On Semisimple Semigroups Characterized in Terms Interval valued Q-Fuzzy interior ideals with thresholds ($\bar{\alpha}, \bar{\beta}$)

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Abstract: In this article, we provide relationship be- tween interval valued Q-fuzzy interior ideals with thresh- olds ($\bar{\alpha}, \bar{\beta}$) and interval valued Q-fuzzy ideals with thresh- olds ($\bar{\alpha}, \bar{\beta}$). In the goal results, we proceed to characterize the simisimple semigroup by using interval valued Q-fuzzy interior ideals with thresholds ($\bar{\alpha}, \bar{\beta}$).

Keywords Interval-valued Q-fuzzy ideals with thresholds ($\bar{\alpha}, \bar{\beta}$), Interval-valued Q-fuzzy interior ideals with thresholds ($\bar{\alpha}, \bar{\beta}$), left regular, regular, intra-regular and semisimple semigroup

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1 Introduction and preliminaries

As a generalization of fuzzy set interval valued fuzzy set was conceptualized by Zadeh in 1975[16]. This concept is not only used in mathematics and logic but also in medical science [5], image processing [3] and decision making method [18] etc. In 1994, Biswas [4] used the ideal of interval valued fuzzy sets to interval valued subgroups. In 2006, Narayanan and Manikantan [13] were studied interval valued fuzzy subsemigroups and types interval valued fuzzy ideals in semigroups. In 2014, Aslam et al. [2], gave the concept interval valued $(\overline{\alpha}, \overline{\beta})$-fuzzy ideals of LA-semigroups where $\overline{\alpha}, \overline{\beta} \in \{\overline{\epsilon}, \overline{\epsilon}, \vee \overline{q}\}$ and he characterized regular LA-semigroups by using interval valued $(\overline{\alpha}, \overline{\beta})$-fuzzy ideals. In 2017, Murugads et al. [11] studied interval valued Q-fuzzy subsemigroup of ordered semigroup.

In the same year Abdullah et al. [1] gave the definition of $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy subsemigroups where $\overline{\alpha} \prec \overline{\beta}$, which are generalization of interval valued fuzzy subsemigroups and they characterized regular semigroups in terms of $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy subsemigroups. In 2019, Murugads and Arikrishnan [12] gave concept of interval valued Q-fuzzy ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ where $\overline{\alpha} \prec \overline{\beta}$ and characterized regular semigroups in terms of interval valued Q-fuzzy ideal with thresholds $(\overline{\alpha}, \overline{\beta})$.

In this article, we provide relationship between interval valued Q-fuzzy interior ideals with thresholds $(\overline{\alpha}, \overline{\beta})$ and interval valued Q-fuzzy ideals with thresholds $(\overline{\alpha}, \overline{\beta})$. In the goal results, we proceed to characterize the simisimple semigroup by using interval valued Q-fuzzy interior ideals with thresholds $(\overline{\alpha}, \overline{\beta})$.

2 Preliminaries

In this topic, we give some basic definitions which will be helpful in next topic.

By a subsemigroup of a semigroup S we mean a non-empty subset K of S such that $K^2 \subseteq K$. A non-empty subset K of a semigroup S is called a *left* (right) ideal of S if $SK \subseteq K$ ($KS \subseteq K$). By an *ideal* K of a semigroup S we mean a left ideal and a right ideal of S. A subsemigroup K of a semigroup S is called an *interior ideal* of S if $SKS \subseteq K$. A semigroup S is called *left* (right) regular if for each $u \in S$, there exists $a \in S$ such that $u = au^2$ ($u = u^2a$). A semigroup S is said to be *intra-regular* if for each $u \in S$, there exist $a, b \in S$ such that $u = au^2b$. A semigroup S is called *semisimple* if every ideal of S is an idempotent. It is evident that S is semisimple if and only if $u \in (SuS)(SuS)$ for every $u \in S$, that is there exist $w, y, z \in S$ such that $u = wuyuz$.

