Solution of Quadratic Integral Equations by using Numerical Methods

WASAN AJEEL AHMOOD
Department of Al-Quran Science
Al-Iraqia University
Faculty of Education for Women, Baghdad
IRAQ

Abstract: - In my paper, three different forms of QIE will be discussed by using ADM. The first of Volterra type while the second of Fredholm type and the third the general form of the first one. In each case, the existence of a unique solution will be proved. Convergence analysis will be discussed and the maximum absolute truncated error of Adomians series solution will be estimated. The repeated trapezoidal method is used for comparison. In the end of this paper, a numerical implementation technique is used to overcome the difficulties appear in evaluating integrals.

Key-Words: - Integral Equations, Adomian decomposition method, and proposed numerical method.

Received: July 8, 2023. Revised: October 18, 2023. Accepted: December 21, 2023. Published: February 7, 2024.

1 Introduction

Integral equations of various types play an important role in many branches of functional analysis and in their applications in physics, economics and other fields. In particular, quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory and the traffic theory, [1], [2], [3], [4].

In mathematics, integral equations are equations in which an unknown function appears under an integral sign. Integral equations are important in many applications. Problems in which integral equations are encountered include radiative transfer, and the oscillation of a string, membrane, or axle.

The class of quadratic integral equations contains, as a special case, numerous integral equations encountered in the theory of radiative transfer, the queuing theory, the kinetic theory of gases and the theory of neutron transport [5].

The authors in [6] Abel's integral equation, one of the very first integral equations seriously studied, and the corresponding integral operator (investigated by Niels Henrik Abel in 1823 and by Liouville in 1832 as a fractional power of the operator of anti-derivation) have never ceased to inspire mathematicians to investigate and to generalize them.

The author in [7] presented an existence theorem for a nonlinear quadratic integral equation of fractional orders, arising in the queuing theory and biology, in the Banach space of real functions defined and continuous on a bounded and closed interval. The concept of measure of noncompactness and a fixed-point theorem due to Darbo are the main tool in carrying out our proof

Quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Especially,
the so-called quadratic integral equation of Chandrasekher type can be very often encountered in many applications [8], [9], [10], [11].

The authors were provided in [12] correct proof of a slightly modified version of the mentioned result.

The main tool used in our proof is the technique associated with the Hausdorff measure of noncompactness.

The authors were studied in [13] a nonlinear quadratic integral equation of Volterra type in the Banach space of real functions defined and continuous on a bounded and closed interval. With the help of a suitable measure of noncompactness, also showed that the mentioned integral equation has monotonic solutions.

2 The Problem with The Solution Algorithm:

In my paper, I used two methods to solve QIE of Volterra type, ADM and repeated trapezoidal [14], and their results are compared.

Let n,m be two real numbers $\geq 1, I=[0,T], T\in R^{+}$ and E=C(I) the space of continuous functions defined on I with norm

$$\|x\| = \max_{t\in J} \mid x(t) \mid .$$

Consider the nonlinear QIE,
x(t)=p(t)+$ax^{n}\int_0^t k(t,s)x^{m}(s)ds$.
Where

$$|x(t)| < b, \forall\, t \in I$$

The solution algorithm of above equation and by using the domain decomposition method is:

$$x_0(t) = p(t), \quad \dots (1)$$

$$x_i(t) = aA_{i-1}(t)\int_0^t k(t,s)B_{i-1}(s)ds., i \geq 1. \quad \dots (2)$$

Where, A_i and $B_i$are Adomian polynomials of the non-linear tearms x^n and x^m respectively, which take the form

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(\sum_{i=0}^{\infty} \lambda^i\ x_i\right)\right]_{\lambda=0} \quad \dots (3)$$

Finally,

$$x(t) = \sum_{i=0}^{\infty} x_i(t) \quad \dots (4)$$

3 Convergence analysis

Theorem 3.1.(**Existence and uniqueness**): Let $p\epsilon C(I)$ and $|k(t,s)| < k$.
If

$$T < \frac{1}{akb^{n+m-1}(n+m)}$$

Then the QIE eq.(1) has a unique solution $x\ \in C(I)$.

