Table 1 Weibull and Weibull-epsilon fit result of solar radiation dataset

Dist.	Par.	Est.	Std. E	$-LL$ (AIC)	KS ($p\ val$)	Remark
Weibull	Shape	6.083	0.274	590.2	0.134	Poor fit
	Scale	6.380	0.057	(1184)	(0.000)	
Weibull -Epsilon	α	2.766	0.168	544.9	0.071	Good fit
	λ	0.080	0.002	(1096)	(0.063)	
	δ	8.515	0.157			

Dist. = Distribution, Par. = Parameter, Est. = Estimate, Std. E = Standard Error, $-LL$ = negative log-likelihood function value, KS = Kolmogorov-Smirnov statistic value, $p\ val$ = p-value

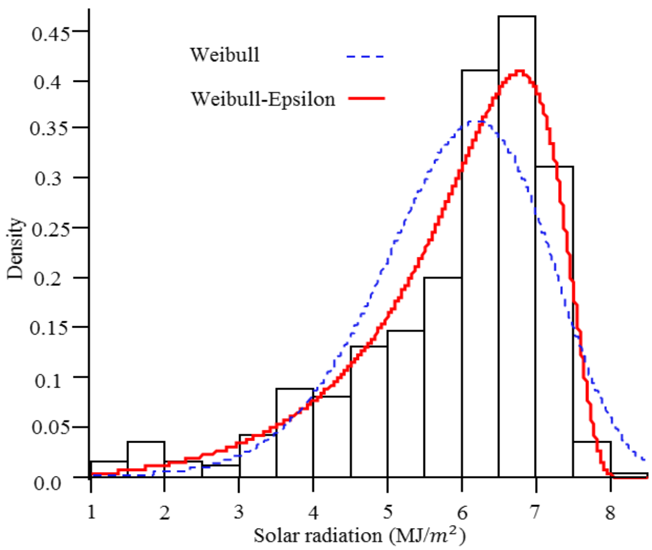

Figure 4 Weibull and Weibull-epsilon fit of solar radiation data

8.2.2 Breaking Stress Data

The results of fitting the Weibull and Weibull-epsilon distributions to the breaking stress of carbon fibers dataset is presented in Table 2 below, along with the fit result of the Cauchy-Weibull-Logistic distribution [30]. The fitted density curves superimposed over the histogram of the data are presented in Figure 5.

Table 2 3-P CW{L}*, Weibull and Weibull-epsilon fit result of breaking stress dataset

Dist.	Par.	Est.	Std. E	$-LL$ (AIC)	KS ($p\ val$)	Remark
3-P CW{L}	β	2.144	0.722	86.99	0.057	Good fit
	k	7.932	1.889	(180.0)	(0.983)	
	λ	2.953	0.108			
Weibull	Shape	3.441	0.331	86.08	0.082	Good fit
	Scale	3.062	0.115	(178.2)	(0.761)	
Weibull-Epsilon	α	2.514	0.245	85.93	0.088	Good fit
	λ	0.223	0.008	(177.8)	(0.655)	
	δ	3866	1532			

Dist. = Distribution, Par. = Parameter, Est. = Estimate, Std. E = Standard Error, $-LL$ = negative log-likelihood function value, KS = Kolmogorov-Smirnov statistic value, $p\ val$ = p-value, 3-P CW{L} = 3 parameter Cauchy-Weibull-Logistic distribution adopted from [30]

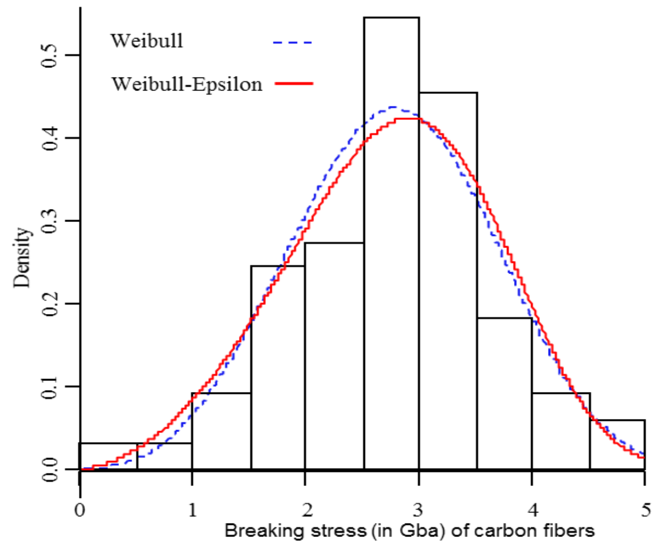

Figure 5 Weibull and Weibull-epsilon fit of breaking stress (in Gba) of carbon fibers

9 Discussion

The plots of the density function of the Weibull-epsilon distribution introduced in this study shows flexibility in taking any form of shape – J-shape, reversed J-shape, positively and negatively skewed shapes. This suggests that it can be used in modeling datasets from different data generating processes. The varying shapes of the plots of its hazard rate function also show it has a potential for modeling in lifetime and reliability studies.

