Weighted Inverse Nakagami Distribution

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Abstract: - In this paper, a weighted distribution based on Inverse Nakagami distribution which is reciprocal of Nakagami-$m$ distribution is suggested. The proposed model is capable of explaining systems with high failure rates within a short interval of time. The proposed model also performs well for relatively small sample sizes. The various statistical properties of the proposed model are presented such as moments, mean and variance, skewness and kurtosis, reliability functions and Shannon entropy. The maximum likelihood as well as moment estimators are explored. A simulation study compares the performance of the estimators against the true value of the parameters. Finally, the proposed distribution is fitted to a real dataset.

Key-Words: - The Inverse Nakagami distribution, The Nakagami distribution, Weighted distribution.

1 Introduction

With growing dependency of human civilization on tele-communication, developing and designing a robust system of communication and understanding the uncertainty related to network signal is always a field of interest for network engineers as well as statisticians. Nakagami (1960) after a series of experiment provided a general formula to describe how intensity of rapid fading are distributed [1]. The Nakagami-$m$ distribution is extensively used in describing multipath signal fading in a network system [2] as well as image processing [3], hydrology engineering [4] and reliability studies [5]. The distribution is a special case to many standard distributions in the literature such as Gamma, Rayleigh, Weibull, Chi-Square and Exponential [6].

Corresponding to a non-negative random variable $X$, the probability density function (pdf) of Nakagami-$m$ distribution (NKG) is given by

$$f(x) = \frac{2}{\Gamma(m)} \left(\frac{m}{w}\right)^m x^{2m-1} e^{-\frac{m}{w}x^2}$$

where $m > 0.5$ is the shape parameter and $w > 0$ is the spread parameter.

Extending the application of a standard distribution which is already defined in the literature by means of various algebraic operations is a common practice. One such operation is taking the inverse of a distribution which provides better flexibility and wider applicability to the base distribution [7]. As for example, the inverse of a standard normal or a uniform distribution becomes bimodal [8]. Inverse Chi square, Inverse Gamma [9] [10], Inverse Weibull and [11] Inverse Rayleigh [12] are some other examples of inverse distribution defined in the literature.

Louzada et. al. (2018) defined the Inverse Nakagami (INK) distribution taking the inverse of a Nakagami-$m$ variate [13]. The INK distribution proved to be a better fit in a system with high failure rate.

Corresponding to a non-negative random variable $Y = \frac{1}{X}$ where $X$ is distributed as Nakagami-$m$, the pdf of INK can be written as

$$f(y) = \frac{2}{\Gamma(m)} \left(\frac{m}{w}\right)^m y^{2m-1} e^{-\frac{m}{w}y^2}$$

which is a special case of Inverse Rayleigh, Inverse Chi and the Inverse half-normal distribution.
The study of weighted distribution gives a new dimension to standard distributions by exploring different characteristics of standard distribution and adding flexibility to the distribution in terms of fitting the distribution to data [14].

If \( f(x) \) is the pdf of a non-negative random variable \( X \) and we consider a weight function \( w(x) \), then the weighted density function corresponding to \( f(x) \) is obtained by [15]

\[
g(x) = \frac{w(x)f(x)}{E[w(x)]} \tag{3}
\]

In this paper, a weighted distribution is proposed based on Inverse Nakagami distribution. The proposed distribution is named Weighted Inverse Nakagami (WINK) distribution. The various statistical properties of the proposed WINK distribution are discussed such as, \( r \)th order raw moments, corresponding central moments, mean, variance, skewness and kurtosis, mode and Shannon entropy. Various reliability properties such as hazard rate, survival functions and reverse hazard rate function are also discussed in detail.

Parameter estimation of the proposed WINK distribution are discussed under maximum likelihood and method of moment estimation. The maximum likelihood estimator of the spread parameter \( w \) is found out to be an unbiased estimator. The performance of both maximum likelihood and method of moment estimator are compared based on simulated data. Finally, the proposed WINK distribution is fitted to a real data sets related to duration till mortality (survival time) of patient with severe head injury due to road mishap. Our proposed WINK distribution performs well in describing a system with high failure rate within a short period of time with few extreme observations.