For any $m_i \in [0,1]$, $i \in \mathcal{A}$, define

$$\bigvee_{i \in \mathcal{A}} m_i := \sup_{i \in \mathcal{A}} \{m_i\} \quad \text{and} \quad \bigwedge_{i \in \mathcal{A}} m_i := \inf_{i \in \mathcal{A}} \{m_i\}.$$

We see that for any $m, n \in [0,1]$, we have

$$m \vee n = \max\{m, n\} \quad \text{and} \quad m \wedge n = \min\{m, n\}.$$

We use $\mathcal{C}$ to denote the set of all closed subintervals in $[0,1]$, i.e.,

$$\mathcal{C} = \{\overline{m} := [m^-, m^+] \mid 0 \leq m^- \leq m^+ \leq 1\}.$$

We note that $[m, m] = \{m\}$ for all $m \in [0,1]$. For $m = 0$ or 1 we shall denote $\overline{0} = [0,0] = \{0\}$ and $\overline{1} = [1,1] = \{1\}$.

For any two interval numbers $\overline{m}$ and $\overline{n}$ in $\mathcal{C}$, define the operations "$\preceq$", "$=$", "$\curlywedge$" "$\curlyvee$" as follows:

1. $\overline{m} \preceq \overline{n}$ if and only if $m^- \leq n^-$ and $m^+ \leq n^+$
2. $\overline{m} = \overline{n}$ if and only if $m^- = n^-$ and $m^+ = n^+$
3. $\overline{m} \curlywedge \overline{n} = [(m^- \wedge n^-), (m^+ \wedge n^+)]$

4. $\overline{m} \curlyvee \overline{n} = [(m^- \vee n^-), (m^+ \vee n^+)]$.
If $\overline{m} \succeq \overline{n}$, we mean $\overline{n} \preceq \overline{m}$.

The following proposition is a tool used to prove the section 4 and 5.

Proposition 2.1. *[6] For any elements $\overline{m}, \overline{n}$ and $\overline{p}$ in $\mathcal{C}$, the following properties are true:*

1. $\overline{m} \curlywedge \overline{m} = \overline{m}$ *and* $\overline{m} \curlyvee \overline{m} = \overline{m}$,
2. $\overline{m} \curlywedge \overline{n} = \overline{n} \curlywedge \overline{m}$ *and* $\overline{m} \curlyvee \overline{n} = \overline{n} \curlyvee \overline{m}$,
3. $(\overline{m} \curlywedge \overline{n}) \curlywedge \overline{p} = \overline{m} \curlywedge (\overline{n} \curlywedge \overline{p})$ *and* $(\overline{m} \curlyvee \overline{n}) \curlyvee \overline{p} = \overline{m} \curlyvee (\overline{n} \curlyvee \overline{p})$,
4. $(\overline{m} \curlywedge \overline{n}) \curlyvee \overline{p} = (\overline{m} \curlyvee \overline{p}) \curlywedge (\overline{n} \curlyvee \overline{p})$ *and* $(\overline{m} \curlyvee \overline{n}) \curlywedge \overline{p} = (\overline{m} \curlywedge \overline{p}) \curlyvee (\overline{n} \curlywedge \overline{p})$,
5. *If* $\overline{m} \preceq \overline{n}$, *then* $\overline{m} \curlywedge \overline{p} \preceq \overline{n} \curlywedge \overline{p}$ *and* $\overline{m} \curlyvee \overline{p} \preceq \overline{n} \curlyvee \overline{p}$.

For each interval $\{\overline{m}_i := [m_i^-, m_i^+] \mid i \in \mathcal{A}\}$ be a family of closed subintervals of $[0, 1]$. Define $\underset{i\in\mathcal{A}}{\curlywedge} \overline{m}_i = [\underset{i\in\mathcal{A}}{\wedge} m_i^-, \underset{i\in\mathcal{A}}{\wedge} m_i^+]$ and $\underset{i\in\mathcal{A}}{\curlyvee} \overline{m}_i = [\underset{i\in\mathcal{A}}{\vee} m_i^-, \underset{i\in\mathcal{A}}{\vee} m_i^+]$.