Proof:

The mapping F: E →E is defined as,

$$Fx = \mathrm{p(t)} + ax^n(t)\int_0^t k(t,s)x^m(s)ds.. \quad \dots (5)$$

Let

$$x, y \in E$$

then,

$$F_x - F_y = ax^n(t)\int_0^t k(t,s)x^m(s)ds - ay^n(t)\int_0^t k(t,s)y^m(s)ds =$$
$$= ax^n(t)\int_0^t k(t,s)x^m(s)ds - ay^n(t)\int_0^t k(t,s)x^m(s)ds + ay^n(t)\int_0^t k(t,s)x^m(s)ds - ay^n(t)\int_0^t k(t,s)y^m(s)ds.$$

$$= a[x^n - y^n]\int_0^t k(t,s)x^m(s)ds + ay^n\int_0^t k(t,s)[x^m - y^m]ds.$$

$$= a(x-y)(x^{n-1}+yx^{n-2}+y^2x^{n-3}+\cdots + y^{n-1})\int_0^t k(t,s)x^m(s)ds + ay^n\int_0^t k(t,s)[x-y](x^{m-1} + yx^{m-2}+y^2x^{m-3}+\cdots + y^{m-1})ds.$$

This implies that:

$$\|F_y-F_z\| \le \left(a \max_{t\in J}|x-y||x^{n-1}+yx^{n-2} + y^2x^{n-3}+\cdots + y^{n-1}|\int_0^t |k(t,s)|\,|x^m|ds\right) + \left(a \max_{t\in J}|y^n|\int_0^t |k(t,s)|\,|x - y||x^{m-1}+yx^{m-2}+y^2x^{m-3} + \cdots + y^{m-1}|ds\right)$$

$$\le anb^{n-1}kb^mT\max_{t\in J}|x-y| + ab^nkmb^{m-1}T\max_{t\in J}|x-y|$$
$$\le akTb^{n+m-1}(n+m)\|x-y\|$$
$$\le \alpha\|x-y\|$$

Under the condition

$$0<\alpha = akTb^{n+m-1}(n+m)<1,$$

The mapping F is contraction and hence for

$$T<\frac{1}{akb^{n+m-1}(n+m)}$$

There exists a unique solution $x \in C(I)$
Of the QIE (1) and this completes the proof.

Theorem 3.2. (Proof of convergence):
the series solution

$$\mathrm{y(t)} = \sum_{\mathrm{i=0}}^{\infty}\mathrm{y_i(t)}$$

of the non-linear fractional differential equation using Adomain decomposition method converges if $|y_1(t)|<c$, c is a positive constant.

Proof

Define the sequence $\{S_p\}$, such that $S_p=\sum_{\mathrm{i=0}}^{\mathrm{p}}\mathrm{y_i(t)}$
Is the sequence of partial sums from the series $\sum_{\mathrm{i=0}}^{\infty}\mathrm{y_i(t)}$ since,

$$f\left(t,\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^ny(t),\dots,I^{n-\xi_m}y(t)\right)=S_p$$

So, we can write

$$f\left(t,\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^nS_p,\dots,I^{n-\xi_m}S_p\right)=\sum_{i=0}^{p}A_i(t)$$

From equations (4) and (5), we have:

$$\sum_{i=0}^{\infty}y_i(t)=\mathrm{p(t)}+\sum_{i=0}^{\infty}A_{i-1}$$

Let Sp and Sq be to arbitrary partial sums with p is greater than q, one can get

$$S_p=\sum_{i=0}^{p}y_i(t)=\mathrm{p(t)}+\sum_{i=0}^{p}A_{i-1}$$

And

$$S_q=\sum_{i=0}^{q}y_i(t)=\mathrm{p(t)}+\sum_{i=0}^{q}A_{i-1}$$

Now, we are going to prove that, the Cauchy sequence $\{S_p\}$ in this Banach space E,