The paper is organized as follows. Section 2 introduces the properties of the proposed distribution. Section 3 discuss the estimation of the parameters of our suggested distribution. In Section 4 we have compared the performance of the estimators based on simulated data. Section 5 explores the applicability of our proposed distribution in fitting two real life data sets. Section 6 summarizes the study.

2 Weighted Inverse Nakagami
Consider a random variable \( X \) which is distributed as (2) and a weight function \( w(x) = x^a \), then from (3) we have the pdf of WINK distribution as

\[
g(x) = \frac{2}{\Gamma\left(m - \frac{a}{2}\right)} \frac{m^{a-1}}{w} x^{a-2m-1} e^{-\frac{m}{wx^2}} \tag{4}
\]

for all \( X > 0, m > \frac{a}{2} \) and \( w > 0 \). Some important distributions can be obtained from (4) which are discussed in section 2.8.

Putting \( a=1 \) and \( 2 \) in (4) we get respectively the Length Biased Inverse Nakagami (LBINK) and Area Biased Inversed Nakagami (ABINK) distribution as follows

\[
g_1(x) = \frac{2}{\Gamma(m - \frac{1}{2})} \frac{m^{-\frac{1}{2}}}{w} x^{-2m} e^{-\frac{m}{wx^2}} \tag{5}
\]

\[
g_2(x) = \frac{2}{\Gamma(m - 1)} \frac{m^{-1}}{w} x^{1-2m} e^{-\frac{m}{wx^2}} \tag{6}
\]

The corresponding cumulative distribution function (CDF) of WINK distribution with parameter \( m \) and \( w \) is obtained as

\[
G_x(x) = \frac{\Gamma(m - \frac{a}{2}, \frac{m}{wx^2})}{\Gamma(m - \frac{a}{2})}; x > 0 \tag{7}
\]

Where, \( \Gamma(m, x) = \int_x^{\infty} e^{-t} t^{m-1} dt \) is the upper incomplete gamma integral.

![Figure 1: Density of the WINK distribution for weight = 1 and 2](image)
The mean of WINK distribution is given by (9) and (13) gives the variance.

2.1.2 Skewness and Kurtosis
The skewness and kurtosis of WINK distribution are respectively found to be

$$
\beta_1 = \frac{2\kappa_3 \kappa_1 + \kappa_0 \kappa_3 - 3\kappa_0 \kappa_1 \kappa_2}{\kappa_0 \kappa_2 - \kappa_1^3}
$$

$$
\beta_2 = \frac{[\kappa_0 \kappa_4 - 4\kappa_0 \kappa_1 \kappa_2 + 6\kappa_0 \kappa_1^2 \kappa_2 - 3\kappa_1^4]}{[\kappa_0 \kappa_2 - \kappa_1^3]^2}
$$

2.2 Moment Generating Function
To derive the moment generating function (MGF) of WINK distribution, Taylor's expansion is used as follows

$$
M_X(t) = E[e^{tx}] = \int_0^\infty e^{tx} f(x)dx
$$

$$
= \int_0^\infty \left[ 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} \right] f(x)dx
$$

$$
= \sum_{r=0}^{\infty} \frac{t^r x^r}{r!} f(x)dx
$$

$$
= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty x^r f(x)dx
$$

$$
= \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{m^r}{w^r} \Gamma\left(m - \frac{ax}{2}\right) \frac{1}{\Gamma\left(m - \frac{a}{2}\right)}
$$

2.3 Characteristic Function
Extending the concept of (18) we can derive the characteristic function of WINK distribution as

$$
\Phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left( \frac{m^r}{w^r} \Gamma\left(m - \frac{ax}{2}\right) \frac{1}{\Gamma\left(m - \frac{a}{2}\right)} \right)
$$

2.4 Survival Function
The survival function or reliability function gives the probability that an observation will not fail until x and is given by

$$
S(x) = 1 - \frac{\Gamma\left(m - \frac{a}{2}\right) \frac{m}{w x^2}}{\Gamma\left(m - \frac{a}{2}\right)}
$$

$$
= \frac{\gamma\left(m - \frac{a}{2}\right) \frac{m}{w x^2}}{\Gamma\left(m - \frac{a}{2}\right)}
$$
Where, \( y(m,x) = \int_0^x e^{-t} t^{m-1} dt \) is the lower incomplete gamma integral.