Definition 2.1. Let S be a semigroup and Q be a non-empty set. A Q-fuzzy subset (Q-fuzzy set) of a set T is a function $f : S \times Q \to [0, 1]$

Definition 2.2. [15] Let T be a non-empty set. An interval valued fuzzy subset (shortly, IVF subset) of T is a function $\overline{f} : T \to \mathcal{C}$

Definition 2.3. [11] Let S be a semigroup and Q be a non-empty set. An interval valued Q-fuzzy subset (shortly, IVQF subset) of T is a function $\overline{f} : S \times Q \to \mathcal{C}$

Definition 2.4. [11] Let K be a non-empty subset of a semigroup S and Q be a non-empty set . An interval valued characteristic function $\overline{\lambda}_K$ of K is defined to be a function $\overline{\lambda}_K : S \times Q \to \mathcal{C}$ by

$$\overline{\lambda}_K(u, q) = \begin{cases} \overline{1} & \text{if} \quad u \in K \\ \overline{0} & \text{if} \quad u \notin K \end{cases}$$

for all $u \in T$.

For two IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroups S, define

(1) $\overline{f} \sqsubseteq \overline{g} \Leftrightarrow \overline{f}(u, q) \preceq \overline{g}(u, q)$ for all $u \in S$ and $q \in Q$,

(2) $\overline{f} = \overline{g} \Leftrightarrow \overline{f} \sqsubseteq \overline{g}$ and $\overline{g} \sqsubseteq \overline{f}$,

(3) $(\overline{f} \sqcap \overline{g})(u, q) = \overline{f}(u, q) \curlywedge \overline{g}(u, q)$ for all $u \in S$ and $q \in Q$.

For two IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroup S. Then the product $\overline{f} \circ \overline{g}$ is defined as follows for all $u \in S$ and $q \in Q$,

$$(\overline{f} \circ \overline{g})(u, q) = \begin{cases} \underset{(y,z)\in F_u}{\curlyvee} \{\overline{f}(y, q) \curlywedge \overline{g}(z, q)\} & \text{if} \quad F_u \neq \emptyset, \\ \overline{0} & \text{if} \quad F_u = \emptyset, \end{cases}$$

where $F_u := \{(y, z) \in S \times S \mid u = yz\}$.

Next, we shall give definitions of various types of IVQF subsemigroup of a semigroups.

Definition 2.5. [12] An IVF subset $\overline{f}$ of a semigroup S is said to be

(1) an *IVQF subsemigroup* of S if $\overline{f}(uv, q) \succeq \overline{f}(u, q) \curlywedge \overline{f}(v, q)$ for all $u, v \in S$ and $q \in Q$,

(2) an *IVQF left (right) ideal* of S if $\overline{f}(uv, q) \succeq \overline{f}(v, q)$ ($\overline{f}(uv, q) \succeq \overline{f}(u, q)$) for all $u, v \in S$ and $q \in Q$. An *IVQF ideal* of S if it is both an IVQF left ideal and an IVQF right ideal of S,

(3) an *IVQF generalized bi-ideal* of S if $\overline{f}(uvw, q) \succeq \overline{f}(u, q) \curlywedge \overline{f}(w, q)$ for all $u, v, w \in S$ and $q \in Q$,

(4) an *IVQF bi-ideal* of S if $\overline{f}$ is an IVQF subsemigroup of S and $\overline{f}(uvw, q) \succeq \overline{f}(u, q) \curlywedge \overline{f}(w, q)$ for all $u, v, w \in S$ and $q \in Q$,

(5) an *IVQF interior ideal* of S if if $\overline{f}$ is an IVQF subsemigroup of S and $\overline{f}(uav, q) \succeq \overline{f}(a, q)$ for all $a, u, v \in S$ and $q \in Q$,

(6) an *IVQF quasi-ideal* of S if $\overline{f}(u, q) \succeq (\overline{\mathcal{S}} \circ \overline{f})(u, q) \curlywedge (\overline{f} \circ \overline{\mathcal{S}})(u, q)$, for all $u \in S$ and $q \in Q$ where $\overline{\mathrm{S}}$ is an IVQF subset of S mapping every element of S on $\overline{1}$.

