 + h^2\right) \qquad (47)$$

$$\left\| e^n \right\| \le C\left(\Delta t^2 + h^2\right) \qquad (48)$$

It is hereby shown that

$$\left\| U^k - u^k \right\| = O\left(\Delta t^2 + h^2\right) \qquad (49)$$

We assume that the exact solution is bounded and that the Newton linearization is performed about U^k. In addition, the solution profile satisfies

$$u(x,t) \in C^{4,2}\left([a,b] x \lfloor 0,T \rfloor\right) \qquad (50)$$

for all $0 \le t^n \le T$ where C is independent of h and Δt.

3 Results and Discussion

Equations (1-3) are solved with the following initial and boundary conditions:

IC

$$u(x,0) = -\sin(\pi x),$$
$$x \in [-4, 4] \qquad (51)$$

BC

$$u(-4,t) = u(4,t) = 0,$$
$$u_{xx}(-4,t) = u_{xx}(4,t) = 0 \qquad ...(52)$$

where

$$f(u) = u^3 - u$$

Validation of the numerical results obtained herein is accomplished by comparing them with those available in scientific literature.

The problem domain is divided into

$M_i = 40, 80, 160$, with equal grid space $h_i = \frac{8}{M_i}$.

Since the exact solution is not known, it has been replaced by numerical values for $M_i = 160$. We present graphs obtained for different values of γ for ; $\Delta t = 0.001,\ at\ t = 0, 0.05, 0.1, 0.15,\ and\ 0.2$ for $M_i = 80$ and $\gamma = 0.0001, \ 0.01$.

