On the Reciprocal Sums of Multiples-of-\(p\)-indexed Fibonacci Numbers

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Abstract: In this paper we derive some identities related to the reciprocal sums of multiples-of-\(p\)-indexed Fibonacci numbers.

Key–Words: Fibonacci numbers, reciprocal, floor function.

1 Introduction

The classical Fibonacci numbers, denoted by \(F_n\), are generated from the recurrence relation

\[ F_n = F_{n-1} + F_{n-2} \quad (n \geq 2), \]

with initial condition \(F_0 = 0\) and \(F_1 = 1\). Over the decades, numerous results on the properties and applications of the Fibonacci numbers have been reported [4].

Recently Ohtsuka and Nakamura [6] found interesting properties of the Fibonacci numbers and proved Theorem 1 below, where \([\cdot]\) indicates the floor function and \(\mathbb{N}_e (\mathbb{N}_o, \text{respectively})\) denotes the set of positive even (odd, respectively) integers.

Theorem 1 Let \(n \geq 1\). Then

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right] = \begin{cases} F_n - F_{n-1}, & \text{if } n \in \mathbb{N}_e; \\
F_n - F_{n-1} - 1, & \text{if } n \in \mathbb{N}_o. \end{cases}
\]

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F^2_k} \right)^{-1} \right] = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \in \mathbb{N}_e; \\
F_{n-1}F_n, & \text{if } n \in \mathbb{N}_o. \end{cases}
\]

After the work of Ohtsuka and Nakamura [6], diverse results in the same direction have appeared in the literature [1–3], [5], [7–9]. In particular, Wang and Zhang [8], [9] considered the even/odd-indexed Fibonacci numbers and the Fibonacci 3-subsequences. According to the results of [8], [9], Theorem 2 and Theorem 3 below hold.

Theorem 2 Let \(n \geq 1\). Then

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F_{2k}} \right)^{-1} \right] = F_{2n} - F_{2n-2} - 1, \quad (3)
\]

Theorem 3 For \(n \geq 1\), we have

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F_{3k}} \right)^{-1} \right] = \begin{cases} 2F_{3n-2}, & \text{if } n \in \mathbb{N}_e; \\
2F_{3n-2} - 1, & \text{if } n \in \mathbb{N}_o. \end{cases}
\]

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F^2_{3k}} \right)^{-1} \right] = \begin{cases} F^2_{3n} - F^2_{3n-3}, & \text{if } n \in \mathbb{N}_e; \\
F_{3n} - F^2_{3n-3} - 1, & \text{if } n \in \mathbb{N}_o. \end{cases}
\]

Before going further, we note that the following identities can be easily proved:

\[ F_{4n-2} = F^2_{2n} - F^2_{2n-2}, \]
\[ 2F_{3n-2} = F_{3n} - F_{3n-3}. \]

The purpose of this paper is to generalize Theorem 1–Theorem 3. More precisely, we obtain identities related to the numbers

\[ \left[ \left( \sum_{k=n}^{\infty} \frac{1}{F_{pk}} \right)^{-1} \right], \left[ \left( \sum_{k=n}^{\infty} \frac{1}{F^2_{pk}} \right)^{-1} \right], \quad p = 1, 2, 3, \cdots. \]

2 Main Results

First, we present two lemmas which will be used to prove our main results.

Lemma 4 [4]

\[ F_mF_n - F_{m+k}F_{n-k} = (-1)^{n-k}F_{m+k-n}F_k. \]
Lemma 5 below is obtained by letting $n = k + 1$ and interchanging the roles of $k, m$ in Lemma 4.

**Lemma 5**

$$F_{m+k} = F_k F_{m+1} + F_{k-1} F_m.$$

**Proposition 6**

$$\frac{1}{F_{pn} - F_{pn-p}} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}}, \quad \text{if } p \in \mathbb{N}_e \text{ or } p, n \in \mathbb{N}_o; \quad (7)$$

$$\sum_{k=n}^{\infty} \frac{1}{F_{pk}} < \frac{1}{F_{pn} - F_{pn-p}}, \quad \text{if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e. \quad (8)$$