The thought of an IVQF subsemigroup with thresholds $(\overline{\alpha}, \overline{\beta})$ where $\overline{\alpha} \prec \overline{\beta}$ as follows:

Definition 2.6. [12] An IVF subset $\overline{f}$ of a semigroup S and $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha}, \overline{\beta} \in \mathcal{C}$ is said to be

(1) an *IVQF subsemigroup with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}(uv, q) \curlyvee \overline{\alpha} \succeq \overline{f}(u, q) \curlywedge \overline{f}(v, q) \curlywedge \overline{\beta}$ for all $u, v \in S$ and $q \in Q$,

(2) an *IVQF left (right) ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}(uv, q) \curlyvee \overline{\alpha} \succeq \overline{f}(v, q) \curlywedge \overline{\beta}$ ($\overline{f}(uv, q) \curlyvee \overline{\alpha} \succeq \overline{f}(u, q) \curlywedge \overline{\beta}$) for all $u, v \in S$ and $q \in Q$. An *IVQF ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if it is both an IVF left ideal and an IVF right ideal of S,

(3) an *IVQF generalized bi-ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}(uvw, q) \curlyvee \overline{\alpha} \succeq \overline{f}(u, q) \curlywedge \overline{f}(w, q) \curlywedge \overline{\beta}$ for all $u, v, w \in S$ and $q \in Q$,

(4) an *IVQF bi-ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}$ is an IVQF subsemigroup with thresholds $(\overline{\alpha}, \overline{\beta})$ of S and $\overline{f}(uvw, q) \curlyvee \overline{\alpha} \succeq \overline{f}(u, q) \curlywedge \overline{f}(w, q) \curlywedge \overline{\beta}$ for all $u, v, w \in S$ and $q \in Q$,

(5) an *IVQF interior ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}$ is an IVQF subsemigroup with thresholds $(\overline{\alpha}, \overline{\beta})$ of S and $\overline{f}(uav, q) \curlyvee \overline{\alpha} \succeq \overline{f}(a, q) \curlywedge \overline{\beta}$ for all $a, u, v \in S$ and $q \in Q$,

(6) an *IVQF quasi-ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ of S if $\overline{f}(u, q) \curlyvee \overline{\alpha} \succeq (\overline{\mathcal{S}} \circ \overline{f})(u, q) \curlywedge (\overline{f} \circ \overline{\mathcal{S}})(u, q) \curlywedge \overline{\beta}$, for all $u \in S$ and $q \in Q$.

Remark 2.1. [12] It is clear to see that every IVQF bi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ is an IVQF generalized bi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S, every IVQF ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ is an IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S and IVQF quasi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ is an IVQF bi-ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S.

The following theorem is easy to prove.

Theorem 2.2. *[12] Every IVQF ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of a semigroup S is an IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S.*

Example 2.1. Consider a semigroup $S = \{0, a, b, c\}$ and Q be any non-empty set

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Let $\overline{f}$ be an IVQF subset of S such that $\overline{f}(0, q) = [0.7, 0.8], \overline{f}(a, q) = [0.4, 0.5], \overline{f}(b, q) = [0.6, 0.7]$ $\overline{f}(c, q) = \overline{0}$ and let $\overline{\alpha} = [0.3, 0.3]$, $\overline{\beta} = [0.5, 0.5]$. Then $\overline{f}$ is not an IVQF interior ideal with $(\overline{\alpha}, \overline{\beta})$ of S . But the $\overline{\lambda}$ is an IVQF ideal with $(\overline{\alpha}, \overline{\beta})$ of S, because $\overline{f}(bc, q) \curlyvee \overline{\alpha} = \overline{f}(a, q) \curlyvee \overline{\alpha} = [0.4, 0.5] \not\succeq [0.5, 0.5] = \overline{f}(b, q) \curlywedge \overline{\beta}$. Thus $\overline{f}$ is not an IVQF right ideal subsemigroup with $(\overline{\alpha}, \overline{\beta})$ of S.

The following theorem show that the IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ and IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ coincide for some types of semigroups. The proof of this theorem is straightforward and simple.

Lemma 2.3. *Let S be a semigroup. If S is left (right) regular, then every IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of S is thresholds $(\overline{\alpha},\overline{\beta})$-IVF ideal of S.*