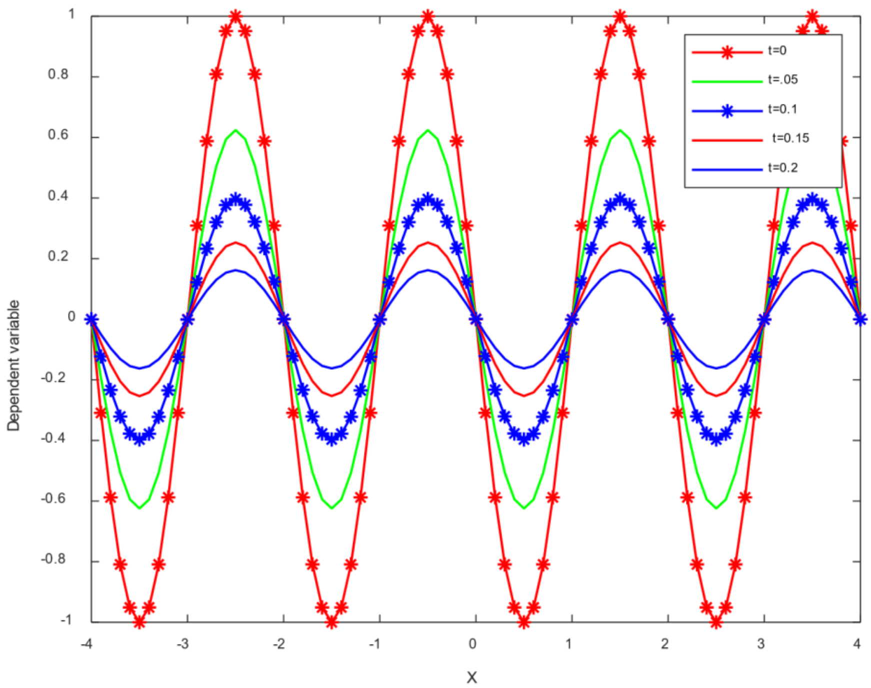

Fig. 1: Solution profiles for

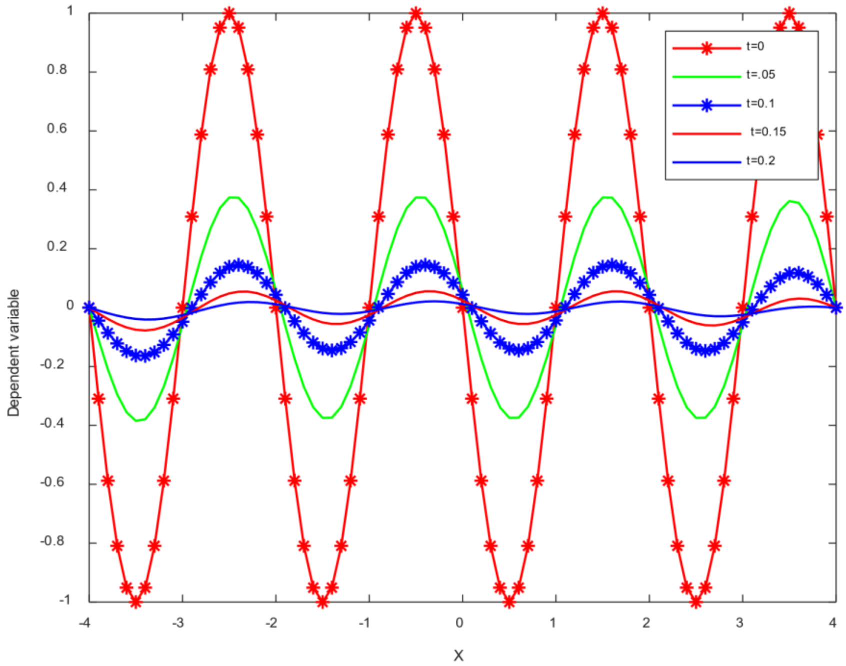

Fig. 2: Solution profiles for

Figs (1-2) show a remarkable decay in the solution profiles as γ increases. This observation confirms literature results [24, 25].

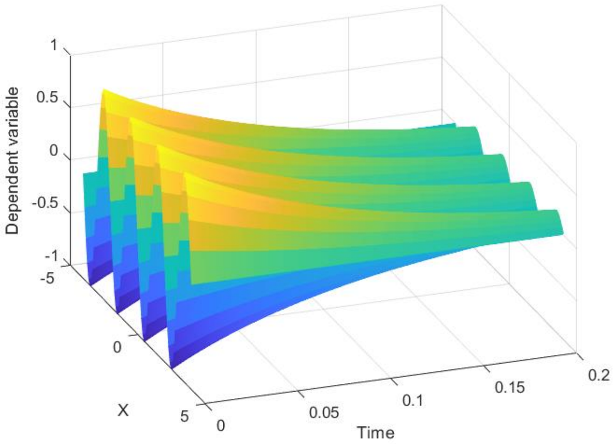

Fig. 3: 3D Solution profiles for

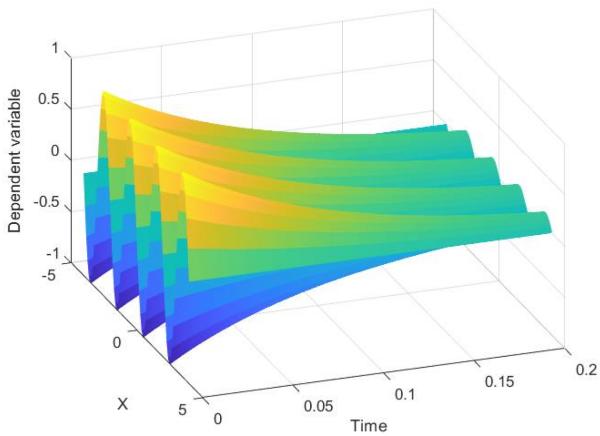

Fig. 4: 3D Solution profiles $\gamma = .001$

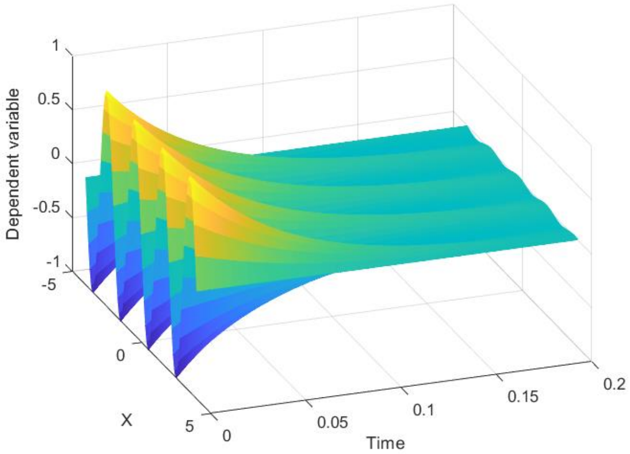

Fig. 5: 3D Solution profiles for

Further confirmation of this behavior can be found in the 3D graphs (Figs (3-5)) of the solution for different values of γ. A noticeable change occurs for $\gamma = 0.01$, thus confirming the stabilizing nature of the fourth derivative in the EFK equation [24].

