Some new Gronwall-Bihari type inequalities and its applications in the analysis for solutions to fractional differential equations

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Abstract: In this paper, we derive some generalizations of certain Gronwall-Bihari with weakly singular kernels for functions in one variable, which provide explicit bounds on unknown functions.To show the feasibility of the obtained inequalities, two illustrative examples are also introduced.

Key–Words: Gronwall-type inequality, fractional differential equation, qualitative analysis, quantitative analysis; mean value Theorem, Fractional Derivative, Riemann-Liouville.

1 Introduction and preliminaries

It is well known that the Gronwall-Bellman inequality $[1,8]$ and their generalizations can provide explicit bounds for solutions to differential and integral equations as well as difference equations.Many authors have researched various inequalities and investigated the boundedness, global existence, uniqueness, stability, and continuous dependence on the initial value and parameters of solutions to differential equations, integral equations see $[2-5,10]$. However, we notice that the existing results in the literature are inadequate for researching the qualitative and quantitative properties of solutions to some fractional integral equations see $[9-11,13,17-18]$.

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order see $[13]$. However, in this branch of Mathematics we are not looking at the usual integer order but at the non-integer order integrals and derivatives. These are called fractional derivatives and fractional integrals. The first appearance of the concept of a fractional derivative is found in a letter written to Guillaume de l'Hôpital by Gottfried Wilhelm Leibniz in 1695. As far as the existence of such a theory is concerned, the foundations of the subject were laid by Liouville in a paper from1832. The autodidact Oliver Heaviside introduce the practical use of fractional differential operators in electrical transmission line analysis circa 1890. Many authors have established a variety of inequalities for those fractional integral and derivative operators, some of which have turned out to be useful in analyzing solutions of certain fractional integral and differential equations, for example, we refer the reader to $[9-10,17-18]$ and the references therein.

In $[12]$, the authors proved the following results :

Theorem 1 *Let $k,\lambda \in \mathbb{R}^+$. Also, let h and u be nonnegative and locally integrable functions defined on $[0,X)$ with $X \leq +\infty$. Further, let $\phi(x)$ be a nonnegative, non-decreasing, and continuous function on $[0,X)$ which is bounded on $[0,X)$, that is, $\phi(x) \leq M$ for all $x \in [0,X)$ and some $M \in \mathbb{R}^+$. Suppose that the functions h, u, and ϕ satisfy the following inequality:*

$$u(x) \leq h(x)+k\phi(x)\int_0^x (x-\rho)^{\frac{\lambda}{k}-1} u(\rho)\,d\rho,\ x\in[0,X). \tag{1.1}$$

Then

$$u(x) \leq h(x)+\sum_{n=1}^{\infty}\frac{\{k\phi(x)\Gamma_k(\lambda)\}^n}{\Gamma_k(n\lambda)}\int_0^x (x-\rho)^{n\frac{\lambda}{k}-1}h(\rho)d\rho\,,\, x\in[0,X)\ . \tag{1.2}$$

Corollary 1 *Let $k,\lambda \in \mathbb{R}^+$ Also, let h and u be nonnegative and locally integrable functions defined on $[1,X)$ with $X \leq +\infty$. Further, let $\phi(x)$ be a nonnegative, nondecreasing, and continuous function on $[0,X)$ which is bounded on $[1,X)$ that is, $\phi(x) \leq M$ for all $x \in [1,X)$ and some $M \in \mathbb{R}^+$. Suppose that the functions h, u, and ϕ satisfy the following inequality:*

$$u(x) \leq h(x) + k\phi(x)\int_0^x \left(\ln\frac{x}{\rho}\right)^{\frac{\lambda}{k}-1} u(\rho)\frac{d\rho}{\rho}, \quad (x\in[1,X)). \tag{1.3}$$

Then

$$u(x) \leq h(x) + \sum_{n=1}^{\infty} \frac{\{k\phi(x)\Gamma_k(\lambda)\}^n}{\Gamma_k(n\lambda)} \int_1^x \left(\ln\frac{x}{\rho}\right)^{n\frac{\lambda}{k}-1} h(\rho)\frac{d\rho}{\rho}, \quad (x\in[1,X)). \tag{1.4}$$

In this paper, we establish some new Gronwall-type inequalities associated with Riemann-Liouville $k-$ and Hadamard k-fractional derivatives which generalize some result given in [12] .We also present some nonlinear integral inequalities with singular kernels of Bihari type,we apply the results established to research boundedness, uniqueness for the solution to some certain integral equations.