2.5 Hazard Function

The hazard function of WINK distribution is obtained as

\[
h(x) = \frac{2}{y(m - \frac{a}{2}, \frac{m}{w})} \left( \frac{m}{w} \right)^{m-a/2} x^{a-2m-1} e^{-\frac{m}{wx}}
\]  

(21)

And the corresponding reverse hazard rate function of WINK distribution is

\[
\phi(x) = \frac{2}{\Gamma(m - \frac{a}{2}, \frac{m}{w})} \left( \frac{m}{w} \right)^{m-a/2} x^{a-2m-1} e^{-\frac{m}{wx}}
\]  

(22)

2.6 Mode

Consider the pdf of WINK in (4), taking logarithm we get

\[
\log[g(x)] = \log(2) + \left( m - \frac{a}{2} \right) \log(m) - \log(w) + \left( a - 2m - 1 \right) \log(x) - \left( \frac{m}{wx^2} \right)
\]  

(23)

Differentiating (23) w.r.t. \( x \) and equating to zero, we get the mode of WINK distribution as

\[
x_0 = \sqrt{\frac{2m}{2m - a + 1}w}
\]  

(24)

2.7 Shannon’s Entropy

Entropy has an important role to play in the theory of information which gives a measure of uncertainty associated with a random variable. Shannon’s entropy being the most popular among them. The Shannon’s entropy of WINK distribution is obtained by solving the equation

\[
H(x) = -E[\log(g(x))]
\]

\[
= -\log C - \left( a - 2m - 1 \right) E[\log(x)] + \left( \frac{m}{w} \right) E\left[ \frac{1}{X^2} \right]
\]

Where, \( C = \frac{2}{\Gamma(m - \frac{a}{2}, \frac{m}{w})} \left( \frac{m}{w} \right)^{m-a/2} \)

Also,

\[
E[\log(x)] = \int_0^\infty \log(x) f(x) dx
\]

\[
= \frac{1}{2} \log \left( \frac{m}{w} \right)
\]

And,

\[
E\left[ \frac{1}{X^2} \right] = \left( \frac{w}{m} \right) \left( m - \frac{a}{2} \right)
\]

Finally,

\[
H(x) = \left( m - \frac{a}{2} \right) + \log \left[ \frac{\Gamma\left( m - \frac{a}{2}, \frac{m}{w} \right)}{2} \right]
\]  

(25)

2.8 Related Distributions

There are many standard distributions that are directly or indirectly related to our proposed WINK distribution. These relationships with different distributions is useful to generate random samples from our proposed WINK distribution and to compare efficiency of proposed distribution.

Theorem 1 If \( X \) is distributed as WINK variate with parameters \( m, w \) and \( a=0 \), then \( Y = \frac{1}{X} \) is distributed.
as Nakagami variate with scale parameter $m$ and shape parameter $w$.

**Theorem 2** If $X$ is distributed as WINK variate with parameters $m$, $w$ and $a$, then $Y = \frac{1}{X^2}$ is distributed as Gamma variate with shape parameter $\left( m - \frac{a}{2} \right)$ and rate parameter $\theta = \frac{m}{w}$.

**Theorem 3** If $X$ is distributed as WINK variate with parameters $m$, $w$ and $a$, then $Y = X^2$ is distributed as Inverse Gamma variate with shape parameter $\left( m - \frac{a}{2} \right)$ and scale parameter $\theta = \frac{m}{w}$.

**Theorem 4** If $X$ is a WINK variate with parameters $m=2$, $a=2$ and $w=1$, then $Y=X$ is an inverse exponential variate with parameter $\lambda$.

**Theorem 5** If $X$ is a WINK variate with parameters $m=p$, $a$ and $w=2p$, then $Y=X^2$ is an inverse Chi square variate with $\left( p - \frac{a}{2} \right)$ degrees of freedom.