$$S_p-S_q=\sum_{i=0}^{p}A_{i-1}-\sum_{i=q+1}^{p}A_{i-1}=\sum_{i=q}^{p-1}A_i$$

$$=f\left(t,\textstyle\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^nS_{p-1},\dots,I^{n-\xi_m}S_{p-1}\right)-f\left(t,\textstyle\sum_{j=0}^{n-1}c_j\frac{t^j}{j!}+I^nS_{q-1},\dots,I^{n-\xi_m}S_{q-1}\right)$$

$$|S_p - S_q| = \left| f\left(t, \sum_{j=0}^{n-1} c_j \frac{t^j}{j!} + I^n S_{p-1}, \dots, I^{n-\xi_m} S_{p-1}\right) - f\left(t, \sum_{j=0}^{n-1} c_j \frac{t^j}{j!} + I^n S_{q-1}, \dots, I^{n-\xi_m} S_{q-1}\right) \right|$$

$$\leq L \sum_{i=0}^{m} \left| I^{n-\xi_1} S_{p-1} - I^{n-\xi_1} S_{q-1} \right|$$

$$\leq L \frac{1}{\Gamma(n-\xi_i)} \int_0^t (t-\tau)^{n-\xi_i-1} \left| S_{p-1} - S_{q-1} \right| d\tau.$$

$$e^{-Nt} |S_p - S_q| \leq L \sum_{i=0}^{m} \frac{1}{\Gamma(n-\xi_i)} \int_0^t e^{-N(t-\tau)} (t-\tau)^{n-\xi_i-1} e^{-N\tau} \left| (S_{p-1} - S_{q-1} \right| d\tau$$

$$\|S_p - S_q\| \leq L \sum_{i=0}^{m} \frac{1}{N^{n-\xi_i}} \|S_{p-1} - S_{q-1}\| \leq \beta \|S_{p-1} - S_{q-1}\|$$

Let p=q+1 then,

$$\|S_{q+1} - S_q\| \leq \beta \|S_q - S_{q-1}\| \leq \beta^2 \|S_{q-1} - S_{q-2}\| \leq \dots \leq \beta^q \|S_1 - S_0\|$$

$$\|S_p - S_q\| \leq \|S_{q+1} - S_q\| \leq \|S_{q+2} - S_{q+1}\| + \dots + \|S_p - S_{p-1}\|$$

$$\leq \beta^q + \beta^{q+1} + \dots + \beta^{p-1} \|S_1 - S_0\|$$
$$\leq \beta^q [1 + \beta + \dots + \beta^{p-q-1}] \|S_1 - S_0\|$$
$$\leq \beta^q \left[\frac{1-\beta^{p-q}}{1-\beta}\right] \|y_1\|$$

Since, $0 < \beta = L \sum_{i=0}^{m} \frac{1}{N^{n-\xi_i}} < 1$, and $p > q$ $then, (1-\beta^{p-q}) \leq 1$. Consequently,

$$\|S_p - S_q\| \leq \left[\frac{\beta^q}{1-\beta}\right] \|y_1\| \leq \left[\frac{\beta^q}{1-\beta}\right] \max_{t \in J} |y_1(t)|$$

But, $|y_1(t)| < c$,and as $q \to \infty$ then, $\|S_p - S_q\| \to 0$ And hence, $\{S_p\}$ is a Cauchy sequence in this Banach space E so, the series $\sum_{i=0}^{\infty} y_i(t)$ convergence.

Theorem 3.3. (Error analysis):
The maximum absolute truncation error of the solution equation (4) to the QIE (1) is estimated to be,

$$\max_{t \in J} \left| x(t) - \sum_{i=0}^{q} x_i(t) \right| \leq \frac{\alpha^q}{1-\alpha} \max_{t \in J} |x_1(t)|$$

Proof
From theorem (4), we have

$$\|S_p - S_q\| \leq \frac{\alpha^q}{1-\alpha} (\max_{t \in J}) |x_1(t)|$$

but,

$$S_p = \sum_{i=0}^{p} x_i,$$

as

$$p \to \infty, \quad S_p \to x(t),$$

so,

$$\|x - S_q\| \leq \frac{\alpha^q}{1-\alpha} (\max_{t \in J}) |x_1(t)|$$

So, the maximum absolute truncation error in the interval *I* is,

$$\max_{t \in J} \left| x(t) - \sum_{i=0}^{q} x_i(t) \right| \leq \frac{\alpha^q}{1-\alpha} \max_{t \in J} |x_1(t)|$$

This is completes the proof.