Now, some important properties for the modified Riemann-Liouville derivative and fractional integral are listed as follows :

Definition 1 *The Riemann-Liouville fractional integral of order α on the interval $[0,x]$ is defined by*

$$(I^\alpha f) := \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1} f(\tau)d\tau \quad (x>0), \tag{1.5}$$

where

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1}\exp(-s)\,ds,$$

which is well defined for $\alpha > 0$.

Definition 2 *The modified Riemann-Liouville derivative of order α is defined by*

$$(D_x^\alpha f)(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{d}{dx}\int_0^x (x-\zeta)^{-\alpha}(f(\zeta)-f(0))\,d\zeta, & 0<\alpha<1 \\ \left(f^{(n)}(x)\right)^{(\alpha-n)}, & n\leq\alpha<n+1, \quad n\geq 1 \end{cases} \tag{1.6}$$

The Hadamard fractional integral ${}_HD^{\mu}_{1,x}f$ of order $\mu>0$ is defined by

$${}_HD^{\mu}_{1,x}f := \frac{1}{\Gamma(\mu)}\int_1^x \left(\ln\frac{x}{\tau}\right)^{\mu-1} f(\tau)\frac{d\tau}{\tau} \quad (x>1) \tag{1.7}$$

The Hadamard fractional derivative ${}_HD^{\mu}_{1,x}f$ of order $\mu>0$ is defined by

$${}_HD^{\mu}_{1,x}f := \frac{1}{\Gamma(n-\mu)}\left(x\frac{d}{dx}\right)^n \int_1^x \left(\ln\frac{x}{\tau}\right)^{n-\mu-1} f(\tau)\frac{d\tau}{\tau} \quad (x>1) \tag{1.8}$$

$$[n=[\mu]+1;\ x>0),$$

Here and in the following, let $\mathbb{C},\mathbb{R},\mathbb{R}^+,\mathbb{R}_+,\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers,positive real numbers, nonnegative real numbers, positive- integers, and non-positive integer, respectively.

Dĭaz and Pariguan [6] introduced k-gamma function Γ_k defined by

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}} t^{z-1}dt \qquad [[\Re(z)]>0; k\in\mathbb{R}^+) \tag{1.9}$$

which has the following relationships:

$$\Gamma_k(z+k) = z\Gamma_k(z),\ \Gamma_k(k)=1 \tag{1.10}$$

and

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right). \tag{1.11}$$

Also, $k-beta$ function $B_k(\alpha,\beta)$ is defined by

$$B_k(\alpha,\beta) = \begin{cases} \frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}dt & (\min\{\Re(\alpha),\Re(\beta)\}>0) \\ \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} & \left(\alpha,\beta\in\mathbb{C}\diagdown k\mathbb{Z}_0^-\right), \end{cases} \tag{1.12}$$

where $k\mathbb{Z}_0^-$ denotes the set of k-multiples of the elements in $\mathbb{Z}_0^-$.

Among many generalizations of the Mittag-Leffler function, one of them is recalled (see [15, 16]) :

$$E_{\lambda,\beta} = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\lambda n+\beta)} \qquad (\lambda,\beta\in\mathbb{C};\ \Re(\lambda)>0), \tag{1.13}$$

which is further generalized and called k-Mittag-Leffler function as follows:

$$E_{k,\lambda,\beta} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\lambda n+\beta)} \; (\lambda,\ \beta \in \mathbb{C};\ \Re(\lambda) > 0;\ k \in \mathbb{R}_+).$$

Lemma 1 ([17]) *Suppose $0 < \alpha < 1$, f is a continuous function, then*

$$D_t^{\alpha}\left(I^{\alpha} f(t)\right) = f(t).$$

Lemma 2 ([7]) *Suppose that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p}\varepsilon^{\frac{q-p}{p}} a + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}.$$

for any $\varepsilon > 0$.