Proof. Suppose that $\overline{f}$ is an IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of S and let $u,v\in S$ and $q\in Q$. Since S is left regular, there exists $k\in S$ such that $u=ku^2$. Thus, $\overline{f}(uv,q)\curlyvee\overline{\alpha}=\overline{f}((ku^2)v,q)\curlyvee\overline{\alpha}=\overline{f}(kuuv,q)\curlyvee\overline{\alpha}=\overline{f}((ku)uv,q)\curlyvee\overline{\alpha}\succeq\overline{f}(u,q)\curlywedge\overline{\beta}$. Hence $\overline{f}$ is an IVQF right ideals with $(\overline{\alpha},\overline{\beta})$ of S. Similarly, we can show that $\overline{f}$ is an IVQF left ideals with $(\overline{\alpha},\overline{\beta})$ of S. Thus $\overline{f}$ is an IVQF ideals with $(\overline{\alpha},\overline{\beta})$ of S. □

Lemma 2.4. *Let S be a semigroup. If S is intra-regular, then every IVQF interior ideals with $(\overline{\alpha},\overline{\beta})$ of S is an IVQF ideal thresholds $(\overline{\alpha},\overline{\beta})$ of S.*

Proof. Suppose that $\overline{f}$ is an IVQF interior ideals with $(\overline{\alpha},\overline{\beta})$ of semigroup S and let $u,v\in S$ and $q\in Q$. Since S is intra-regular, there exist $x,y\in S$ such that $u=xu^2y$. Thus, $\overline{f}(uv,q)\curlyvee\overline{\alpha}=\overline{f}((xu^2y)v,q)\curlyvee\overline{\alpha}=\overline{f}((xuuy)v,q)\curlyvee\overline{\alpha}=\overline{f}((xu)u(yv),q)\curlyvee\overline{\alpha}\succeq\overline{f}(u,q)\curlywedge\overline{\beta}$. Hence $\overline{f}$ is an IVQF right ideals with $(\overline{\alpha},\overline{\beta})$ of S. Similarly, we can show that $\overline{f}$ is an IVQF left ideals with $(\overline{\alpha},\overline{\beta})$ of S. Thus $\overline{f}$ is an IVQF ideals with $(\overline{\alpha},\overline{\beta})$ of S. □

Lemma 2.5. *Let S be a semigroup. If S is semisimple, then every IVQF interior ideals with $(\overline{\alpha},\overline{\beta})$ of S is is an IVQF ideal thresholds $(\overline{\alpha},\overline{\beta})$ of S.*

Proof. Suppose that $\overline{f}$ is an IVQF interior ideals with $(\overline{\alpha},\overline{\beta})$ of S and let $u,v\in S$ and $q\in Q$. Since S is semisimple, there exist $x,y,z\in S$ such that $u=xuyuz$. Thus, $\overline{f}(uv,q)\curlyvee\overline{\alpha}=\overline{f}((xuyuz)v,q)\curlyvee\overline{\alpha}=\overline{f}((xuy)u(zv),q)\curlyvee\overline{\alpha}\succeq\overline{f}(u,q)\curlywedge\overline{\beta}$. Hence $\overline{f}$ is an IVQF right ideals with $(\overline{\alpha},\overline{\beta})$ ideal of S. Similarly, we can show that $\overline{f}$ is an IVQF left ideals with $(\overline{\alpha},\overline{\beta})$ of S. Thus $\overline{f}$ is an IVQF ideals with $(\overline{\alpha},\overline{\beta})$ of S. □

By Lemma 2.3, 2.4 and 2.5 we have Theorem 2.6.

Theorem 2.6. *In left (right) regular, intra-regular and semisimple semigroup, the IVQF interior ideals with $(\overline{\alpha},\overline{\beta})$ and the is an IVQF ideal thresholds $(\overline{\alpha},\overline{\beta})$ coincide.*

In this ensuing theorem is present relationship between types ideals of a semigroup S and the interval valued characteristic function.

Theorem 2.7. *[12] If K is a left ideal (right ideal generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) of S, then characteristic function $\overline{\chi}_K$ is an IVQF left ideal (right ideal, generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) with thresholds $(\overline{\alpha},\overline{\beta})$ of S for all $\overline{\alpha}\prec\overline{\beta}$ and $\overline{\alpha},\overline{\beta}\in\mathcal{C}$.*

3 Characterize semsimiple semigroups in terms IVQF interior ideal with thresholds $(\overline{\alpha},\overline{\beta})$ and IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$.

In this topic, we will characterize a semsimiple semigroup in terms of IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ and IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$.

In 2019, [12] Murugads and Arikrishnan propose symbols of IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ for use characterizes a semigroup in terms IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ of semigroup.

