A Two-Parameter Exponential-Akash Distribution: Theory and Application

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Abstract: In this research, we developed a new distribution obtained from Akash distribution using the exponentiation method. The new distribution is named Two-Parameter Exponential Akash (TPEA) Distribution, which is a hybrid distribution where a shape parameter is introduced to the Akash distribution. Some statistical properties of the proposed hybrid distribution include reliability function, hazard rate function, reversed hazard function, moments and its related measures, moment generating function and order statistics are derived. Method of maximum likelihood estimation was used to estimate the distribution parameters. Also, shapes of the density, distribution, survival and hazard functions of the developed distribution were illustrated and presented. We examined the consistency and performance of the parameters estimates of the proposed distribution, using a real life data set and compared the results with some extant distributions. The results of the comparison include: goodness of fit statistics and criteria, standard error and p-value. This reveals that the new distribution has better fit and representation of the data set than other distributions. We hope that this new distribution will serve as an alternative model in some areas like Medicine, Agriculture, Finance, Insurance and lifetime analysis.

Keywords: Exponentiation method, Goodness of fit, Order statistics, Shape Parameter, Standard Error

1 Introduction

Statistics theory and application are the bedrock of analysis of lifetime data in real-life phenomena, whereby the nature of lifetime data in different fields of study such as Engineering, Medical science, Finance, Quality control, Insurance etc; requires probability distribution that will determine the shape and follow the distribution of the lifetime data to give better fit to the dataset. In view of this, various authors and researchers in literature have developed several distributions for modelling and analyzing lifetime datasets. Uwaeme *et al.* [9] used exponentiation method to improve on Pranav distribution and proposed extended Pranav distribution. Also, they used a lifetime dataset of strength of glass of the aircraft window and the results obtained revealed that their distribution has better fit and overwhelmed than other distributions compared in their work.

Furthermore, examples of other proposed distributions with one parameter for modelling lifetime datasets are: Gupta and Kundu [2] generalized exponential distribution, Shanker [6] Akash distribution, Shanker [7] Shanker distribution, Shukla [8] Pranav distribution, Abebe and Shukla [1] Discrete Pranav distribution among others. Meanwhile, there is need to have a robust, flexible and better fit distribution to lifetime datasets than some of the distributions mentioned above due to their status as parent distributions.

In this work, we propose a two-parameter exponential-Akash (TPEA) distribution using exponentiation method in conjunction with Akash distribution as a parent distribution. We established its flexibility and ability to provide a better representation and fit for real life datasets.

2 Material and Methods

2.1 TPEA distribution

The exponentiation method was initiated by Mudholkar and Srivastava [4] where they introduced exponentiated Weibull family known as extension of the Weibull family. Let a random variable Y be an exponentiated distribution with probability density function (pdf) and cumulative distribution function (cdf) given respectively by

$$g(y) = w(y).c[W(y)]^{c-1} \quad y > 0, c > 0 \tag{1}$$

and

$$G(y) = [W(y)]^c \tag{2}$$

where, $w(y)$ and $W(y)$ are the pdf and cdf of any univariate continuous distribution. Though, in this work, $w(y)$ and $W(y)$ are the pdf and cdf of the one-parameter Akash distribution. This was recently investigated by Shanker [6] and are defined by

$$w(y) = \frac{\theta^3}{\theta^2+2}(1+y^2)e^{-\theta y} \tag{3}$$

And

$$W(y) = 1 - \left[1 + \frac{\theta y(\theta y+2)}{\theta^2+2}\right]e^{-\theta y} \tag{4}$$

Now, we obtain both the pdf and cdf of the Two-Parameter exponential Akash distribution by substituting equations (3) and (4) into (1) and (2) as given below:

$$g(y) = \frac{\theta^3}{\theta^2+2}(1+y^2)e^{-\theta y}.c[U]^{c-1} \tag{5}$$

and

$$G(y) = \int_0^{W(y)} Ce^{cy}dy = [U]^c \tag{6}$$

where, $U = W(y) = 1 - \left[1 + \frac{\theta y(\theta y+2)}{\theta^2+2}\right]e^{-\theta y}$, $y > 0, c > 0$ and $\theta > 0$, c is the additional shape parameter and θ is the existing scale parameter. Therefore, (5) and (6) become the pdf and cdf of the TPEA distribution; and the graphical illustrations of the pdf and cdf alongside with the survival and hazard function of the proposed distribution are depicted below.