**Theorem 6** If $X$ is a WINK variate with parameters $m=2$, $a=2$ and $w$, then $Y=X$ is an inverse Weibull variate with parameters $\left( 2, \frac{2}{w} \right)$.

**Theorem 7** If $X$ is a WINK variate with parameters $m=2$, $a=2$ and $w$, then $Y=X$ is an inverse Rayleigh variate with parameter $\frac{2}{w}$.

### 3 Inferential Procedures

Our proposed distribution has three parameters, the scale parameter $m$, the spread parameter $w$ and the weight parameter $a$. For simplicity, throughout the estimation of the parameters, we will keep the weight parameter fixed at 1, 2 and 3. The shape parameter and spread parameter are estimated using both method of moments (MME) and maximum likelihood (MLE). Their performance is compared based on a simulated dataset.

#### 3.1 Maximum Likelihood Estimation

The maximum likelihood estimation procedure is preferred over all other classical method of statistical inference because of its asymptotic properties. Let, $X_1, X_2, ..., X_n$ be a random sample from WINK $(m, w, a)$. In such a case, the likelihood function corresponding to (4) is given by

$$L(x; m, w, a) = \left[ \frac{2}{\Gamma \left( m - \frac{a}{2} \right)} \right]^n \left( \frac{m}{w} \right) \left( m - \frac{a}{2} \right)^n \prod_{i=1}^{n} x_i^{a-2m-1} e^{- \frac{m}{w} \sum_{i=1}^{n} \frac{1}{x_i^2}}$$

The corresponding log likelihood function is

$$\log L(x; m, w, a) = n \log(2) - n \log \Gamma \left( m - \frac{a}{2} \right) + n \left( m - \frac{a}{2} \right) \log \left( \frac{m}{w} \right) + (a - 2m - 1) \sum \log x_i - \left( \frac{m}{w} \right) \sum \frac{1}{x_i^2}$$

Differentiating (27) with respect to $m$ and $w$ and equating to zero, we get the likelihood equations as

$$\frac{\partial}{\partial m} \log L = 0$$

$$\frac{\partial}{\partial w} \log L = 0$$

Where, $\psi(r) = \frac{\partial}{\partial r} \log \Gamma(r) = \frac{\Gamma'(r)}{\Gamma(r)}$, is the digamma function.

Equation (29) requires knowledge of $m$ and $a$. (28) is solved using numerical method for particular values of $a = 1$, 2 and 3. The values are then replaced in (29) to get an estimate of $w$. It can be shown that (29) gives an unbiased estimate of $w$.

The MLE estimates are asymptotically normally distributed with a joint bivariate normal distribution $\hat{m}, \hat{w} \sim N_2 \left( m, w, \frac{1}{I(m, w)} \right)$ for $n \to \infty$, where $I(m, w)$ is the Fisher information matrix given by

$$I(m, w) = n \left[ \psi_1 \left( m - \frac{a}{2} \right) - \frac{2m + a}{2m^2} \frac{a}{2mw} - \frac{2m - a}{2w^2} \right]$$

$$\psi_1(r) = \frac{\partial}{\partial r} \psi(r)$$ is the tri-gamma function.

#### 3.1.1 Bias Correction

Let $logL$ be the log Likelihood function corresponding to a random sample of size $n$ from a distribution with $p$-dimensional parameter vector $\Theta = [\theta_1, \theta_2, ..., \theta_p]$. When the sample is
independent but not identically distributed, then the bias in estimating the parameters is given by [16]

\[
Bias(\hat{\theta}_s) = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} t_{ij} t_{jk} \left[ t_{ijk} + 0.5 t_{ijk} \right] + O(n^{-2}); \ s = 1,2, ..., p
\]  
(31)

Where,

\[
t_{ij} = E \left[ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log L \right]
\]  
(32)

\[
t_{ijk} = E \left[ \frac{\partial^3}{\partial \theta_i \partial \theta_j \partial \theta_k} \log L \right]
\]  
(33)

\[
t_{i,j,k} = E \left[ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log L \cdot \frac{\partial}{\partial \theta_k} \log L \right]
\]  
(34)

\[
t_{ij}^k = \frac{\partial}{\partial \theta_k} t_{ij}; \ i,j,k = 1,2, ..., p
\]  
(35)

t_{ij} is the (i, j)th element of the Fisher Information matrix (30) and t_{ij} is the (i, j)th element of the variance-covariance matrix T.

