Comparing Stephensen Hybrid Method with Other Techniques for Solving Nonlinear Equations

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Abstract: - In the previous article [5], a simple method known as Stephensen's hybrid method was introduced. This method is a modified version of the well-known Stephensen method, which retains all its key features, but with some slight adjustments. The hybrid Stephensen method also converges at a second-order rate. In this article, we will compare Stephensen's hybrid method with some previous methods. Finally, by providing some real examples of non-algebraic functions, we will demonstrate that the new method is an effective approach to finding the root of an equation.

Key-Words: - convergence, numerical computation, Newton method, Stephensen method, Stephensen hybrid hethod, nonlinear equation, iteration formula

Received: June 25, 2022. Revised: January 18, 2023. Accepted: February 19, 2023. Published: April 11, 2023.

1 Introduction

Solving the nonlinear equation is one of the important problems in engineering and many sciences, including mathematics. To solve the nonlinear equation, you can use famous methods such as halving method, displacement method, Newton's method, etc. [2, 3]. The subject of our discussion in this article is not the explanation of these methods, because the explanation and interpretation of these methods can be easily found in most books of numerical calculations.

Newton's method is a practical method that can find new methods by generalizing this method. This method is also known as the Newton-Raphson method. The relatively good convergence of this method and its simplicity compared to other methods have made it very popular among researchers and by generalizing this method, they can achieve other methods. In this article, we will first give a brief explanation about Newton's method and then we will compare several methods.

2 Newton Iteration Formula

We assume that the function f is a non-linear, smooth function $f : R \to R$, and has a simple root x^*, i.e. $f(x^*) = 0$. We also assume that x_0 is an initial approximation of the solution of equation $f(x) = 0$. This first approximation may not be appropriate and that is normal because it is a quick and initial guess. So we have to find a better approximation and use Newton's repetition formula from the initial guess x_0 to reach the real answer x^*. It should be noted that the derivative of function f should not be zero at point x_0. Then we get the point x_1 from the relation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

In order for the obtained approximation to be close to the answer, we must continue this process.

If we continue the process for the second time, the point x_2 is obtained using the relation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

which is again a condition for finding point x_2, the derivative not being zero at point x_1. If we continue this process, the obtained points will get closer and closer to the real answer.

With the condition $f'(x_n) \neq 0$, the process of n will be in the form of $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, where x_{n+1} is an approximation of the solution of equation $f(x) = 0$. If ε is a very small, constant, positive and predetermined value, we continue this process until at least one of the following conditions is met.

$$|x_{n+1} - x_n| < \varepsilon \qquad \text{(i)}$$

$$|f(x_{n+1})| < \varepsilon \qquad \text{(ii)}$$

We call each of these conditions *stop condition*. The smaller the value of ε is, the closer the value of x_{n+1} will be to the root of the equation, which is the true answer. So, the formula for repeating

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (1)$$

by $n = 0,1,2,...$ is Newton's iteration formula, and this formula has convergence of the second order.

3 Some Iteration Formulas Using Newton's Formula

Various methods can be used to find Newton's repetition formula (1) and expand it. One of these famous ways is to use the tangent line and replace the slope of the line and the derivative of the function, which is in many books. [1,6]

Another way to find Newton's formula is to use definite integral methods

$$\int_{x_n}^{x} f'(t)dt = f(x) - f(x_n) \qquad (2)$$

using the rectangular method [5]. At the end, we will reach Newton $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ iteration formula.

If we use the midpoint method in the definite integral (2), after placing and simplifying, we will arrive at the repetition formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - \frac{f(x_n)}{2f'(x_n)})} \qquad (3)$$

If we use the trapezoidal method in the definite integral, we will reach the repetition formula

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n - \frac{f(x_n)}{f'(x_n)})} \qquad (4)$$

, which is the modified Newton's repetition formula. [6,7]

In fact, with different generalizations and substitutions in Newton's formula, we will arrive at new iteration formulas, some of which are superior to others.