Also, other associating functions such as reliability, hazard rate and reversed hazard rate functions are obtain and discuss as follows:

2.2 The Survival Function

The reliability function of the TPEA distribution is defined as the probability that a system or an object still survives prior the time t; and it's given by

$$SF(y) = 1 - P(Y \leq y) = 1 - G(y) \tag{7}$$

$$SF(y) = 1 - [U]^c \tag{8}$$

2.3 The Hazard Rate Function

The hazard function is also known as risk function and is defined as conditional probability of failure rate that the probability of an object fail within a survived time t; and its expression is given by

$$HF(y) = \frac{g(y)}{1-G(y)} = \frac{\frac{\theta^3}{\theta^2+2}(1+y^2)e^{-\theta y}.c[U]^{c-1}}{1-[U]^c} \tag{9}$$

2,4 The Reversed Hazard Function

The reversed hazard function of the TPEA distribution is defined as the ratio between the density function to its cumulative density function in which we expressed as

$$RHF(y) = \frac{g(y)}{G(y)} = \frac{\frac{\theta^3}{\theta^2+2}(1+y^2)e^{-\theta y}.c[U]^{c-1}}{1-[U]^c} \tag{10}$$

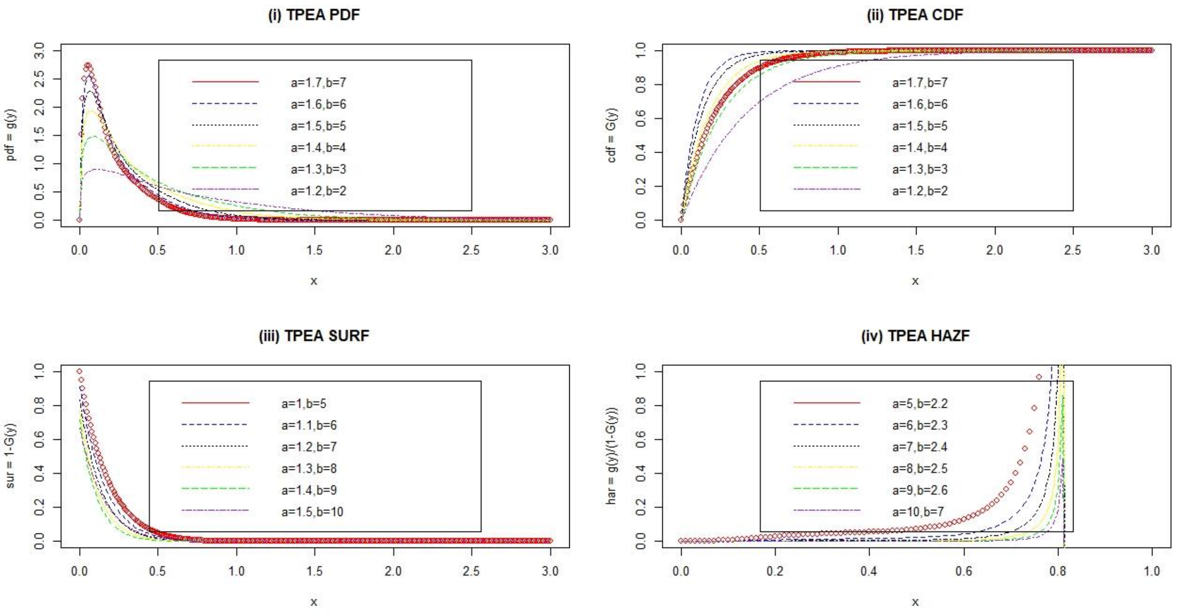

Figure 1: The density, distribution, survival and hazard function plot of the TPEA distribution.

3 Moments and Generating Function

Both the moments and generating function of the propose distribution are derived and obtained mathematically in this section.