In case the sample is not independent, then (31) can be rewritten as [17],

\[
Bias(\hat{\theta}_s) = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} t_{ij}^k \left[ t_{ijk} - 0.5 t_{ijk} \right] + O(n^{-2}); \ s = 1,2, ..., p
\]  
(36)

Let, \( a_{ij}^{(k)} = t_{ij}^k - 0.5 t_{ijk} \)

And \( A = [A^{(1)} | A^{(2)} | ... | A^{(p)}] \)

Where, \( A^{(k)} = \{a_{ij}^{(k)}\}; \ i,j,k = 1,2, ..., p \)

Then (36) can be rewritten as

\[
Bias(\hat{\theta}) = T^{-1} A vec(T^{-1})
\]  
(37)

And the bias corrected MLE is given by

\[
\hat{\theta}^c = \hat{\theta} - Bias(\hat{\theta})
\]  
(38)

Since the MLE of \( w \) is an unbiased estimate, therefore the correction is applied only on the MLE of \( m \). After some simplification, the bias corrected MLE (CMLE) for shape parameter \( m \) is obtained as

\[
m^c = \hat{m} - \frac{2\hat{m} \left( \hat{m}^3 \psi_2 \left( \frac{\hat{m} - a}{2} \right) + \hat{m} + a \right)}{n \left( 2\hat{m} \psi_1 \left( \frac{\hat{m} - a}{2} \right) - 2\hat{m} - a \right)^2}
\]  
(39)

3.2 Method of Moments

Using (10) and (12) and assuming the weight parameter \( a = 1 \) (LBINK), we have

\[
\mu_2 = \left( \frac{m}{w} \right) \frac{1}{(m - \frac{a}{2})}
\]  
(40)

\[
\mu_4 = \left( \frac{m}{w} \right)^2 \frac{1}{(m - \frac{a}{2}) (m - \frac{a}{2})}
\]  
(41)

Solving (40) and (41) we get

\[
\hat{m} = \frac{\mu_4}{\mu_4 - \mu_2^2} + \frac{3}{2}
\]  
(42)

and

\[
\hat{w} = \frac{1}{\mu_2} + \frac{3\mu_4 - \mu_2^2}{2\mu_2^2 \mu_4}
\]  
(43)

Substituting the population moments \( \mu_2 \) and \( \mu_4 \) by corresponding sample moments \( m_2 \) and \( m_4 \), we can estimate the parameters of LBINK.

Similarly, for weight parameter \( a = 2 \) (ABINK) we have

\[
\hat{m} = \frac{\mu_4}{\mu_4 - \mu_2^2} + 2
\]  
(44)

\[
\hat{w} = \frac{1}{\mu_2} + \frac{\mu_4 - \mu_2^2}{2\mu_2^2 \mu_4}
\]  
(45)

And for weight parameter \( a = 3 \) we have

\[
\hat{m} = \frac{\mu_4}{\mu_4 - \mu_2^2} + \frac{5}{2}
\]  
(46)

\[
\hat{w} = \frac{1}{\mu_2} + \frac{5\mu_4 - \mu_2^2}{2\mu_2^2 \mu_4}
\]  
(47)

In general, for any integer value of the weight parameter \( a = r \) (r = 1,2,3, ...) we have

\[
\hat{m} = \frac{\mu_4}{\mu_4 - \mu_2^2} + \frac{r + 2}{2}
\]  
(48)

\[
\hat{w} = \frac{1}{\mu_2} + \frac{(r + 2) \mu_4 - \mu_2^2}{2\mu_2^2 \mu_4}
\]  
(49)

Substituting the sample moments in (48) and (49) we get

\[
\hat{m} = \frac{m_4}{m_4 - m_2^2} + \frac{r + 2}{2}
\]  
(50)

\[
\hat{w} = \frac{1}{m_2} + \frac{(r + 2) m_4 - m_2^2}{2m_2 m_4}
\]  
(51)

Where, \( m_2 = \frac{1}{n} \sum x^2 \) and \( m_4 = \frac{1}{n} \sum x^4 \).