In another method, Stephensen's method can be used, and it is actually a generalization of Newton's method. In Newton's method, having a derivative for the desired function makes the work a little difficult, and with a simple substitution instead of the derivative, Stephensen's iteration formula can be reached.

To find Stephensen's iteration method, first, in Newton's iteration formula for the first-order derivative of the function $f(x)$ at the point x with the step length h from the three main approximation formulas, i.e. the forward difference formula

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \qquad (5)$$

or the backward difference formula

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} \qquad (6)$$

or We use the central difference formula

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \qquad (7).$$

If we use the first order derivative of the function $f(x)$ at the point x_n with the step length $h = f(x_n)$ in the forward difference formula (5), we have

$$f'(x_n) \approx \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}.$$

By inserting in Newton's iteration formula (1), we get Stephensen's iteration formula

$$x_{n+1} = x_n - \frac{(f(x_n))^2}{f(x_n + f(x_n)) - f(x_n)} \qquad (8)$$

By replacing each of the differential formulas, we will reach one of the following repetition formulas.

$$x_{n+1} = x_n - \frac{2(f(x_n))^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \qquad (9)$$

$$x_{n+1} = x_n - \frac{(f(x_n))^2}{f(x_n) - f(x_n - f(x_n))} \qquad (10)$$

Each of these iteration formulas has convergence of order higher than two. Now we go to Stephensen's hybrid method. The method of finding the iteration formula of this method is fully explained in [5].

The iteration formula of Stephensen's hybrid method is as

$$x_{n+1} = x_n - 2(f(x_n))^2\left(\frac{w}{f(x_n + f(x_n)) - f(x_n)} + \frac{1-w}{f(x_n) - f(x_n - f(x_n))}\right)$$

(11).

where $0 \le w \le 1$ is a real parameter and by changing this parameter, the convergence of this method can be improved or worsened.
It can be said that the repetition formulas given here are almost part of the simple iteration formulas for obtaining the root of a non-linear equation.

4 Error Comparison In the Presented Methods

Here are some examples of non-linear functions to compare the error.
We want to get the roots of functions f_1, f_2, f_3 and f_4. In table (1) these functions are presented along with their actual solution to compare the amount of error in different methods. [4,5]

Table 1: several functions and their roots

No.	Nonlinear functions	Zero
1	$f_1(x) = \sin^2 x - x + 1$	1.897194306
2	$f_2(x) = 2x^3 - x - 2$	1.165373043
3	$f_3(x) = x^3 - e^x - 1$	2.081116467
4	$f_4(x) = 4\cos x - 3x + 2$	1.177565812

Below is the graph of the error function of each function f_1, f_2, f_3 and f_4 to compare different methods. In each of the graphs, the red color represents Newton's method, the pink color represents the error graph of Stephensen's method, the green color represents the error graph of Stephensen's hybrid method with $w = 0.2$, the purple color represents Stephensen's hybrid method with $w = 0.7$, and the yellow color represents The error graph of Stephensen's hybrid method with $w = 0.5$ is given.
This test has been considered for all functions as $\varepsilon = 10^{-1000}$ and initial guess $x_0 = 1$ and has been coded and graphed using Maple software. The condition for stopping these methods is $|x_{n+1} - x_n| < 10^{-1000}$.

All computations were done using MAPLE using 128 digit floating point arithmetics (Digits:= 128)

Newton Method - iteration formulas (1)

Steffensen Method - iteration formulas (8)

Hybrid Steffensen Method $w = 0.2$ - iteration formulas (11)

Hybrid Steffensen Method $w = 0.7$ - iteration formulas (11)

Steffensen Method $w = 0.5$ - iteration formulas (11)

In each graph, the horizontal axis indicates the number of iterations and the vertical axis indicates the error rate of each method.