Lemma 3 ([17]) *Let $\alpha > 0, a(t), b(t), u(t)$ be continuous functions defined on $t \geq 0$.Then for $t \geq 0$,*

$$D_t^{\alpha} u(t) \leq a(t) + b(t)u(t).$$

Implies

$$u(t) \leq u(0)\exp\left\{\int_0^{\frac{t^{\alpha}}{\Gamma(1+\alpha)}} b\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right\} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} a(\tau) \times \exp\left\{-\int_{\frac{\tau^{\alpha}}{\Gamma(1+\alpha)}}^{\frac{t^{\alpha}}{\Gamma(1+\alpha)}} b\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}) ds\right\} d\tau.$$

Definition 3 ([5]) *A function $w : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathfrak{F}$, if it satisfies the following conditions*

$$w(x) > 0, \text{is non-decreasing and continuous for } x \geq 0,$$
$$\frac{1}{a}w(x) \leq w\left(\frac{x}{a}\right), \quad \text{for } a > 0.$$

2 Main Results

Theorem 2 *Suppose that $k, \lambda \in \mathbb{R}^+$, h , u are nonnegative locally integrable functions defined on $[0, X)$ with $X \leq +\infty$. Further, let $\phi(x)$be a nonnegative, nondecreasing, and continuous function on $[0, X)$ which is bounded on $[0, X)$, that is, $\phi(x) \leq M$ for all $x \in [0, X)$ and some $M \in \mathbb{R}^+$. Suppose that the functions h, u, and ϕ satisfy the following inequality:*

$$u^p(x) \leq h(x) + k\,\phi(x)\int_0^x (x-\rho)^{\frac{\lambda}{k}-1} u^q(\rho)\,d\rho, \quad (x \in [0, X)). \tag{2.1}$$

where $p \neq 0$, $p \geq q > 0$, are constants. Then

$$u(x) \leq \left\{\widetilde{h}(x) + \sum_{n=1}^{\infty} \frac{\left\{k\widetilde{\phi}(x)\Gamma_k(\lambda)\right\}^n}{\Gamma_k(n\lambda)} \int_0^x (x-\rho)^{n\frac{\lambda}{k}-1} \widetilde{h}(\rho)d\rho\right\}^{\frac{1}{p}}, \quad x \in [0, X), \tag{2.2}$$

where

$$\widetilde{h}(x) = h(x) + \frac{k^2}{\lambda}\frac{p-q}{p}\varepsilon^{\frac{q}{p}} x^{\frac{\lambda}{k}}\phi(x), \widetilde{\phi}(x) = \frac{q}{p}\varepsilon^{\frac{q-p}{p}}\phi(x). \tag{2.3}$$

Proof. *Denote the right-hand side of (2.1) by $z(x)$. Then we have*

$$u(x) \leq z^{\frac{1}{p}}(x), \qquad (x \in [0, X)). \tag{2.4}$$

So it follows that

$$z(x) \leq h(x) + k\phi(x)\int_0^x (x-\rho)^{\frac{\lambda}{k}-1} z^{\frac{q}{p}}(\rho)\,d\rho, \quad (x \in [0, X)). \tag{2.5}$$

Using Lemma 2, we obtain that

$$z(x) \leq h(x) + k\phi(x)\int_0^x (x-\rho)^{\frac{\lambda}{k}-1}\left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}} z(\rho) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}\right)d\rho, \quad (x \in [0, X)). \tag{2.6}$$

The inequality (2.6) can be rewritten as

$$z(x) \leq \widetilde{h}(x) + k\widetilde{\phi}(x)\int_0^x (x-\rho)^{\frac{\lambda}{k}-1} z(\rho)d\rho \tag{2.7}$$

where and $\widetilde{h}$ and $\widetilde{k}$ are defined as in (2.3).

Applying Theorem 1 to (2.7), we can get the desired inequality (2.2). ■

Remark 1 *If $p = q = 1$, then Theorem 2 reduces to Theorem 1.*

Theorem 3 *Let k, λ, p, q are defined as in Theorem 2. Also, let h and u be nonnegative and locally integrable functions defined on $[1, X)$ with $X \leq +\infty$. Further, let $\phi(x)$ be a nonnegative, nondecreasing, and continuous function on $[0, X)$ which is bounded on $[1, X)$,that is, $\phi(x) \leq M$ for all $x \in [1, X)$ and some $M \in \mathbb{R}^{+}$. Suppose that the functions h, u, and ϕ satisfy the following inequality:*

$$u^{p}(x) \leq h(x) + k\phi(x) \int_{0}^{x} \left(\ln \frac{x}{\rho}\right)^{\frac{\lambda}{k}-1} u^{q}(\rho) \frac{d\rho}{\rho} \quad (x \in [1, X)). \tag{2.8}$$