4 Comparison of Estimators

In this section the performance of maximum likelihood estimator (MLE) and moment estimator (MME) are compared based on randomly generated
1000 samples of size \( n = 20, 50, 100 \) and 1000 for different values of the parameters.

The performance of the estimators is studied by computing their mean relative error (MRE) and mean square error (MSE) as follows

\[
MRE = \frac{1}{N} \sum_{r=1}^{N} \frac{\hat{\theta}_{r,j}}{\theta}; \quad MSE = \frac{1}{N} \sum_{r=1}^{N} (\hat{\theta}_{r,j} - \theta)^2
\]

Where \( r = 1,2 \) and \( N \) is the number of iterations.

Figure 5 to 16 presents the comparison of MLE and MME estimator for particular values of the parameters. Figure 17 and 18 presents the MRE and MSE for both MLE and MLE. MRE value close to 1 indicates a good estimation whereas in case of MSE, a value close to zero is desirable.

Examining Figure 5 to 16 it can be inferred that the MLE performs consistently better than MME. For weight \( (\alpha) = 1 \) (LBINK) MME is highly bias compared to weight \( (\alpha) = 2 \) and 3. Further, MLE for the spread parameter \( w \) approaches normality much faster compare to MLE of shape parameter \( m \).
Figure 10: comparison of MLE and MME for sample size 50 and weight =3

Figure 11: Comparison of MLE and MME for sample size 100 and weight = 1

Figure 12: Comparison of MLE and MME for sample size 100 and weight = 2

Figure 13: Comparison of MLE and MME for sample size 100 and weight = 3

Figure 14: Comparison of MLE and MME for sample size 1000 and weight = 1

Figure 15: Comparison of MLE and MME for sample size 1000 and weight = 2
From Figure 17 and 18, we can infer that MLE of $w$ is an unbiased estimator. Whereas, MLE of $m$ is positively biased and approaches the true value for higher sample sizes. In all different combinations of parameter values, MLE is found out to be performing better than MME. However, except for LBINK, the MME of $w$ is also found to be unbiased.

5 Applications

In this section the proposed model is fitted to a real life dataset pertaining to mortality of patients with severe head injury due to road mishap taken from a Private Hospital of Guwahati, India. The patients are followed for a period of one year form the date of admission and a total of 180 cases are considered in this study. On the spot mortality as well as mortality after a period of one year are ignored.

The outcomes are compared with Inverse Nakagami, Nakagami, Inverse Gamma, Inverse Exponential, Inverse Chi Square, Rayleigh and Inverse Weibull distribution. To compare the performance of WINK distribution with these distributions, different discrimination criterions are constructed under the log likelihood function. The discrimination criterions are:

Akaike Information Criterion

$AIC = -2l(\hat{\theta}, x) + 2k$

Bayesian Information Criterion

$BIC = -2l(\hat{\theta}, x) + \log(n)k$

Corrected Akaike Information Criterion

$AICC = AIC + \frac{2k(k+1)}{n-k-1}$

Hannan-Quinn Information Criterion

$HQIC = -2l(\hat{\theta}, x) + 2k\log(\log(n))$

Consistent Akaike Information Criterion

$CAIC = AIC + k\log(n) - k$

Where,

$n =$ sample size.

$k =$ number of parameters to be estimated.

$l(\hat{\theta}, x) =$ maximized log likelihood function.

Table 1 gives the survival time of 180 patients with severe head injury. Table 2 gives the MLE of the parameters of WINK distribution for weight parameter $a=1, 2$ and $3$, their standard deviation and 95% confidence intervals. Table 3 presents the results of AIC, BIC, AICC, HQIC and CAIC criteria. Table 4 presents the results of Goodness of fit test carried out for different probability distributions in describing the observed dataset.
Table 2: MLE, Standard deviation and 95% Confidence Interval for \( m \) and \( w \) for \( a = 1, 2, 3 \).