Table 1. Errors and Norms of numerical solutions $\gamma = .0001$

Errors and Norms	T = 0.15, M=40	T=0.2, M=80
MAE (Mean Average Error)	1.26824765E-02	6.93729071E-03
RMS (Root Mean Square Error)	1.44082419E-02	7.75260508E-03
L_2 *Norm*	8.51144948E-02	4.86844676E-03
L_∞ Norm	1.20001897E-02	3.40842515E-03

Table 1. shows the decrease of errors and norms as the grid points increase and the solution evolves with time

We make further extensions on the validation of the above results by solving equations (1-3) for $x \in [-4,4]$ for the following initial and boundary conditions. Numerical solution for these set of initial and boundary conditions are implemented for

$$h = .025,\ \Delta t = 0.001$$

$$at\ t = 0.25, 1, 1.75,\ 2.5,\ 4.5\ \ for \gamma = 0.0001$$

Figs. (6-7) display the transient solution profiles

$IC1$

$$u(x,0) = -10^{-3}\exp(-x^3), \quad x \in [-4,4] \qquad ...(53)$$

$IC2$

$$u(x,0) = 10^{-3}\exp(-x^3), \quad x \in [-4,4]$$

BC

$$u(-4,t) = u(4,t) = 1, \quad u_{xx}(-4,t) = u_{xx}(4,t) = 0 \qquad ...(54)$$

$$u(-4,t) = u(4,t) = 1, \quad u_{xx}(-4,t) = u_{xx}(4,t) = 0 \qquad ...(55)$$

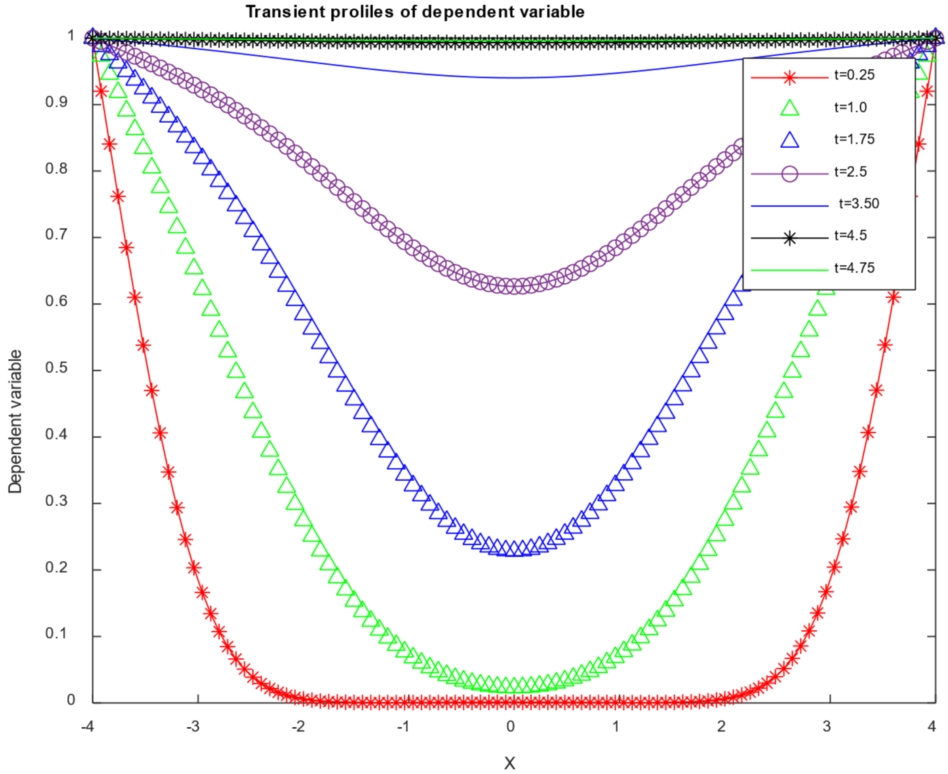

Fig. 6: Solution profiles at different times

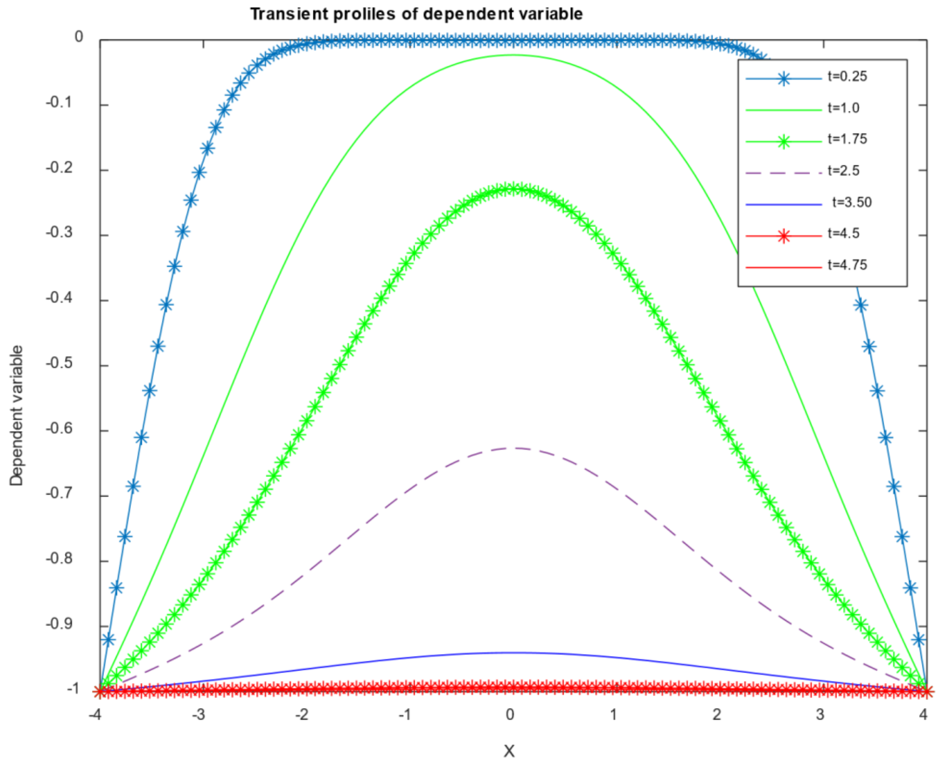

Fig. 7: Transient solution profiles