Then

$$u(x) \leq \left\{ \widetilde{h}(x) + \sum_{n=1}^{\infty} \frac{\left\{k\widetilde{\phi}(x)\Gamma_{k}(\lambda)\right\}^{n}}{\Gamma_{k}(n\lambda)} \int_{1}^{x} \left(\ln \frac{x}{\rho}\right)^{n\frac{\lambda}{k}-1} \widetilde{h}(\rho) \frac{d\rho}{\rho} \right\}^{\frac{1}{p}}, \quad (x \in [1, X)), \tag{2.9}$$

where

$$\widetilde{h}(x) = h(x) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}} k\phi(x) \int_{0}^{x} \left(\ln \frac{x}{\rho}\right)^{\frac{\lambda}{k}-1} \frac{d\rho}{\rho}, \qquad \widehat{\phi}(x) = \frac{q}{p}\varepsilon^{\frac{q-p}{p}} \phi(x). \tag{2.10}$$

Proof. *Denote the right-hand side of (2.8) by $z(x)$. Then we have*

$$u(x) \leq z^{\frac{1}{p}}(x), \qquad (x \in [0, X)).$$

So it follows that

$$z(x) \leq h(x) + k\phi(x) \int_{0}^{x} \left(\ln \frac{x}{\rho}\right)^{\frac{\lambda}{k}-1} z^{\frac{q}{p}}(\rho) \frac{d\rho}{\rho}, \quad (x \in [0, X)).$$

Using Lemma 2, we obtain that

$$z(x) \leq h(x) + k\phi(x) \int_{0}^{x} \left(\ln \frac{x}{\rho}\right)^{\frac{\lambda}{k}-1} \left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}} z(\rho) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}} \right) \frac{d\rho}{\rho}, \quad (x \in [0, X)).$$

The last inequality can be rewritten as

$$z(x) \leq \widetilde{h}(x) + k\widetilde{\phi}(x) \int_{0}^{x} (x-\rho)^{\frac{\lambda}{k}-1} z(\rho) \frac{d\rho}{\rho},$$

where $\widetilde{h}$ and $\widetilde{\phi}$ are defined as in (2.10).

Applying Corollary 1 to the above inequality, we can get the desired inequality (2.9). ■

Remark 2 *If $p = q = 1$, then Theorem 3 reduces to Corollary 1.*

Theorem 4 *Suppose that $0 < \alpha < 1$ and $u, \phi, h \in C(\mathbb{R}_{+}, \mathbb{R}_{+})$. Furrther, let $g : \mathbb{R}_{+} \to \mathbb{R}_{+}$ is a differentiable increasing function on $]0, +\infty[$ with continuous nonincreasing first derivative g'on $]0, +\infty[$. If*

$$u^{p}(x) \leq h(x) + \frac{1}{\Gamma(\alpha)}\phi(x) \int_{0}^{x} (x-\rho)^{\alpha-1} g(u^{q}(\rho)) d\rho, \quad (x \in [0, X)) \tag{2.11}$$

Then

$$u(x) \leq \Big\{h(x) + \frac{1}{\Gamma(\alpha)}\phi(x) \int_{0}^{x} (x-\rho)^{\alpha-1} \widehat{h}(\rho) \times \exp\left(-\int_{\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}}^{\frac{x^{\alpha}}{\Gamma(1+\alpha)}} \widehat{\phi}\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}} \right) ds \right) d\rho\Big\}^{\frac{1}{p}}, \tag{2.12}$$

for any $p \neq 0$, $p \geq q > 0$, where

$$\widehat{h}(x) = g\left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}\right), \tag{2.13}$$

$$\widehat{\phi}(x) = \frac{q}{p}\varepsilon^{\frac{q-p}{p}} \phi(x) g'\left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}\right).$$

Proof. Define a function $v(x)$ by

$$v(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-\rho)^{\alpha-1} g(u^{q}(\rho)) d\rho, \tag{2.14}$$

then

$$u(x) \leq (h(x) + \phi(x)v(x))^{\frac{1}{p}} \quad (x \in [0, X)). \tag{2.15}$$

By Lemma 2, we get for any $\varepsilon > 0$,

$$u(t) \leq \frac{q}{p}\varepsilon^{\frac{q-p}{p}} (h(x) + \phi(x)v(x)) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}, \tag{2.16}$$

Applying Lemma 1 to (2.14) and using (2.16), we have

$$D_x^{\alpha} v(x) \leq g\left((h(x) + \phi(x)v(x))^{\frac{q}{p}} \right), \qquad (2.17)$$

$$D_x^{\alpha} v(x) \leq g\left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}} (h(x) + \phi(x)v(x)) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}\right) \qquad (2.18)$$