**Proof:** Consider

$$X_1 = \frac{1}{F_{pn} - F_{pn-p}} - \frac{1}{F_{pn+p} - F_{pn}} = \frac{1}{(F_{pn} - F_{pn-p})(F_{pn+p} - F_{pn})F_{pn}},$$

where, by Lemma 4

$$\hat{X}_1 = F_{pn+p}F_{pn+p} - F^2_{pn} = (-1)^{m-p-1}F^2_{p}.$$

If $p \in \mathbb{N}_e$ or $p, n \in \mathbb{N}_o$, then $X_1 < 0$ and

$$\frac{1}{F_{pn} - F_{pn-p}} - \frac{1}{F_{pn+p} - F_{pn}} < \frac{1}{F_{pn}}.$$

Repeatedly applying the above inequality, we have

$$\frac{1}{F_{pn} - F_{pn-p}} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}}, \quad \text{if } p \in \mathbb{N}_e \text{ or } p, n \in \mathbb{N}_o.$$

Similarly, if $p \in \mathbb{N}_o$ and $n \in \mathbb{N}_e$, then $X_1 > 0$ and we obtain

$$\sum_{k=n}^{\infty} \frac{1}{F_{pk}} < \frac{1}{F_{pn} - F_{pn-p}}, \quad \text{if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e.$$

Hence the proof is completed. □

**Proposition 7**

$$\sum_{k=n}^{\infty} \frac{1}{F_{pk}} < \frac{1}{F_{pn} - F_{pn-p} - 1}, \quad \text{if } p \in \mathbb{N}_e \text{ or } p, n \in \mathbb{N}_o. \quad (9)$$

**Proof:** Consider

$$\begin{align*}
X_2 &= \frac{1}{F_{pn} - F_{pn-p} - 1} - \frac{1}{F_{pn+p} - F_{pn+p} - 1} - \frac{1}{F_{pn} - F_{pn+p}} \\
&= \frac{1}{F_{pn} - F_{pn+p} - 1} \\
&\quad \times \frac{(F_{pn} - F_{pn-p} - 1)(F_{pn+p} - F_{pn+p} - 1)}{F_{pn}F_{pn+p}}.
\end{align*}$$

where, by Lemma 4

$$\hat{X}_2 = (F_{pn+p+2} - 1)(F_{pn+p}F_{pn+p} - F^2_{pn}) + (F_{pn+p} + 1)(F_{pn+p}F_{pn+p} - F^2_{pn}) - F_{pn+p}F_{pn-p} - F_{pn+p}F_{pn} + F_{pn+p}F_{pn+p+2} = (-1)^{m-p-1}F^2_{p}(F_{pn+p+2} - 1) + (-1)^{m-p}F^2_{p}(F_{pn+p} + 1) - F_{pn+p}F_{pn-p} - F_{pn+p}F_{pn} + F_{pn+p}F_{pn+p+2}.$$

Now assume that $p \in \mathbb{N}_e$. We can easily show that $\hat{X}_2 > 0$ for $n = 1$. Hence let $n \geq 2$. By Lemma 5, we have

$$\begin{align*}
\hat{X}_2 &= -F^2_{p}(F_{pn+p+2} + F_{pn+p}) - F_{pn+p}F_{pn} - F_{pn} - F_{pn+p}F_{pn+p+2} \\
&\quad - F_{pn+p}F_{pn+p+2} - F_{pn+p}F_{pn} - F_{pn+p}F_{pn+2} \\
&\quad - F_{pn+p}F_{pn} - F_{pn+p}F_{pn} - F_{pn} - F_{pn+p}.
\end{align*}$$

Since, for $n \geq 2$

$$F_{pn+p+1} - F^2_{p} \geq F_{p}F_{pn},$$

then

$$\begin{align*}
\hat{X}_2 &\geq \frac{1}{F_{pn+p+1} - F_{p}F_{pn}} - \frac{1}{F_{p}F_{pn+p} - F_{pn+p}} \\
&\quad - F_{pn+p}F_{pn} - F_{pn+p}F_{pn} + F_{pn+p}F_{pn+1}F_{p} \\
&\quad - F_{p}F_{pn+p+2} + (F_{p}F_{p-1}F_{pn} + F_{p}F_{p-1}F_{pn}) \\
&\quad + F_{p}F_{p-1}F_{pn} - F_{p}F_{p-1}F_{pn+1}F_{p} \\
&\quad - F_{p}F_{p-1}F_{pn} < 0.
\end{align*}$$

If $p, n \in \mathbb{N}_o$, then

$$\hat{X}_2 = -F^2_{p}(F_{pn+p+2} - F_{pn+p}) - 2 - F_{pn+p}F_{pn}.$$
\[ -F_{pn} - F_{pn+p} + F_{pn+p}F_{pn+2p} \]
\[ > -F_p(F_{pn+2p} + F_{pn-p}) - F_{pn-p}F_{pn} - F_{pn} \]
\[ -F_{pn+p} + F_{pn+p}F_{pn+2p} \]
\[ > 0. \]

Consequently, we have
\[ \frac{1}{F_{pn}} + \frac{1}{F_{pn+p}} < \frac{1}{F_{pn} - F_{pn-p} - 1} + \frac{1}{F_{pn+2p} - F_{pn+p} - 1}, \]
from which we can obtain the inequality
\[ \sum_{k=n}^{\infty} \frac{1}{F_{pk}} < \frac{1}{F_{pn} - F_{pn-p} - 1} \]
if \( p \in \mathbb{N}_e \) or \( p, n \in \mathbb{N}_o \),
and the proof is completed. \( \square \)

Theorem 8 below follows from Proposition 6 and Proposition 7.

**Theorem 8**

\[ \left[ \sum_{k=n}^{\infty} \frac{1}{F_{pk}} \right]^{-1} = F_{pn+p} - F_{pn-p} - 1, \quad \text{if } p \in \mathbb{N}_e \text{ and } n \geq 1. \]  
\[ (10) \]

**Proposition 9**

\[ \frac{1}{F_{pn} - F_{pn-p} + 1} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}}, \quad \text{if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e. \]
\[ (11) \]

**Proof:** Consider
\[ \hat{X}_3 = \frac{1}{F_{pn} - F_{pn-p} + 1} - \frac{1}{F_{pn+2p} - F_{pn+p} + 1} \]
\[ - \frac{1}{F_{pn}} - \frac{1}{F_{pn+p}} \]
\[ = \left( F_{pn}-F_{pn-p}+1 \right) \left( F_{pn+2p} - F_{pn+p} + 1 \right) \]
\[ \times \frac{1}{F_{pn}F_{pn+p}}. \]

where
\[ \hat{X}_3 = \hat{X}_2 + 2(F_{pn} + F_{pn+p})(F_{pn-p} - F_{pn}) \]
\[ + F_{pn+p} - F_{pn+2p} \]
\[ = (-1)^{pn-p-1}F_p(F_{pn+2p} - 1) \]
\[ + (-1)^{pn-1}p^2(F_{pn-p} + 1) \]
\[ - F_{pn-p}F_{pn} - F_{pn-p} + F_{pn+p}F_{pn+2p} \]
\[ + 2(F_{pn} + F_{pn+p})(F_{pn-p} - F_{pn} + F_{pn+p}) \]
\[ - F_{pn+2p}). \]

Here, \( \hat{X}_2 \) is as defined in the proof Proposition 7.
If \( p \in \mathbb{N}_o \) and \( n \in \mathbb{N}_e \), then, by Lemma 4 and Lemma 5
\[ \hat{X}_3 = F_p^2F_{pn+2p} + F_{pn-p}F_{pn} - 2(F_{pn}^2 - F_{pn-p}F_{pn+p}) \]
\[ + 2(F_{pn+p}^2 - F_{pn}F_{pn+2p}) - F_p^2F_{pn-p} \]
\[ - F_{pn} - F_{pn+p} - F_{pn+p}F_{pn+2p} - 2F_p^2 \]
\[ = F_p^2F_{pn+2p} + F_{pn-p}F_{pn} + 2F_p^2 - F_p^2F_{pn-p} \]
\[ - F_{pn} - F_{pn+p} - F_{pn+p}F_{pn+2p} \]
\[ = F_p^2F_{pn+1+p} + F_{pn+1}(F_{pn}) + F_{pn-p}F_{pn} \]
\[ + 2F_p^2 - F_p^2F_{pn-p} - F_{pn} \]
\[ - (F_pF_{pn+1} + F_{p-1}F_{pn}) \]
\[ - (F_pF_{pn+1} + F_{p-1}F_{pn})(F_{2p}F_{pn+1} + F_{2p-1}F_{pn}) \]
For the case where \( p = 1 \) and \( n \in \mathbb{N}_e \), it is easily seen that \( \hat{X}_3 < 0 \). If \( p \geq 3 \) and \( n \in \mathbb{N}_e \), then
\[ \hat{X}_3 < \left( F_p^2F_{2p}F_{pn+1} - F_pF_{2p}F_{pn+1} \right) \]
\[ + (F_{2p}F_{2p-1}F_{pn} - F_{2p-1}F_{2p}F_{pn+1}) \]
\[ + (F_{pn-p}F_{pn} - F_{p-1}F_{2p}F_{pn+1}) \]
\[ + (2F_p^2 - F_p^2F_{pn-p}) \]
\[ < 0. \]