3.1 Moments

As we have discussed earlier, moments of the new distribution is derived and presented as follows:

$$E(Y^z) = \int_0^{W(y)} y^z g(y)\partial y \qquad (11)$$

Now, putting (5) into (11), we obtain (12)

$$E(Y^z) = \int_0^{\infty} y^z \frac{c\,\theta^3}{\theta^2+2}(1+y^2)e^{-\theta y}.[U]^{c-1}\partial y \qquad (12)$$

Multiplying and open the bracket of the expression after the integral sign in (12), we get (13)

$$E(Y^z) = \int_0^{\infty} \frac{C\,\theta^3 y^z}{\theta^2+2} e^{-\theta y}.[U]^{c-1}\partial y + \int_0^{\infty} \frac{c\,\theta^3 y^{z+2}}{\theta^2+2} e^{-\theta y}.c[U]^{c-1}\partial y \qquad (13)$$

Thus, by using binomial expansion in (13), we have

$$[U]^{c-1} = \sum_{l=0}^{\infty}\binom{c-1}{l}(-1)^l[V]\,e^{-\theta y}$$

where, $V = \left[1+\frac{\theta y(\theta y+2)}{\theta^2+2}\right]^l$ also, the binomial expansion of $\left[1+\frac{\theta y}{\theta+1}\right]^l$ yield by

$$V = \sum_{m=0}^{\infty}\binom{l}{m}\left[\frac{\theta y(\theta y+2)}{\theta^2+2}\right]^m$$
$$= \sum_{m=0}^{\infty}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\sum_{p=0}^{n}\binom{n}{p}\frac{\theta^l y^m}{(\theta^2+2)^m}2^n.1^p.\,\theta^{m-n-p}.y^{m-n-p}$$

Hence,

$$[U]^{c-1} = \sum_{l=0}^{\infty}\binom{c-1}{l}(-1)^l\sum_{m=0}^{\infty}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\sum_{p=0}^{n}\binom{n}{p}\frac{\theta^l y^m}{(\theta^2+2)^m}2^n.1^p.\theta^{m-n-p}.y^{m-n-p}$$

By substituting the above expressions into (13) gives the following:

$$E(Y^z) = \sum_{l=0}^{\infty}\binom{c-1}{l}(-1)^l\sum_{m=0}^{\infty}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\sum_{p=0}^{n}\binom{n}{p}.A.\int_0^{\infty} y^{z+2m-n-p}e^{-\theta y(l+1)}\partial y + \sum_{l=0}^{\infty}\binom{c-1}{l}(-1)^l \sum_{m=0}^{\infty}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\sum_{p=0}^{n}\binom{n}{p}.A.\int_0^{\infty} y^{z+2m-n-p+2}e^{-\theta y(l+1)}\partial y$$

where $A = \frac{c.2^n.1^p.\theta^{2m-n-p+3}}{(\theta^2+2)^{m+1}}$

see Uwaeme *et al*. [9].

Since $\int_0^\infty y^s e^{-cy} \partial y = \frac{\Gamma(S+1)}{C^{s+1}}$ and

$$\Gamma(c) = (c-1)!$$

$$E(Y^z) = \sum_{l=0}^{\infty}\binom{c-1}{l}(-1)^l \sum_{m=0}^{\infty}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\sum_{p=0}^{n}\binom{n}{p}.A$$

$$.\theta^{-(z+2m-n-p+1)}.B + \sum_{l=0}^{\infty}\binom{c-1}{l}(-1)^l \sum_{m=0}^{\infty}\binom{l}{m}$$

$$\sum_{n=0}^{m}\binom{m}{n}\sum_{p=0}^{n}\binom{n}{p}.A.\theta^{-(z+2m-n-p+2)}.Q$$

where, $B = \frac{(Z+2m-n-p)!}{(m+1)^{z+2m-n-p+1}}$

and $Q = \frac{(Z+2m-n-p+2)!}{(m+1)^{z+2m-n-p+2}}$

$$R_{l,m,n,p} = \sum_{l=0}^{\infty}\binom{c-1}{l}(-1)^l \sum_{m=0}^{\infty}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\sum_{p=0}^{n}\binom{n}{p}.\frac{2^n.1^p.c.\theta^{-z+3}}{(\theta^2+2)^{m+1}}$$

and

$$S_{l,m,n,p} = \sum_{l=0}^{\infty}\binom{c-1}{l}(-1)^l \sum_{m=0}^{\infty}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\sum_{p=0}^{n}\binom{n}{p}.\frac{2^n.1^p.c.\theta^{-z}}{(\theta^2+2)^{m+1}}$$

The z-th moments of the TPEA distribution is given by

$$E(Y^z) = R_{l,m,n,p}\frac{(Z+2m-n-p)!}{(m+1)^{z+2m-n-p+1}} + S_{l,m,n,p}\frac{(Z+2m-n-p+2)!}{(m+1)^{z+2m-n-p+2}} \quad (14)$$