Applying the mean value Theorem for the function g, then for every $x \geq y > 0$ there exists $c \in \,]\, y, x[$such that

$$g(x) - g(y) = g^{'}(c)(x-y) \leq g^{'}(y)(x-y),$$

then

$$\begin{gathered} D_x^{\alpha} v(x) \leq g(\tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}) \\ + \tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} g^{'}(\tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}})\phi(x)v(x), \end{gathered}$$

The last inequality can be reformulated as

$$D_x^{\alpha} v(x) \leq \widehat{h}(x) + \widehat{\phi}(x)v(x), \qquad (2.19)$$

where $\widehat{h}$ and $\widehat{\phi}$ are defined as in (2.13).
Using Lemma 3 to (2.19), we get

$$\begin{gathered} v(x) \leq \tfrac{1}{\Gamma(\alpha)} \int_0^x (x-\rho)^{\alpha-1} \widehat{h}(\rho) \\ \times \exp\left\{ -\int_{\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}}^{\frac{x^{\alpha}}{\Gamma(1+\alpha)}} \widehat{\phi}\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}} \right) ds \right\} d\rho, \end{gathered} \qquad (2.20)$$

Combining (2.20) and (2.15), we get (2.12). ■

Corollary 2 *Assume that the hypotheses of Theorem 4 hold. If*

$$\begin{gathered} u^p(x) \leq h(x) + \tfrac{1}{\Gamma(\alpha)} \phi(x) \int_0^x (x-\rho)^{\alpha-1} \arctan(u^q(\rho))\, d\rho \\ (x \in [0, X)), \end{gathered}$$

Then

$$\begin{gathered} u(x) \leq \{h(x) + \phi(x) \int_0^x (x-\rho)^{\alpha-1} \widehat{h}(\rho) \\ \times \exp\left(-\int_{\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}}^{\frac{x^{\alpha}}{\Gamma(1+\alpha)}} \widehat{\phi}\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}} \right) ds \right) d\rho\}^{\frac{1}{p}} \quad . \end{gathered}$$

Where

$$\begin{aligned} \widehat{h}(x) &= \arctan(\frac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}) \\ \widehat{\phi}(x) &= \frac{\frac{q}{p}\varepsilon^{\frac{q-p}{p}}\phi(x)}{1 + \left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}\right)^2}. \end{aligned}$$

Corollary 3 *Assume that the hypotheses of Theorem 4 hold. If*

$$\begin{gathered} u^p(x) \leq h(x) + \tfrac{1}{\Gamma(\alpha)} \phi(x) \int_0^x (x-\rho)^{\alpha-1} \log(1 + u^q(\rho))\, d\rho \\ (x \in [0, X)). \end{gathered}$$

Then

$$\begin{gathered} u(x) \leq \{h(x) + \phi(x) \int_0^x (x-\rho)^{\alpha-1} \widehat{h}(\rho) \\ \times \exp\left(-\int_{\frac{\rho^{\alpha}}{\Gamma(1+\alpha)}}^{\frac{x^{\alpha}}{\Gamma(1+\alpha)}} \widehat{\phi}\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}} \right) ds \right) d\rho\}^{\frac{1}{p}}, \end{gathered}$$

where

$$\begin{aligned} \widehat{h}(x) &= \log(1 + \frac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}), \\ \widehat{\phi}(x) &= \frac{\frac{q}{p}\varepsilon^{\frac{q-p}{p}}\phi(x)}{1 + \left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}\right)}. \end{aligned}$$

Theorem 5 *Suppose that* $0 < \alpha < 1$ *and* $u, \phi, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ *. Furrther, let* $S \in C\left(\mathbb{R}_+^2, \mathbb{R}_+\right)$ *be a continuous function such that*

$$0 \leq S(t,x) - S(t,y) \leq L(t,y)(x-y), \quad x \geq y \geq 0, \qquad (2.21)$$

for $t \in [0, X)$,where $L : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a continuous function. If

$$u^p(x) \leq h(x) + \frac{1}{\Gamma(\alpha)}\phi(x) \int_0^x (x-\rho)^{\alpha-1} S(\rho, u^q(\rho)) d\rho. \qquad (2.22)$$

Then

$$u(x) \leq \{h_1(x) + \phi_1(x)\tfrac{1}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1} h_1(\rho) \times \exp\left\{-\int_{\frac{\rho^\alpha}{\Gamma(1+\alpha)}}^{\frac{x^\alpha}{\Gamma(1+\alpha)}} \phi_1\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right) ds\right\} d\rho\}^{\frac{1}{p}}, \tag{2.23}$$

where

$$h_1(x) = S\left(x, \frac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}\right), \tag{2.24}$$
$$\phi_1(x) = \frac{q}{p}\varepsilon^{\frac{q-p}{p}} L\left(x, \frac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}\right)\phi(x).$$

Proof. Let

$$z(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1} S(\rho, u^q(\rho))d\rho$$

so we can get $z(0) = 0$. From (2.22), we have

$$u(x) \leq (h(x) + \phi(x)z(x))^{\frac{1}{p}}. \tag{2.25}$$

By Lemma 2 we obtain for any $\varepsilon > 0$,

$$u^q(x) \leq \frac{q}{p}\varepsilon^{\frac{q-p}{p}}(h(x) + \phi(x)z(x)) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}, \tag{2.26}$$

By Lemma 1, we have

$$D_x^\alpha z(x) = S(x, u^q(x)). \tag{2.27}$$

From Lemma 2 and using (2.21),(2.26), one has for any $\varepsilon > 0$

$$\begin{aligned} S(x, u^q(x)) &\leq S(x, (h(x) + \phi(x)z(x))^{\frac{q}{p}}) \\ &\leq S\left(x, \tfrac{q}{p}\varepsilon^{\frac{q-p}{p}}(h(x) + \phi(x)z(x)) + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}\right) \\ &\leq \tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} L\left(x, \tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}\right)\phi(x)z(x) \\ &\quad + S\left(x, \tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}\right). \end{aligned} \tag{2.28}$$

From (2.27) and (2.28), we have

$$\begin{aligned} D_x^\alpha z(x) &\leq \tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} L\left(x, \tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}\right)\phi(x)z(x) \\ &\quad + S\left(x, \tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} h(x) + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}\right) \\ &= h_1(x) + \phi_1(x)z(x), \end{aligned} \tag{2.29}$$

where $h_1(x)$, $\phi_1(x)$ are defned as in (2.24).

By Lemma 3, we get

$$z(x) \leq \frac{1}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1} h_1(\rho)\exp\left\{-\int_{\frac{\rho^\alpha}{\Gamma(1+\alpha)}}^{\frac{x^\alpha}{\Gamma(1+\alpha)}} \phi_1\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right) ds\right\} d\rho. \tag{2.30}$$

Combining (2.30) and (2.25), we get (2.23). ■

Theorem 6 *Suppose that* $0 < \alpha < 1$ *and* $u, \phi, h \in C(\mathbb{R}_+, \mathbb{R}_+)$.*Furrther, let* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathfrak{F}$ *(see Definition 3), and* $h(t)$ *be nondecreasing function in* $[0, X)$. *If*

$$u(x) \leq h(x) + \tfrac{1}{\Gamma(\alpha)}\phi(x)\int_0^x (x-\rho)^{\alpha-1} g(u(\rho))\,d\rho \quad (x \in [0, X)) \tag{2.31}$$

Then

$$u(x) \leq h(x)\left\{\Omega_n^{-1}\left[\Omega\left(2^{n-1}\right) + \tfrac{1}{n}(1-e^{-nx})K_{n,m}\phi^n(x)\right]\right\}^{\frac{1}{n}}, \quad 0 \leq x \leq X_1 < X, \tag{2.32}$$

where

$$\alpha = \tfrac{1}{1+z},\ z > 0,\ n = \tfrac{1}{\alpha} + r = 1 + z + r, \quad m = \tfrac{1+z+r}{z+r},\ r > 0, \Omega_n(\upsilon) = \int_{\upsilon_0}^{\upsilon} \frac{d\sigma}{g^n\left(\sigma^{\frac{1}{n}}\right)}, \tag{2.33}$$

and

$$K_{n,m} = 2^{n-1}\left\{\tfrac{1}{\Gamma(\alpha)}\tfrac{e^{mx}}{m^{1-\beta m}}\Gamma(1-\beta m)\right\}^{\frac{n}{m}}, \quad \beta = 1 - \alpha = \tfrac{z}{1+z}, \tag{2.34}$$

and $X_1 > 0$ *is such that*

$$\left[\Omega\left(2^{n-1}\right) + \tfrac{1}{n}(1-e^{-nx})K_{n,m}\phi^n(x)\right] \in Dom\left(\Omega_n^{-1}\right), \quad x \in [0, X_1] \tag{2.35}$$

Proof. *The inequality* (2.31) *can be rewritten as*

$$\frac{u(x)}{h(x)} \leq 1 + \frac{1}{h(x)}\frac{1}{\Gamma(\alpha)}\phi(x)\int_0^x (x-\rho)^{\alpha-1} g(u(\rho))\,d\rho\ . \tag{2.36}$$

Since $h(x)$ *is nondecreasing function, we get*

$$\frac{u(x)}{h(x)} \leq 1 + \frac{1}{\Gamma(\alpha)}\phi(x)\int_0^x \frac{1}{h(\rho)}\left[(x-\rho)^{\alpha-1} g(u(\rho))\right] d\rho.$$

Let $z(x) = \frac{u(x)}{h(x)}$.*Since* g *belongs to class* $\mathfrak{F}$, *one has*

$$z(x) \leq 1+\frac{1}{\Gamma(\alpha)}\phi(x)\int_0^x \left[(x-\rho)^{\alpha-1} g(z(\rho))\right] d\rho. \tag{2.37}$$

Obviously $\frac{1}{m}+\frac{1}{n}=1$.*Using the Höder inequality we obtain from* (2.37)

$$\begin{aligned} z(x) &\leq 1+\frac{1}{\Gamma(\alpha)}\phi(x)\int_0^x \left[(x-\rho)^{\alpha-1} e^{\rho}e^{-\rho} g(z(\rho))\right] d\rho \\ &\leq 1+\frac{1}{\Gamma(\alpha)}\phi(x)\left[\int_0^x (x-\rho)^{-\beta m} e^{m\rho} d\rho\right]^{\frac{1}{m}} \left[\int_0^x e^{-n\rho} g(z(\rho))^n d\rho\right]^{\frac{1}{n}}. \end{aligned} \tag{2.38}$$

Since $(A+B)^n \leq 2^{n-1}(A^n+B^n)$ *holds for any* $A \geq 0, B \geq 0$ *and*

$$\int_0^x (x-\rho)^{-\beta m} e^{m\rho} d\rho \leq \frac{e^{mx}}{m^{1-\beta m}}\Gamma(1-\beta m),$$

$1-\beta m = \frac{r}{(1+z)(z+r)} > 0$,*we obtain from* (2.38) *that*

$$z^n(x) \leq 2^{n-1} + K_{n,m}\phi^n(x)\int_0^x e^{-n\rho} g^n(z(\rho))\, d\rho.$$

Let $x^* \in [0,x]$ *be a positive constant chosen, we get*

$$z^n(x) \leq 2^{n-1}+K_{n,m}\phi^n(x^*)\int_0^x e^{-n\rho} g^n(z(\rho))\, d\rho, \tag{2.39}$$

where $K_{n,m}$ *is defined by* (2.34). *Let* $G(x)$ *be the right-hand side of the inequality* (2.39). *Then* $z(x) \leq G^{\frac{1}{n}}(x)$ *and this yields* $g^n(z(x)) \leq g^n\left(G^{\frac{1}{n}}(x)\right)$. *From* (2.39), *we obtain*

$$\frac{G'(x)}{g^n\left(G^{\frac{1}{n}}(x)\right)} \leq \frac{K_{n,m}\phi^n(x^*)e^{-nx}g^n(z(x))}{g^n\left(G^{\frac{1}{n}}(x)\right)},$$

i.e.,

$$\frac{d}{dx}\int_0^{G(x)} \frac{d\sigma}{g^n\left(\sigma^{\frac{1}{n}}\right)} \leq K_{n,m}e^{-nx}\phi^n(x^*),$$

or

$$\frac{d}{dx}\Omega_n(G(x)) \leq K_{n,m}e^{-nx}\phi^n(x^*),$$

where Ω_n *is defined by* (2.33). ■

Integrating this inequality from 0 to x, we obtain

$$\Omega_n(z(x)^n) \leq \Omega_n(2^{n-1})+\phi^n(x^*)K_{n,m}\int_0^x e^{-n\rho}d\rho, \tag{2.40}$$

Letting $x = x^*$ in (2.40) and considering $x^* > 0$ is arbitary, after substituting x^* with x, we get

$$\Omega_n(z(x)^n) \leq \Omega_n(2^{n-1})+\phi^n(x)K_{n,m}\int_0^x e^{-n\rho}d\rho. \tag{2.41}$$

Then

$$z(x) \leq \left\{\Omega_n^{-1}\left[\Omega(2^{n-1}) + \frac{1}{n}(1-e^{-nx})K_{n,m}\phi^n(x)\right]\right\}^{\frac{1}{n}}. \tag{2.42}$$

This completes the proof of Theorem 6.