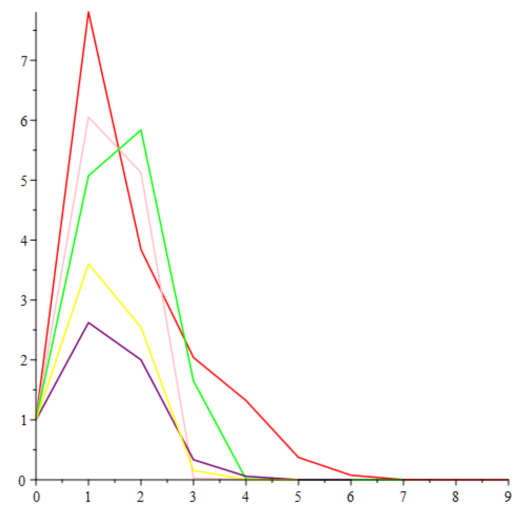

Figure 1: Error plots for function f_1

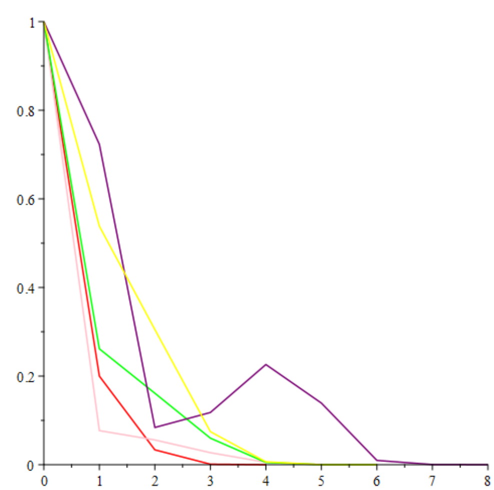

Figure 2: Error plots for function f_2

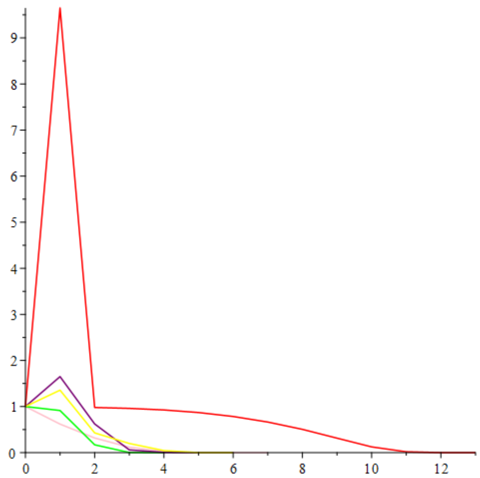

Figure 3: Error plots for function f_3

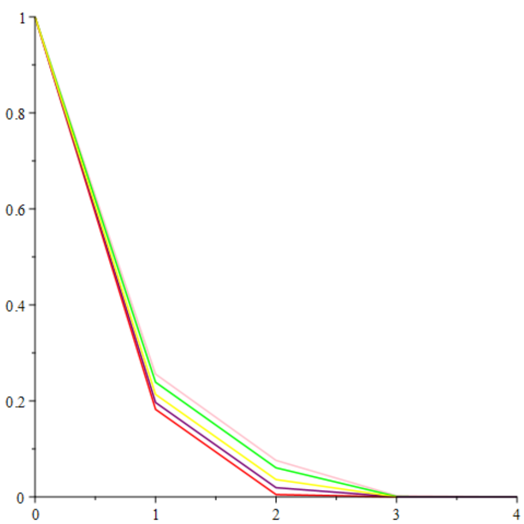

Figure 4: Error plots for function f_4

All computations were done using MAPLE using 128 digit floating point arithmetics (Digits:= 128)

4 Conclusion

In the graphs above, it can be seen that Newton's method has a lot of error in the early stages, but other methods have less error in the early stages, and maybe this is the superiority over Newton's method, as well as the superiority of these other methods over Newton's method, not having a derivative is a function.

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