The Mean

We obtained the mean of the distribution from z-th moments when $z = 1$ from (14) as expressed below:

$$Z_1 = E(Y^1) = \mu_1 = R_{l,m,n,p}\frac{(2m-n-p+1)!}{(m+1)^{2m-n-p+2}} + S_{l,m,n,p}\frac{(2m-n-p+3)!}{(m+1)^{2m-n-p+3}} \quad (15)$$

Similarly, the variance of Y for TPEA distribution is also obtained from z-th central moment of order 2

$$Var(y) = Z_2 - (Z_1)^2 = \mu_2 - (\mu_1)^2$$

$$Vay(y) = \left[R_{l,m,n,p}\frac{(2m-n-p+2)!}{(m+1)^{2m-n-p+3}} + S_{l,m,n,p}\frac{(2m-n-p+4)!}{(m+1)^{2m-n-p+4}}\right] - [Z_1]^2 \quad (16)$$

$$Z_2 = E(Y^2) = \mu_2$$
$$= R_{l,m,n,p}\frac{(2m-n-p+2)!}{(m+1)^{2m-n-p+3}} + S_{l,m,n,p}\frac{(2m-n-p+4)!}{(m+1)^{2m-n-p+4}}$$

In addition, when $z = 3$

$$Z_3 = E(Y^3)$$
$$= R_{l,m,n,p}\frac{(2m-n-p+3)!}{(m+1)^{2m-n-p+4}} + S_{l,m,n,p}\frac{(2m-n-p+5)!}{(m+1)^{2m-n-p+5}}$$

Therefore, explicit form is used to obtain other related measures such as Coefficient of variation CoV(Y) and the Skewness Coefficient SK(Y) as given by

$$CoV(y) = \frac{\left(\sqrt{Z_2-(Z_1)^2}\right)}{Z_1} \quad (17)$$

and

$$SK(y) = \frac{Z_3-(Z_1)^3}{(Z_2-(Z_1)^2)^{3/2}} \quad (18)$$

Also, see Ramadan [5] and leren *et al*. [3]

3.2 Moment Generating Function (MGF)

The MGF of the two parameter exponential Akash distribution is proposed here and defined as a random variable Y having the TPEA distribution. Thus, the MGF of Y, $M_y(t)$ can be obtained by

$$M_y(t) = E(e^{ty}) = \int_0^\infty e^{ty} g(y) \partial y \quad (19)$$

Though, by using the binomial series expansion in (19) and following the steps we took under subsection 3.1 above, we obtain

$$M_y(t) = \sum_{h=0}^{\infty}\left(\frac{t}{\theta}\right)^h \left[R_{l,m,n,p}\frac{(h+2m-n-p)!}{h!(m+1)^{k+2m-n-p+1}} + S_{l,m,n,p}\frac{(h+2m-n-p+2)!}{h!(m+1)^{k+2m-n-p+3}}\right] \quad (20)$$

(20) becomes the MGF of the TPEA distribution.

3.3 Order Statistics

Suppose $Y_1, \ldots, Y_n$ is a random sample from a distribution with density function $g(y)$.

Then, let $Y_{1:p} <, \dots, < Y_{p:n}$ denotes the associating order statistics obtained from the sample. The density function $g_{p:n}$ of the pth order statistics is defined as

$$g_{p:n}(Y) = \frac{n!}{(p-1)!\,(n-p)!} \sum_{t=0}^{n-p} (-1)^t \binom{n-p}{t}.$$

$$g(y) G(y)^{\,t+p-1} \qquad (21)$$

where, $g(y)$ and $G(y)$ are defined in (5) and (6) as the density and distribution function of the TPEA distribution respectively. Furthermore, substituting (5) and (6) into (21) the density function of the pth order statistics $Y_{p:n}$ is given by

$$g_{p:n}(Y) = \frac{n!}{(p-1)!\,(n-p)!} \sum_{t=0}^{n-p} (-1)^t \binom{n-p}{t}$$

$$\left[\frac{C.\ \theta^3}{\theta^2+2}(1+y^2)e^{-\theta y}[U]^{c-1}\right][[U]^c]^{\,t+p-1} \quad (22)$$