4. Numerical Examples:

Example (4.1):

Consider the following non-linear QIE,

$$x(t) = \left(t^3 - \frac{t^{12}}{40}\right) + \frac{1}{5} x(t) \int_0^t (ts) x^2(s) ds,$$

… (21)

And has the exact solution $x(t) = t^3$.

Applying ADM to above equation, we

$$x_0(t) = \left(t^3 - \frac{t^{12}}{40}\right)$$

$$x_i(t) = \frac{1}{5} x_{i-1}(t) \int_0^t (ts) A_{i-1}(s) ds, \quad i \geq 1.$$

Where A_i are Adomain polynomials of the nonlinear term $(x)^2$ and the solution will be,

$$x(t) = \sum_{i=0}^{q} x_i(t)$$

This series solution converges if $T < 1.1362194$.

Table 1 shows a caparison between the absolute error of ADM solution and repeated trapezoidal (RT) solution, while table 2 shows the maximum absolute truncated error at different values of q (when t=1) and table 3 shows the maximum absolute error of RT at different values of h (h is the step size).

Table [1]: Absolute Error

t	Error of ADM (q=5)	Error of RT (h=0.01)
0.1	1.21323*10^{-24}	1.16096*10^{-15}
0.2	2.54434*10^{-18}	1.19317*10^{-12}
0.3	1.26909*10^{-14}	6.88523*10^{-11}
0.4	5.33583*10^{-12}	1.22297*10^{-9}
0.5	5.78484*10^{-10}	1.13921*10^{-8}
0.6	2.66071*10^{-8}	7.05709*10^{-8}
0.7	6.76878*10^{-7}	3.30211*10^{-7}
0.8	0.000011148	1.26135*10^{-6}
0.9	0.000131325	4.15054*10^{-6}
1	0.0011805	0.0000122957

Table [2]: Max. Absolute Error

q	max. error of ADM
5	0.00462792
10	0.000359867
15	0.0000279832
20	2.17598*10^{-6}

Table [3]: Max. absolute error

h	Error of RT
0.1	0.001231
0.01	0.0000122957
0.001	1.22955*10^{-7}

Example (4.2):

Consider the following non-linear QIE,

$$x(t) = \left(t^2 - \frac{t^{11}}{630}\right) + \frac{1}{6} x^2(t) \int_0^t (t - s^2) x^2(s) ds,$$

… (21)

And has the exact solution $x(t) = t^2$.

Applying ADM to above equation, we get:

$$x_0(t) = \left(t^2 - \frac{t^{11}}{630}\right)$$

$$x_i(t) = \frac{1}{6} A_{i-1}(t) \int_0^t (t - s^2) A_{i-1}(s) ds, \quad i \geq 1.$$

Where A_i are Adomain polynomials of the non-linear term $(x)^2$. This series solution converges if $T < 1.0699132$.

Table 4 shows a caparison between the absolute error of ADM solution and repeated trapezoidal (RT) solution, while table 5 shows the maximum absolute truncated error at different values of q (when t=1) and table 6 shows the maximum absolute error of RT at different values of h .