For any IVQF subset $\overline{f}$ of a semigroup S with $\overline{\alpha}\prec\overline{\beta}$ and $\overline{\alpha},\overline{\beta}\in\mathcal{C}$., define

$$\overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q)=(\overline{f}(u,q)\curlywedge\overline{\alpha})\curlyvee\overline{\beta}$$

for all $u\in S$ and $q\in Q$.

For any IVQF subsets $\overline{f}$ and $\overline{g}$ of a semigroup S with $\overline{\alpha}\prec\overline{\beta}$ and $\overline{\alpha},\overline{\beta}\in\mathcal{C}$, define the operation "$\curlywedge^{\overline{\alpha}}_{\overline{\beta}}$" as follows:

$$(\overline{f}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{g})(u,q)=(\overline{f}(u,q)\curlywedge\overline{g}(u,q)\curlywedge\overline{\alpha})\curlyvee\overline{\beta}$$

for all $u\in S$ and $q\in Q$. And define the product $\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{g}$ as follows: for all $u\in S$ and $q\in Q$,

$$(\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{g})(u,q)=((\overline{f}\circ\overline{g})(u,q)\curlywedge\overline{\alpha})\curlyvee\overline{\beta}$$

where

$$(\overline{f}\circ\overline{g})(u,q)=\begin{cases}\curlyvee_{(x,y)\in F_u}\{\overline{f}(x,q)\curlywedge\overline{g}(y,q)\} & \text{if } F_u\neq\emptyset,\\ \overline{0} & \text{if } F_u=\emptyset,\end{cases}$$

where $F_u:=\{(x,y)\in S\times S\mid u=xy\}$.

Remark 3.1. Since $\overline{\chi}$ is an interval valued characteristic, we have

$$\overline{\lambda}_{(\overline{\alpha},\overline{\beta})}(u,q):=\begin{cases}\overline{\beta} & \text{if } u\in K,\\ \overline{\alpha} & \text{if } u\notin K.\end{cases}$$

Lemma 3.1. *[12] Let K and L be non-empty subsets of a semigroup S with $\overline{\alpha}\prec\overline{\beta}$ and $\overline{\alpha},\overline{\beta}\in\mathcal{C}$. Then the following assertions hold:*

(1) $(\overline{\lambda}_K)\curlywedge^{\overline{\alpha}}_{\overline{\beta}}(\overline{\lambda}_L)=(\overline{\lambda}_{K\cap L})_{(\overline{\alpha},\overline{\beta})}$.

(2) $(\overline{\lambda}_K)\circ^{\overline{\alpha}}_{\overline{\beta}}(\overline{\lambda}_L)=(\overline{\lambda}_{KL})_{(\overline{\alpha},\overline{\beta})}$.

On the basis of Lemma 3.2, we can prove Theorem 3.4.

Lemma 3.2. *[12] Let S be a semigroup. If $\overline{f}$ is a $(\overline{\alpha},\overline{\beta})$-IVQF right ideal and $\overline{g}$ is a $(\overline{\alpha},\overline{\beta})$-IVQF left ideal of S, then $\overline{f}\underset{(\overline{\beta},\overline{\alpha})}{\circ}\overline{g}\sqsubseteq\overline{f}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{g}$.*

Lemma 3.3. *[10] For a semigroup S, the following statements are equivalent.*

1. *S is semisimple,*
2. *Every interior ideal K of S is idempotent,*
3. *Every ideal K of S is idempotent,*
4. *For any ideals K and L of S, $K\cap L=KL$*
5. *For any ideal K and any interior ideal L of S, $K\cap L=KL$*
6. *For any interior K and any ideal L of S, $K\cap L=KL$*
7. *For any interior ideals K and L of S, $K\cap L=KL$.*

The following Theorem show an equivalent conditional statement for a semisimple semigroup.

Theorem 3.4. *Let S be a semigroup. Then the following are equivalent:*

1. *S is semisimple,*
2. *$\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{f}=\overline{f}$, for every IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{f}$ of S,*
3. *$\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{f}=\overline{f}$, for every IVQF ideals with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{f}$ of S,*
4. *$\overline{f}\circ^{\overline{\alpha}}_{\overline{\beta}}\overline{g}=\overline{f}\curlywedge^{\overline{\alpha}}_{\overline{\beta}}\overline{g}$, for every IVQF interior ideals with thresholds $(\overline{\alpha},\overline{\beta})$ $\overline{f}$ and $\overline{g}$ of S,*

5. $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} = \overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$, *for every IVQF ideals with thresholds* $(\overline{\alpha}, \overline{\beta})$ $\overline{f}$ *and* $\overline{g}$ *of* S,
6. $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} = \overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$, *for every IVQF interior ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ $\overline{f}$ *of* S *and every IVQF ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ $\overline{g}$ *of* S,
7. $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} = \overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$, *for every IVQF ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ $\overline{f}$ *of* S *and every IVQF interior ideal with thresholds* $(\overline{\alpha}, \overline{\beta})$ $\overline{g}$ *of* S.

Proof. (1) $\Rightarrow$ (2) Suppose that $\overline{f}$ is a IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S. Then $\overline{f}$ is a IVQF subsemigroup with thresholds $(\overline{\alpha}, \overline{\beta})$. We will show that $\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{f} = \overline{f}_{(\overline{t},\overline{s})}$. Let $u \in S$ and $q \in Q$.

If $F_u = \emptyset$, then it is easy to verify that $(\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{f})(u,q) \preceq \overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q)$.

If $F_u \neq \emptyset$, then

$$\begin{aligned}
(\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{f})(u,q) &= (\curlyvee_{(x,y)\in F_u} \{\overline{f}(x,q) \curlywedge \overline{f}(y,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\curlyvee_{(x,y)\in F_u} \{\overline{f}(x,q) \curlywedge \overline{f}(y,q) \curlywedge \overline{\beta}\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\preceq (\curlyvee_{(x,y)\in F_u} \{\overline{f}(xy,q) \curlyvee \overline{\alpha}\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= ((\overline{f}(u,q) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= ((\overline{f}(u,q) \curlyvee \overline{\alpha}) \curlyvee \overline{\alpha}) \curlywedge (\overline{\beta} \curlyvee \overline{\alpha}) \\
&= (\overline{f}(u,q) \curlyvee \overline{\alpha}) \curlywedge (\overline{\beta} \curlyvee \overline{\alpha}) \\
&= (\overline{f}(u,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= \overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q).
\end{aligned}$$

Thus, $(\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{f})(u,q) \preceq \overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q)$. Hence, $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{f} \sqsubseteq \overline{f}_{(\overline{\alpha},\overline{\beta})}$.

Since S is semisimple, we have there exist $w,x,y,z \in S$ such that $u = (xuy)(zuw)$. Thus

$$\begin{aligned}
(\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{f})(u,q) &= (\curlyvee_{(i,j)\in F_u} \{\overline{f}(i,q) \curlywedge \overline{f}(j,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\curlyvee_{(i,j)\in F_{(xuy)(wuz)}} \{\overline{f}(i,q) \curlywedge \overline{f}(j,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\succeq ((\overline{f}(xuy,q) \curlywedge \overline{f}(wuz,q)) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= ((\overline{f}(xuy,q) \curlyvee \overline{\alpha}) \curlywedge (\overline{f}(wuz,q) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\succeq ((\overline{f}(u,q) \curlywedge \overline{\beta}) \curlywedge (\overline{f}(u,q) \curlywedge \overline{\beta}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= ((\overline{f}(u,q) \curlywedge \overline{\beta}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\overline{f}(u,q) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} = \overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q).
\end{aligned}$$

Hence, $(\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{f})(u,q) \succeq \overline{f}_{(\overline{\alpha},\overline{\beta})}(u,q)$, and so $\overline{f}_{(\overline{\alpha},\overline{\beta})} \sqsubseteq \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{f}$. Therefore, $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{f} = \overline{f}_{(\overline{\alpha},\overline{\beta})}$.

(2) $\Rightarrow$ (1) Let K be an interior ideal of S. Then by Theorem 2.7, $\overline{\lambda}_K$ is a IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S. By supposition and Lemma 3.1, we have

$$(\overline{\lambda}_{K^2})_{(\overline{\alpha},\overline{\beta})}(u,q) = ((\overline{\lambda}_K) \circ^{\overline{\alpha}}_{\overline{\beta}} (\overline{\lambda}_K))(u,q) = (\overline{\lambda}_K)_{(\overline{\alpha},\overline{\beta})}(u,q) = \overline{\beta}.$$

Thus $u \in K^2$. Hence $K^2 = K$. By Lemma 3.3, we have S is semisimple.