Hence we have
\[ \frac{1}{F_{pn}-F_{pn-p}+1} - \frac{1}{F_{pn+2p}-F_{pn+p}+1} < \frac{1}{F_{pn}} + \frac{1}{F_{pn+p}}. \]

Repeatedly applying the above inequality, we obtain
\[ \frac{1}{F_{pn}-F_{pn-p}+1} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}}, \quad \text{if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e, \]
and the proof is completed. \( \square \)

From Proposition 6, Proposition 7 and Proposition 9, we obtain the following result.

**Theorem 10** Let \( p \in \mathbb{N}_o \). Then
\[ \left[ \sum_{k=n}^{\infty} \frac{1}{F_{pk}} \right]^{-1} = \begin{cases} F_{pn} - F_{pn-p}, & \text{if } n \in \mathbb{N}_e; \\ F_{pn} - F_{pn-p} - 1, & \text{if } n \in \mathbb{N}_o. \end{cases} \]
\[ (12) \]

**Proposition 11**

\[ \frac{1}{F_{2p}^2 - F_{pn-p}^2} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2}, \quad \text{if } p \in \mathbb{N}_e \text{ or } p, n \in \mathbb{N}_o; (13) \]
\[ \sum_{k=n}^{\infty} \frac{1}{F_{2p}^2 - F_{pn-p}^2} < \frac{1}{F_{2p}^2 - F_{pn-p}^2}, \quad \text{if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e. \]
Proof: Consider

\[ Y_1 = \frac{1}{F_{pm}^2 - F_{pm+2}^2} - \frac{1}{F_{pm+2}^2 - F_{pm}^2} - \frac{1}{F_{pm}^2} \]

\[ = \frac{\hat{Y}_1}{(F_{pm}^2 - F_{pm}^2)(F_{pm+2}^2 - F_{pm}^2)F_{pm}^2}, \]

where, by Lemma 4

\[ \hat{Y}_1 = F_{pm}^2 - F_{pm+2}^2 - F_{pm+2}^4 \]

\[ = (-1)^{m-1} (F_{pm}^2 + F_{pm+2}^2). \]

If \( p \in \mathbb{N}_e \) or \( p, n \in \mathbb{N}_o \), then \( Y_1 < 0 \) and

\[ \frac{1}{F_{pm+2}^2 - F_{pm}^2} - \frac{1}{F_{pm}^2} < \frac{1}{F_{pm+2}^2}. \]

Repeatedly applying the above inequality, we have

\[ \frac{1}{F_{pm}^2 - F_{pm+2}^2} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2}, \text{ if } p \in \mathbb{N}_e \text{ or } p, n \in \mathbb{N}_o. \]

Similarly, if \( p \in \mathbb{N}_o \) and \( n \in \mathbb{N}_e \), then \( Y_1 > 0 \) and we obtain

\[ \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2} < \frac{1}{F_{pm}^2 - F_{pm+2}^2 - 1}, \text{ if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e. \]

Hence the proof is completed. \( \square \)

Proposition 12

\[ \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2} < \frac{1}{F_{pm}^2 - F_{pm+2}^2 - 1}, \text{ if } p \in \mathbb{N}_o \text{ or } p, n \in \mathbb{N}_o. \] (15)