Table [4]: Absolute Error

t	Error of ADM (m=5)	Error of RT (h=0.01)
0.1	5.35416*10^{-26}	5.20417*10^{-18}
0.2	5.61407*10^{-20}	7.07767*10^{-16}
0.3	1.86682*10^{-16}	1.21431*10^{-14}

0.4	$\mathbf{5.88678*10^{-14}}$	$\mathbf{9.09828*10^{-14}}$
0.5	$\mathbf{5.10595*10^{-12}}$	$\mathbf{4.33903*10^{-13}}$
0.6	$\mathbf{1.95748*10^{-10}}$	$\mathbf{1.55492*10^{-12}}$
0.7	$\mathbf{4.27188*10^{-9}}$	$\mathbf{4.5728*10^{-12}}$
0.8	$\mathbf{6.1718*10^{-8}}$	$\mathbf{1.16553*10^{-11}}$
0.9	$\mathbf{6.50635*10^{-7}}$	$\mathbf{2.66075*10^{-11}}$
1	$\mathbf{5.34794*10^{-6}}$	$\mathbf{5.57567*10^{-11}}$

Table [5]: Max. Absolute Error

m	max. error of ADM
5	0.00062496895
10	0.00008230044
15	0.00001083792
20	$\mathbf{1.42722*10^{-6}}$

Table [6]: Max. absolute error

h	Error of RT
0.1	$\mathbf{5.5362*10^{-7}}$
0.01	$\mathbf{5.57567*10^{-11}}$
0.001	$\mathbf{5.66214*10^{-15}}$

6 Conclusion

In this paper, I used the Adomian decomposition method to solve the quadratic integral equations, some new theorems are introduced, which give the sufficient conditions of existence, uniqueness, convergence, and maximum absolute truncation error to Adomian decomposition method series solution when applied to these equations. Some numerical examples are discussed and solved by using the Adomian decomposition method.

We see from the results that the exact error coincides with the approximate error obtained from using the theorems, see for example.

We use a numerical method for comparison, we see that after we overcome the disadvantage of this method.

The two methods which we used to solve quadratic integral equations (ADM and the repeated trapezoidal method), each method has an advantage over the other.

References:

[1] I.K. Argyros, “Quadratic quations and applications to Chandrasekher’s and related equations”. Bull. Aust. Math. Soc., 32 (1985), pp. 275-292.

[2] L.W. Busbridge “The Mathematics of Radiative Transfer”. Cambridge Univesrity Press, Cambridge, MA (1960).

[3] P.B. Bosma, W.A. de Rooij. “Efficient method to calculate Chandrasekhar’s H-functions”. Ser. Astron. Astrophys., 126 (1983), pp. 283-292.

[4] J. Juang, K.-Y Lin, W.-W Lin. “Spectral analysis of some iterations in the Chandrasekhar’s H-function”. Numer. Func. Anal. Optim., 24 (5–6) (2003), pp. 575-586

[5] A.H. Salem, On the quadratic integral equations and their applications, Journal Computers & Mathematics with Applications, Vol. 62, October 2011, pp. 2931-2943.

[6] R. Gorenflo, S. Vessella. “Abel Integral Equations: Analysis and Applications”, Lecture Notes in Mathematics, Springer-Verlag, Berlin (1991)

[7] M. Darwis. “On quadratic integral equation of fractional orders, J. Math. Anal. Appl., 311 (1) (2005), pp. 112-119.

[8] J. Banas, A. Martinon, Monotonic solutions of a quadratic integral equation of Volterra type, Comput. Math. ´Appl. 47 (2004) 271–279.

[9] O J. Banas, M. Lecko, W.G. El-Sayed, Existence theorems of some quadratic integral equations, J. Math. Anal. Appl. 222 (1998) 276–285.

[10] S. Chandrasekher, Radiative Transfer, Dover, New York, 1960.

[11] S. Hu, M. Khavani, W. Zhuang, Integral equations arising in the kinetic theory of gases, Appl. Anal. 34, (1989) 261–266.

[12] J. Banaś, B. Rzepka, “ Monotonic solutions of a quadratic integral equation of fractional order”. J. Math. Anal. Appl., 332 (2007), pp. 1370-1378.

[13] J. Banaś, B. Rzepka, “Monotonic solutions of a quadratic integral equation of Volterra type”. J. Math. Anal. Appl., Vol. 47, (2004), pp. 271-279.

[14] L. M. Delves and J. L. Mohamad, Computational methods for Integral Equations, Cambridge University Press, 1985.