3 Applications

In this section, we present some applications of the inequalities (2.11) in Theorem 4 for studying the boundedness and uniqueness of certain fractional integral equation with the Riemann Liouville (R-L) fractional operator. Consider the following fractional integral equation:

$$u^p(x) = h(x) + I^{\alpha}(F(x,u(x))), \tag{3.1}$$

where $0 < \alpha < 1$, $p \geq 1$ and $F \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $h \in C(\mathbb{R},\mathbb{R})$.

Example 1 *Assume that* $F(x,u(x))$ *satisfies*

$$|F(x,u)| \leq \phi(x)g(|u|), \tag{3.2}$$

where g, ϕ are defined as in Theorem 4, and $\phi(x)$ is nondecreasing function in $x \geq 0$, then we have the following estimate for $u(x)$

$$\begin{aligned} |u(x)| \leq \{&|h(x)| + \tfrac{1}{\Gamma(\alpha)}\phi(x)\textstyle\int_0^x (x-\rho)^{\alpha-1}\widehat{h}(\rho) \\ &\times \exp\left(-\int_{\frac{\rho^\alpha}{\Gamma(1+\alpha)}}^{\frac{x^\alpha}{\Gamma(1+\alpha)}} \widehat{\phi}\left((s\Gamma(1+\alpha))^{\frac{1}{\alpha}}\right) ds\right) d\rho\}^{\frac{1}{p}}, \end{aligned} \tag{3.3}$$

where

$$\widehat{h}(x) = g(\frac{1}{p}\varepsilon^{\frac{1-p}{p}}|h(x)| + \frac{p-1}{p}\varepsilon^{\frac{1}{p}}) \quad (3.4)$$
$$\widehat{\phi}(x) = \frac{1}{p}\varepsilon^{\frac{1-p}{p}} g'(\frac{1}{p}\varepsilon^{\frac{1-p}{p}}|h(x)| + \frac{p-1}{p}\varepsilon^{\frac{1}{p}}) .$$

Proof. According to Definition 1, from (3.1)-(3.2), we have

$$u^p(x) = h(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1}(F(\rho, u(\rho))d\rho, \quad (3.5)$$

$$|u^p(x)| \leq |h(x)| + \frac{1}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1}|(F(\rho, u(\rho))| d\rho, \quad (3.6)$$

$$|u^p(x)| \leq |h(x)| + \frac{1}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1}\phi(\rho) g\,|u(\rho)|\, d\rho,$$

taking into account that ϕ is nondecreasing function, we get

$$|u^p(x)| \leq |h(x)| + \frac{\phi(x)}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1} g(|u(\rho)|)d\rho.$$

Letting $q = 1$, and applying Theorem 4, we get the desired estimate in (3.3). ■

Example 2 *Assume that*

$$|F(x, u) - F(x, \overline{u})| \leq \phi(x) g\left(|u - \overline{u}|\right), \quad (3.7)$$

where g is defined as in Theorem 4 such that $g(0) = 0$ *and* $\phi(x)$ *is nondecreasing functions in* $x \geq 0$. *Then equation* (3.1) *has a unique solution.*

Proof. *Suppose* $u(x)$, $\overline{u}(x)$ *are two solutions of equation* (3.1), *then we have*

$$u(x) = h(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1} F(\rho, u(\rho))d\rho,$$
$$\overline{u}(x) = h(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1} F(\rho, \overline{u}(\rho))d\rho,$$

Furthermore,

$$u(x) - \overline{u}(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1}\left[F(\rho, u(\rho)) - F(\rho, \overline{u}(\rho))\right] d\rho,$$

which implies

$$|u(x) - \overline{u}(x)| \leq \frac{1}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1}\phi(\rho) g\left(|u(\rho) - \overline{u}(\rho)|\right) d\rho.$$

Taking into account that ϕ *is nondecreasing function, we get*

$$|u(x) - \overline{u}(x)| \leq \frac{\phi(x)}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1} g\left(|u(\rho) - \overline{u}(\rho)|\right) d\rho. \quad (3.8)$$

Through a suitable application of Theorem 4 to (3.8) (*with* $p = q = 1$), *we obtain that* $|u(x) - \overline{u}(x)| \leq 0$, *which implies* $u(x) = \overline{u}(x)$. ■

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