Proof: Consider

\[ Y_2 = \frac{1}{F_{pm}^2 - F_{pm+2}^2} - \frac{1}{F_{pm+2}^2 - F_{pm}^2} - 1 - \frac{1}{F_{pm+2}^2} \]

\[ = \frac{\hat{Y}_2}{(F_{pm}^2 - F_{pm+2}^2)(F_{pm}^2 - F_{pm+2}^2)F_{pm+2}^2}, \]

where, by Lemma 4

\[ \hat{Y}_2 = (F_{pm}^2 + F_{pm+2}^2) - F_{pm+2}^4 + F_{pm}^2 + F_{pm}^2 \]

\[ \times (F_{pm}^2 - F_{pm+2}^2) \]

Assume that \( p \in \mathbb{N}_e \). Since

\[ F_{pm+2}^2 > F_{pm}^2 + F_{pm}F_{pm+2}, \]

\[ F_{pm+2}^2F_{pm+2}^2 > F_{pm}^2F_{pm} + F_{pm+2}^2F_{pm+2}^2, \]

then

\[ \hat{Y}_2 = F_{pm}^2 + F_{pm}F_{pm+2} + 1)(F_{pm}^2 + F_{pm}F_{pm+2} - 2) \]

\[ > 0, \]

and so \( Y_2 > 0 \) for \( p \in \mathbb{N}_e \).

If \( p, n \in \mathbb{N}_o \), then we also have \( Y_2 > 0 \).

Consequently, if \( p \in \mathbb{N}_e \) or \( p, n \in \mathbb{N}_o \), we have

\[ \frac{1}{F_{pm}^2} + \frac{1}{F_{pm+2}^2} < \frac{1}{F_{pm} - F_{pm+2}^2 - F_{pm+2}^2 - 1}, \]

from which we obtain the inequality

\[ \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2} < \frac{1}{F_{pm}^2 - F_{pm+2}^2 - 1}, \text{ if } p \in \mathbb{N}_o \text{ or } p, n \in \mathbb{N}_o, \]

and the proof is completed. \( \square \)

From Proposition 11 and Proposition 12, we obtain the following result.

Theorem 13

\[ \left[ \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2} \right]^{-1} = F_{pm}^2 - F_{pm+2}^2, \text{ if } p \in \mathbb{N}_e \text{ and } n \geq 1. \] (16)

Proposition 14

\[ \frac{1}{F_{pm}^2} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2}, \text{ if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e. \] (17)
Proof: Consider

\[
Y_3 = \frac{1}{F_{pm}^2 - F_{pm}^2 - 1} - \frac{1}{F_{pm+2p}^2 - F_{pm+2p}^2 - 1}
\]

and the proof is completed.

\[
\hat{Y}_3 = \hat{Y}_2 + 2(F_{pm}^2 + F_{pm+2p}^2)(F_{pm}^2 - F_{pm}^2 - F_{pm+2p}^2 - 1)
\]

where

\[
\hat{Y}_2 = (F_{pm}^2 - F_{pm}^2 - 1)(F_{pm+2p}^2 - F_{pm+2p}^2 - 1)
\]

Here, \( \hat{Y}_2 \) is as defined in the proof Proposition 12.

If \( p \in \mathbb{N}_o \) and \( n \in \mathbb{N}_e \), then \( \hat{Y}_3 < 0 \) and we have

\[
\frac{1}{F_{pm}^2 - F_{pm}^2 - 1} - \frac{1}{F_{pm+2p}^2 - F_{pm+2p}^2 - 1} < \frac{1}{F_{pm}^2} + \frac{1}{F_{pm+2p}^2}.
\]

Repeatedly applying the above inequality, we obtain

\[
\frac{1}{F_{pm}^2 - F_{pm}^2 - 1} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2}, \quad \text{if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e,
\]

and the proof is completed.

From Proposition 11, Proposition 12 and Proposition 14, we obtain the following result.

**Theorem 15** Let \( p \in \mathbb{N}_o \). Then

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2} \right)^{-1} \right] = \begin{cases} F_{pm}^2 - F_{pm}^2 - 1, & \text{if } n \in \mathbb{N}_e, \\ F_{pm}^2 - F_{pm}^2 + 1, & \text{if } n \in \mathbb{N}_o. \end{cases}
\]

(18)

**References:**