(1) $\Rightarrow$ (4) Let $\overline{f}$ and $\overline{g}$ be IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S. Then by Theorem 2.5, $\overline{f}$ and $\overline{g}$ are IVQF ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S. Thus by Lemma 3.2, $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \sqsubseteq \overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$. On other hand, let $u \in S$ and $q \in Q$. Then there exist $w,x,y,z \in S$ such that $u = (xuy)(zuw)$. Thus

$$\begin{aligned}
(\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q) &= (\curlyvee_{(i,j)\in F_u} \{\overline{f}(i,q) \curlywedge \overline{g}(j,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\curlyvee_{(i,j)\in F_{(xuy)(wuz)}} \{\overline{f}(i,q) \curlywedge \overline{g}(j,q)\} \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\succeq ((\overline{f}(xuy,q) \curlywedge \overline{g}(wuz,q)) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= ((\overline{f}(xuy,q) \curlyvee \overline{\alpha}) \curlywedge (\overline{g}(wuz,q) \curlyvee \overline{\alpha}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&\succeq ((\overline{f}(u,q) \curlywedge \overline{\beta}) \curlywedge (\overline{g}(u,q) \curlywedge \overline{\beta}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (((\overline{f}(u,q) \curlywedge \overline{g}(u,q)) \curlywedge \overline{\beta}) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= ((\overline{f}(u,q) \curlywedge \overline{g}(u,q)) \curlywedge \overline{\beta}) \curlyvee \overline{\alpha} \\
&= (\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q).
\end{aligned}$$

Hence, $(\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q) \succeq (\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g})(u,q)$ and so $\overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g} \sqsubseteq \overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$. Therefore, $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} = \overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$.

(4) $\Rightarrow$ (1) Let K and L be interior ideals of S. Then by Theorem 2.7, $\overline{\lambda}_K$ and $\overline{\lambda}_L$ are IVQF interior ideals with thresholds $(\overline{\alpha}, \overline{\beta})$ of S. By supposition and Lemma 3.1, we have

$$\begin{aligned}
(\overline{\lambda}_{KL})_{(\overline{\alpha},\overline{\beta})}(u,q) &= ((\overline{\lambda}_K) \circ^{\overline{\alpha}}_{\overline{\beta}} (\overline{\lambda}_L))(u,q) \\
&= ((\overline{\lambda}_K) \curlywedge^{\overline{\alpha}}_{\overline{\beta}} (\overline{\lambda}_L))(u,q) \\
&= (\overline{\lambda}_{K\cap L})_{(\overline{\alpha},\overline{\beta})}(uq) = \overline{\beta}.
\end{aligned}$$

Thus, $u \in KL$. Hence, $KL = K \cap L$. By Lemma 3.3, S is semisimple.

(1) $\Rightarrow$ (6) Let $\overline{f}$ and $\overline{g}$ be an IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ and an IVQF ideal with thresholds $(\overline{\alpha}, \overline{\beta})$of S respectively. Then $\overline{g}$ is an IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S. Thus by (4), $\overline{f} \circ^{\overline{\alpha}}_{\overline{\beta}} \overline{g} = \overline{f} \curlywedge^{\overline{\alpha}}_{\overline{\beta}} \overline{g}$.

(6) $\Rightarrow$ (1) Let K, L be an interior ideal and ideal of S respectively. Then by Theorem 2.7, $\overline{\lambda}_K$ and $\overline{\lambda}_L$ is a $\overline{\lambda}_L$ are IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ and $\overline{\lambda}_L$ are IVQF ideals with thresholds $(\overline{\alpha}, \overline{\beta})$ of S respectively. Then by Theorem 2.2, $\overline{\lambda}_L$ is an IVQF interior ideal with thresholds $(\overline{\alpha}, \overline{\beta})$ of S. Similarly from (4) $\Rightarrow$ (1), we have S is semisimple.

So, (1) $\Leftrightarrow$ (3), (1) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (7) are Straightforward. □

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