Meanwhile, both the pdf of the minimum and maximum (Y_1 and Y_n) when $p = 1$ and $p = n$ order statistics of the TPEA distribution are presented as follows:

$$g_{1:n}(Y) = n \textstyle\sum_{t=0}^{n-p} (-1)^t \binom{n-1}{t}$$

$$\left[\frac{C.\ \theta^3}{\theta^2+2}(1+y^2)e^{-\theta y}[U]^{c-1}\right][[U]^c]^{\,t} \qquad (23)$$

and

$$g_{n:n}(Y) = n\left[\frac{c.\theta^3}{\theta^2+2}(1+y^2)e^{-\theta y}[U]^{c-1}\right]$$
$$[[U]^c]^{\,n-1} \qquad (24)$$

4 Estimation of Parameter Through Maximum Likelihood Estimates

Let $Y_1, \dots, Y_n$ be a random sample of size n from the TPEA distribution with the log-likelihood function of parameters written as

$$LoL_{TPEA}(c,\theta) = \textstyle\sum_{i=1}^{n} \log g(y_i) \qquad (25)$$

$$= \sum_{i=1}^{n} log\left\{\frac{C.\ \theta^3}{\theta^2+2}(1+y_i{}^2)\left[1-\left[1+\frac{\theta y_i(\theta y^2{}_i+2)}{\theta^2+2}\right]e^{-\theta y_i}\right]^{c-1} e^{-\theta y_i}\right\}$$

$$= log\left\{\left(\frac{C.\ \theta^3}{\theta^2+2}\right)^n \sum_{i=1}^{n}(1+y_i{}^2)e^{-\theta\sum y_i}\sum_{i=1}^{n}\left[1-\left[1+\frac{\theta y_i(\theta y^2{}_i+2)}{\theta^2+2}\right]e^{-\theta y_i}\right]^{c-1}\right\}$$

$$LoL = n[\ln(c) + 2\ln(\theta) - \ln(\theta^2+2)] +$$

$$\sum_{i=1}^{n} ln(y_i{}^2) - \theta\sum_{i=1}^{n} y_i + (c-1)$$

$$\textstyle\sum_{i=1}^{n} \ln(U_i(\theta)) \qquad (26)$$

where, $U_i(\theta) = \left[1-\left[1+\frac{\theta y_i(\theta y^2{}_i+2)}{\theta^2+2}\right]e^{-\theta y_i}\right]$

We maximize the log-likelihood by solving the nonlinear equations using partial differential system to differentiate (20) with respect to θ as we have it in the next line:

$$\frac{\partial LoL}{\partial\theta} = [n[\ln(c) + 2\ln(\theta) - \ln(\theta^2+2)]$$

$$+\ln(1) - \theta\sum_{i=1}^{n} y_i + (c-1)\sum_{i=1}^{n}\ln(U_i(\theta))$$

That is

$$\partial\big(n[\ln(c) + 2\ln(\theta) - \ln(\theta^2+2)]\big)$$
$$= \partial\big[n[2\ln(\theta) - \ln s]\big]$$

where, $s = (\theta^2+2)$

$$= \frac{2n}{\theta} - \frac{2n\theta}{\theta^2+2} = \frac{4n}{\theta^2+2}$$

More so,

$$\partial\{\textstyle\sum_{i=1}^{n}\ln(U_i(\theta))\} = \sum_{i=1}^{n}\frac{\partial(U_i(\theta))}{(U_i(\theta))}$$

$$\partial\big(U_i(\theta)\big) = \partial\left[1-\left[1+\frac{\theta y_i(\theta y^2{}_i+2)}{\theta^2+2}\right]e^{-\theta y_i}\right]$$
$$= \partial(1 - j.\,e^{-\theta y_i})$$

Since

$j = 1 + \frac{\theta y_i(\theta y^2{}_i+2)}{\theta^2+2}$. Therefore, by applying quotient rule, $\partial j = 0 + \frac{\partial r}{\partial s}$

where, $r = \theta y_i\left(\theta y^2{}_i + 2\right)$

and $s = (\theta^2 + 2)$, then

$$\partial\left(U_i(\theta)\right) = \frac{\theta^2 y_i^2(\theta^2 y_i - 2\theta) - \theta y_i(4\theta^2 + 6\theta) - 4}{(\theta^2 + 2)^2}$$

Taking the partial differentiation of LoL with respect to c and θ results to

$$\frac{\partial LoL}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n} \ln(U_i(\theta)) \qquad (27)$$

and

$$\frac{\partial LoL}{\partial \theta} = \frac{4n\theta+(\theta^2+2)}{\theta(\theta^2+2)} - \sum_{i=1}^{n} y_i + \frac{(c-1)\sum_{i=1}^{n}\left(\frac{\theta^2 y_i^2(\theta^2 y_i-2\theta)-\theta y_i(4\theta^2+6\theta)-4}{(\theta^2+2)^2}\right)}{\left[1-\left[1+\frac{\theta y_i(\theta y^2{}_i+2)}{\theta^2+2}\right]e^{-\theta y_i}\right]} \qquad (28)$$

4.1 Application to Real Data

The data used contains 182 distances from the seismological measuring station to the epicenter of the earthquake (in km) as the variable of interest reported by the Descriptive statistics in Table 1 has the boxplot, histogram plot, density plot and empirical cumulative density function plot of the data as shown below:

Table 1: Descriptive statistics of the Earthquake Data

Mean	*Median*	*Mode*	*Variance*	*Skewness*	*Kurtosis*	*Minimum*	*Maximum*
45.6000	23.4000	25.0000	3865.1200	2.8900	9.4300	0.5000	370.0000

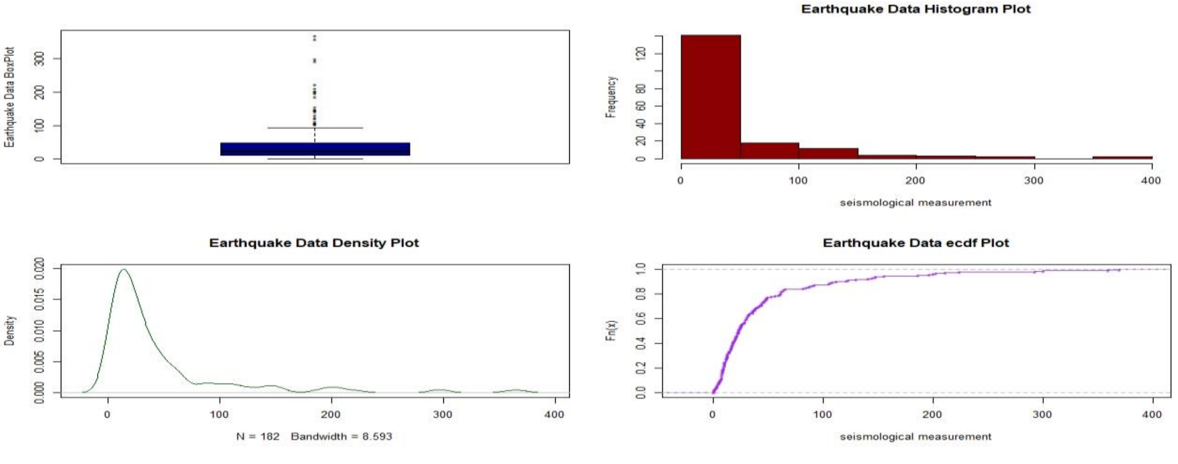

Figure 2: The boxplot, histogram, density and ecdf plot of the data set.

Table 2: Consists the MLE, Standard Error (in parenthesis) and Goodness of fit Statistics.

Model	*Parameter Estimates*	*Standard Error*	*W**	*A**	*KS*	*P-Value*
TPEAD	**θ = 0.0309**	**(0.0029)**	**0.3864**	**2.2642**	**0.4248**	**< 2.2e-16**
	c = 0.2789	**(0.0262)**				
EXPD	θ = 0.0347	(0.0032)	0.3566	2.0939	0.4857	< 2.2e-16
	α = 0.1965	(0.0181)				
EPD	α = 0.0219	(0.0016)	0.5290	3.0798	0.1268	0.0058

SHD	$\theta = 0.0439$	(0.0023)	0.6524	3.7865	0.1673	7.506e-05
DPD	$\theta = 0.0439$	(0.0023)	0.6524	3.7865	0.1673	7.506e-05
AKD	$\theta = 0.0657$	(0.0028)	0.7265	4.2114	0.3117	8.882e-16
PD	$\theta = 0.0879$	(0.0033)	0.7789	4.5126	0.4140	< 2.2e-16

Table 3: Contains Model Criteria Statistics.

Model	***-2LogL***	***AIC***	***CAIC***	***HQIC***	***BIC***
TPEAD	**758.9490**	**1521.8980**	**1521.9650**	**1524.4960**	**1528.3060**
EPD	877.2364	1756.4730	1756.4950	1757.7720	1759.6770
EXPD	884.0670	1772.1360	1772.2030	1774.7330	1778.5440
SHD	925.7356	1853.4710	1853.4930	1854.7700	1856.6750
DPD	925.7356	1853.4710	1853.4930	1854.7700	1856.6750
AKD	1000.3200	2002.6410	2002.6630	2002.6630	2005.8450
PD	1096.7100	2195.4190	2195.442	2196.7180	2198.6230

Table 1 shows the descriptive statistics of the data used which the graphical representation depicts in figure 2. It reveals that the skewness of the data i.e the data is right skewed according to its density plot. Then, Table 2, contains the values of the MLEs, associating standard error and the goodness of fit, while, Table 3 consists of model selection criteria of the TPEA model and other competing models. Hence, figure 3 is the estimated pdf and cdf plots of all the distributions compared in this study.

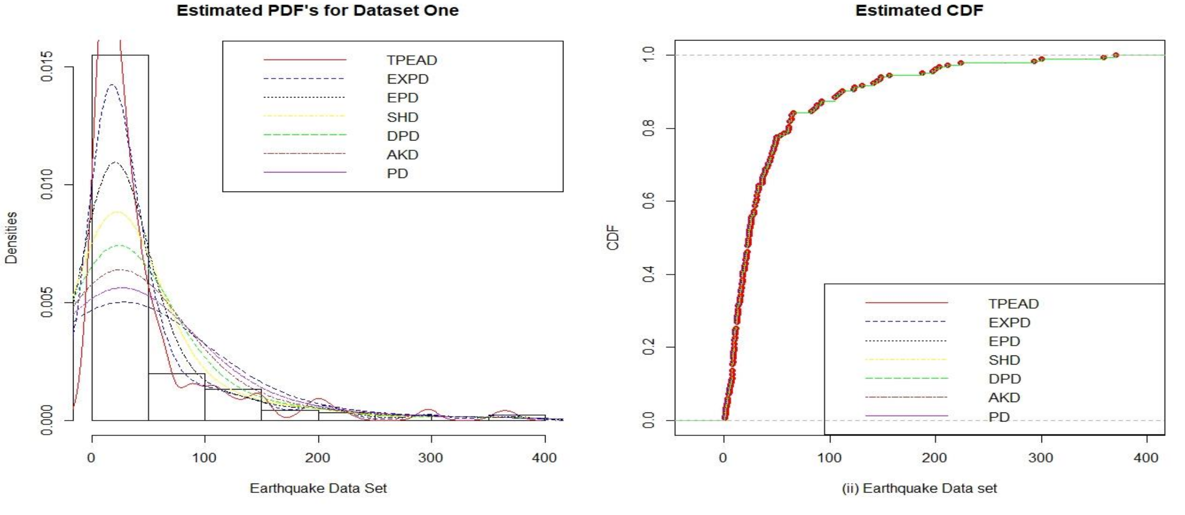

Figure 3: Estimated PDF and CDF of the TPEA and extant distributions with the data set.

5 Discussion

A two-parameter exponential-Akash distribution has been successively proposed. It has its base from exponential method and Akash distribution. Some of its statistical properties which include survival, hazard, reversed hazard, moments, moment generating function, the mean, variance, coefficient of variation, skewness and order statistics were properly discussed. Also, we are able to estimate the model parameters using the method of maximum likelihood estimation. A real-life data set is used and presented for an illustration to show the goodness of fit and model criteria statistics of the two-parameter exponential-Akash over Exponential, Extended Pranav, Shanker, Discrete-Pranav, Akash and Pranav distributions.

5.1 Conclusion

The results in Tables 2 and 3 indicate that the Two-parameter exponential-Akash (TPEA) has the lowest value of MLEs, standard error, goodness of fit and model criteria statistics. The lower the values of the measures of goodness of fit and model selection criteria, the better the distribution/model. Hence, the values of TPEA in Table 2 and 3 show its better performance over other distributions compared in this study. Therefore TPEA can be used in modeling lifetime data sets.

Conflict of interest. No conflict of interest

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