The results in Table 1 show that the Weibull-epsilon distribution provides a good fit to the solar radiation dataset while the Weibull distribution does not. This is depicted in Figure 4. This shows that the distribution can be applicable in modeling solar power for informed deployment of solar panels that will attain maximum energy generating efficiency. The likelihood gain per data point of the Weibull-epsilon distribution is 13.2 % over the Weibull distribution.

The results in Table 2 for the breaking stress of carbon fibers (Gba) data show that the parameter estimates of both the Weibull and Weibull-epsilon distributions are very small; that is, they are precise estimates. This implies that they are very close to their true parameter values. The overall goodness-of-fit results also show that both distributions are compatible with the data. The fit of the distribution to the data shown in Figure 5 gives a pictorial evidence that the models are compatible with the dataset.

Table 2 also shows that the Weibull-epsilon distribution fit performs better than the Weibull and Cauchy-Weibull-Logistic distributions with a percent gain in likelihood per data point of 0.23 % and 1.62 %, respectively.

10 Conclusion

A new probability distribution, called the Weibull-epsilon distribution, is constructed in this study. It is shown to be compatible with solar radiation and breaking stress of carbon fibers datasets. It brings about improvement in fit when compared with the Weibull distribution for the two datasets considered. It is also better than the Cauchy-Weibull-Logistic distribution in fit to the breaking stress of carbon fibers datasets.

The distribution can assume different shapes within its parameter range indicating it holds a good potential in modeling different lifetime data generating processes. The bathtub shapes of its hazard rate function also show the model can be used in reliability studies. This flexibility in shape creates avenue for further research interest in applications, particularly in reliability studies in engineering, social and biological systems, mortality studies in demography and renewable energy modeling.

This study also provide scope for further research on the properties of the distribution, particularly the derivation of moments, and using the distribution as a base for generating other distributions.

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Appendix 1

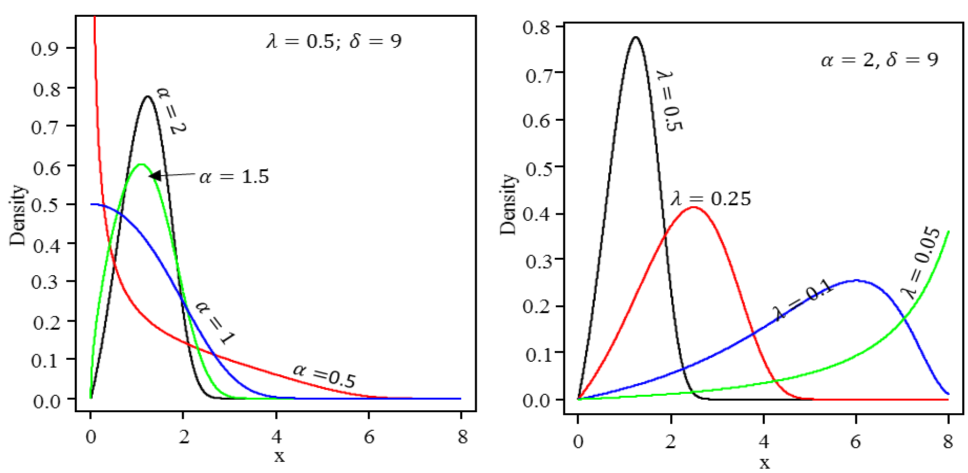

Figure 1 Weibull-Epsilon Density Plots at various parameter values

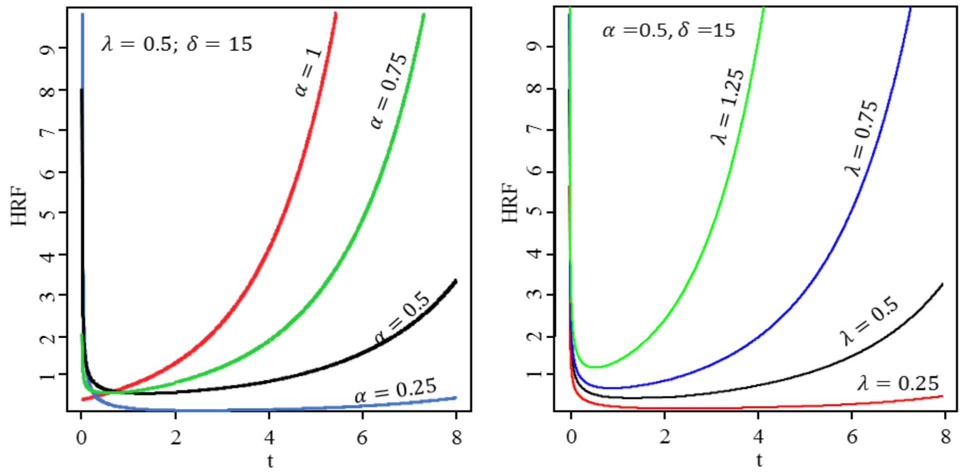

Figure 2 Weibull-epsilon hazard rate function plots

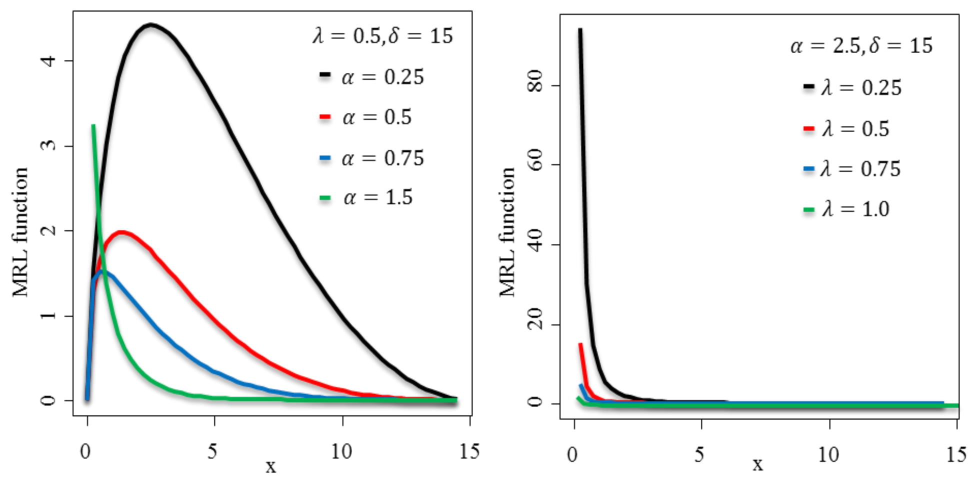

Figure 3 Mean Residual Life function of the Weibull-epsilon distribution at parameter values

Appendix 2

$$E(X-x|X>x)=\frac{1}{S(x)}\int_{x}^{\infty} S(u)\,du$$

$$=\frac{1}{exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}}\int_{x}^{\delta} exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}du$$

But

$$\int_{x}^{\delta} exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}du=-exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\cdot\frac{1}{\frac{d}{du}\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}}\Bigg|_{x}^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\cdot\frac{1}{\alpha\cdot\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha-1}\cdot\frac{\lambda}{2}\delta\cdot\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta-1}\cdot\frac{1\cdot(\delta-u)--1\cdot(u+\delta)}{(\delta-u)^2}}\Bigg|_x^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\cdot\frac{1}{\alpha\lambda\frac{\delta}{2}\frac{2\delta}{(\delta-u)^2}\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta-1}\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha-1}}\Bigg|_x^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\cdot\frac{1}{\alpha\lambda\frac{\delta}{2}\frac{2\delta}{(\delta-u)^2}\frac{\delta-u}{u+\delta}\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha-1}}\Bigg|_x^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\cdot\frac{1}{\alpha\lambda\frac{\delta^2}{\delta^2-u^2}\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha-1}}\Bigg|_x^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-u^2}{\alpha\lambda\delta^2}\left(\frac{u+\delta}{\delta-u}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}\Bigg|_x^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{\delta+\delta}{\delta-\delta}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-\delta^2}{\alpha\lambda\delta^2}\left(\frac{\delta+\delta}{\delta-\delta}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{\delta+\delta}{\delta-\delta}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$

$$+ exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-x^2}{\alpha\lambda\delta^2}\left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$

$$= -exp\{-[\infty-1]^{\alpha}\}\cdot 0\cdot 0\cdot[\infty-1]^{1-\alpha} + exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-x^2}{\alpha\lambda\delta^2}\left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$

$$= exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-x^2}{\alpha\lambda\delta^2}\left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$

Therefore,

$$E(X-x|X>x) = \frac{1}{exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}}exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-x^2}{\alpha\lambda\delta^2}\left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$

$$= \frac{\delta^2-x^2}{\alpha\lambda\delta^2}\left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$