As expected, both of them not only agree with literature results [24], but also agree with the initial and boundary conditions. Further confirmation of the numerical results is displayed in 3D profiles shown in Figs. (8-9).

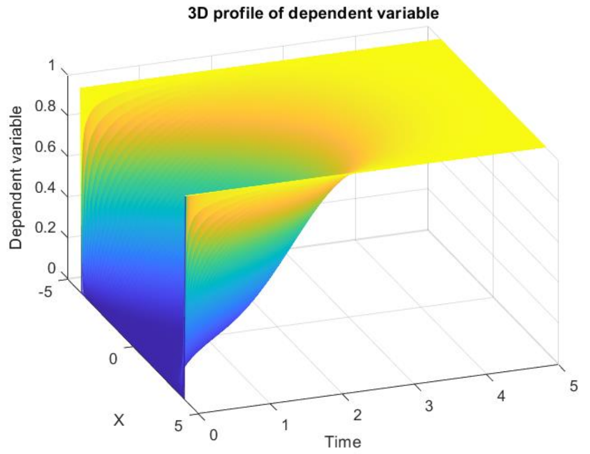

Fig. 8: 3D Plot: Solutions approach 1 as time increases

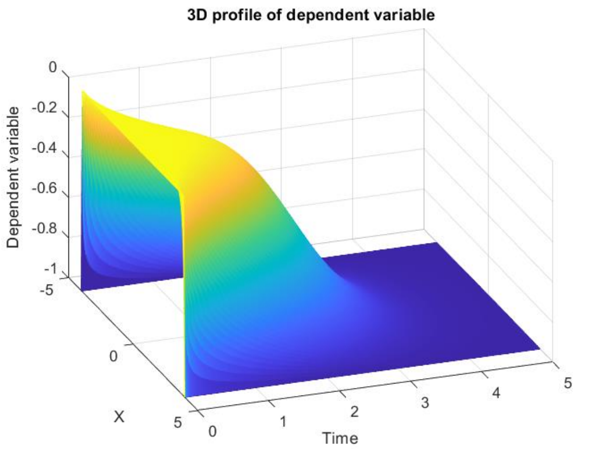

Fig. 9: 3D Plot: Solutions approach -1 as time increases

For the given initial and boundary conditions, the numerical solutions decay and approach 1 and -1 as time increases [24], indicating that the EFK equation approaches stability for this set of conditions.

4 Conclusion

In the work reported herein, a finite difference (FD) based numerical method is applied to produce the numerical solutions of a nonlinear fourth-order partial differential equation. This technique provides approximate solutions which were found to be very close to those available in scientific literature [12], [13], [21], [24], [25]. Both the applicability and the accuracy of the method are confirmed by tables and figures. A major advantage of this technique is its straightforward FD discretization as well as the simplicity in coding resulting in a representation of the discretized equations in a tri-diagonal matrix form. These characteristics should enhance further application to more demanding scenarios involving higher order and nonlinear partial differential equations.

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