<table>
<thead>
<tr>
<th>( a )</th>
<th>( \theta )</th>
<th>MLE</th>
<th>SD</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( m )</td>
<td>0.9699</td>
<td>0.0397</td>
<td>0.8921 1.0476</td>
</tr>
<tr>
<td></td>
<td>( w )</td>
<td>0.8651</td>
<td>0.1028</td>
<td>0.6633 1.0660</td>
</tr>
<tr>
<td>2</td>
<td>( m )</td>
<td>1.4685</td>
<td>0.0395</td>
<td>1.3907 1.5462</td>
</tr>
<tr>
<td></td>
<td>( w )</td>
<td>1.3158</td>
<td>0.1644</td>
<td>0.9937 1.6380</td>
</tr>
<tr>
<td>3</td>
<td>( m )</td>
<td>1.9676</td>
<td>0.0397</td>
<td>1.8899 2.0453</td>
</tr>
<tr>
<td></td>
<td>( w )</td>
<td>1.7667</td>
<td>0.2263</td>
<td>1.3223 2.2111</td>
</tr>
</tbody>
</table>

Table 3: Results of different discrimination criteria for different Probability distribution considering the data set of 180 patients.

<table>
<thead>
<tr>
<th>Test</th>
<th>WINK</th>
<th>Inverse</th>
<th>Nakagami</th>
<th>Inverse</th>
<th>Gamma</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIC</td>
<td>927.44</td>
<td>954.15</td>
<td>1270.37</td>
<td>952.78</td>
<td></td>
</tr>
<tr>
<td>BIC</td>
<td>933.83</td>
<td>960.53</td>
<td>1276.76</td>
<td>959.17</td>
<td></td>
</tr>
<tr>
<td>AICC</td>
<td>933.83</td>
<td>960.53</td>
<td>1276.76</td>
<td>959.17</td>
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<td>955.37</td>
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<tr>
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<td>1278.76</td>
<td>955.94</td>
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</tr>
</tbody>
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<th>Chi Square</th>
<th>Rayleigh</th>
<th>Weibull</th>
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<td>1047.44</td>
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</table>

Table 4: Result of Goodness of fit statistics for different probability distribution considering the data set of 180 patients.

<table>
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<tr>
<th>Statistics</th>
<th>WINK</th>
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<th>Nakagami</th>
<th>Inverse</th>
<th>Gamma</th>
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<td>0.2513</td>
<td>0.3477</td>
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<tr>
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<tr>
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<td>9.5435</td>
<td>37.3121</td>
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</table>

<table>
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<th>Statistics</th>
<th>Inverse</th>
<th>Exponential</th>
<th>Chi Square</th>
<th>Rayleigh</th>
<th>Weibull</th>
</tr>
</thead>
<tbody>
<tr>
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<td>16.0385</td>
<td>83.6791</td>
<td>7.5007</td>
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</tbody>
</table>

From Table 3 we can clearly state that our proposed distribution under different discrimination methods is performing better than other well-known distributions. The results of various goodness-of-fit test in Table 4 also confirm our claim. Therefore, the practical importance of WINK distribution is observed in situations where we experience high amount of recurrent observations within a small interval with a very few observations lying away from the centre.

6 Conclusion

In this paper, the Weighted Inverse Nakagami (WINK) distribution is proposed as a generalization of Inverse Nakagami distribution. The distribution can be used to explore data with high amount of recurrent observations within a small interval of duration with some observations far away from the mode of the distribution. The various statistical properties of the proposed distribution such as moments, mode, entropy and reliability properties are discussed.

The parameters of the distribution are estimated using both MLE and MME procedure and it was found from simulation study that under different combination of parameter values, the maximum likelihood estimators performed better than moment estimators. A bias correction is applied to the MLE of shape parameter \( m \) to reduce the positive biasness of the estimator. Finally, the distribution is fitted to a dataset with high failure rate.

The proposed distribution can be extended and applied in a large number of practical situations. System with high initial failure is very common in practice.
References:


