On Errors in Euler's Formula for Solving ODEs

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Abstract: - Euler introduced the formula $y = \exp(\omega x)$ for solving ODEs of the form $Ay^{''} + By^{'} + Cy = 0$. It is now a procedure that can be found at the basis of numerous mathematical theories, and has countless applications in several fields. In this contribution, we demonstrate that this formula is invalid as a tool for solving such equations. We determine the correct one through quadrature, and establish it to be $y = a\{\exp(\omega[x+\phi]) - \exp(-\omega[x+\phi])\}/(2\,\omega)$, or simply $y = a\sin(i\omega[x+\phi])/(i\,\omega)$.

Key-Words: - Ordinary Differential Equations, Partial Differential Equations, Linear Algebra, Complex Analysis

1 Introduction

Leonhard Euler (1707-1783), was undoubtedly a titan of his time. He made contributions to a wide variety of fields, including Mathematics, Physics, Astronomy, Geography, Logic and Engineering.

Mistakes are beginning to emerge in this great scholar's works. The latest flaw was pointed out by Tarek Elgindi in his study of Euler's fluid equations, in two papers [1] and [2], published online.

We address his solutions for the equation

$$Ay^{''} + By^{'} + Cy = 0, \tag{1}$$

where A, B and C are constants, and $y = y(x)$. It can be written in the simple form

$$y^{''} + \lambda\, y = 0. \tag{2}$$

Euler proposed the formula

$$y = e^{\omega x}, \tag{3}$$

for solving it, and got

$$y = C_1 e^{\omega x} + C_2 e^{-\omega x}, \tag{4}$$

for $\lambda = -\omega^2$ with C_1 and C_2 being constants.

For the case for $\lambda = \omega^2$ he got

$$y = C_3\cos(\omega x) + C_4\sin(\omega x), \tag{5}$$

with the constants C_1 and C_2. These solutions can be found in many books on differential equations, including the latest and earliest: [3], [4], [5], [6] and [7], and in application texts [8], [9] and [10].

The case $\lambda = 0$ is considered trivial, because the solution thereof can easily be arrived at, and is

$$y = C_5 x + C_6, \tag{6}$$

but cannot be arrived at using Euler's formula. The parameters C_5 and C_6 too are constants.

We will demonstrate in Section 2 that the other two case solutions, (4) and (5), have errors. The solution in (6) does not have any, because it was obtained through quadrature.

In Section 3 we determine the quadrature solution for (2). That is, we use rules of integral calculus, and not a guessed formula.

Section 4 is on a numerical experiment, wherein we test the validity of our solution.

2 The Error

For the three results in (4), (5) and (6) to be solutions of (2), then they have to agree at the case $\lambda = 0$. This requires that (4) and (5) each has to be paired and compared with (6), and evaluated at $\lambda = 0$.

2.1 Comparing (4) with (6)

Consider

$$C_1 e^{\omega x} + C_2 e^{-\omega x} = C_5 x + C_6, \qquad (7)$$

for $\lambda = 0$.

Substituting the condition $\lambda = 0$, that is, $\omega = 0$, yields

$$C_1 + C_2 = C_5 x + C_6, \qquad (8)$$

which is a contradiction.

2.2 Comparing (5) with (6)

Consider

$$C_3 \cos(\omega x) + C_4 \sin(\omega x) = C_5 x + C_6, \qquad (9)$$

for $\lambda = 0$. Substituting the condition gives

$$C_3 = C_5 x + C_6. \qquad (10)$$

This too is a contradiction.

3 The Quadrature Solution

To solve (2) through quadrature, we begin by expressing it in the form

$$y' y'' = -\lambda\, y y'. \qquad (11)$$

We now introduce Leibzniz's notation, in order to gain more control on the equation. That is,

$$\frac{\mathrm{d}}{\mathrm{d}\,x} \frac{(y')^2}{2} = -\lambda \frac{\mathrm{d}}{\mathrm{d}\,x} \frac{y^2}{2}. \qquad (12)$$

Next we introduce the integral signs:

$$\int \frac{\mathrm{d}}{\mathrm{d}\,x} \frac{(y')^2}{2} d\,x = -\lambda \int \frac{\mathrm{d}}{\mathrm{d}\,x} \frac{y^2}{2} d\,x, \qquad (13)$$

so that

$$\frac{(y')^2}{2} = -\lambda \frac{y^2}{2} + E, \qquad (14)$$

where E is a constant of integration; a result of the first reduction of order. For the second order reduction, we first separate the variables, in the process introducing integral signs. That is,

$$\int \frac{d\,y}{\sqrt{-\lambda\, y^2 + 2\,\mathrm{E}}} = \int d\,x. \qquad (15)$$

Integration of this result leads to

$$\frac{1}{\sqrt{\lambda}} \arcsin\left(\frac{y}{\sqrt{2\,\mathrm{E}/\lambda}}\right) = x + \phi, \qquad (16)$$

where ϕ is a second constant of integration. This means the quadrature solution for (2) is

$$y = \frac{2\,\mathrm{E}}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\,[x + \phi]). \qquad (17)$$

4 Numerical Experiments

To test the validity of our quadrature solution, we deduce its special case forms. That is, the cases $\lambda = -\omega^2$, $\lambda = \omega^2$ and $\lambda = 0$. These special cases solutions are then paired and compared for consistency.

4.1 The case $\lambda = -\omega^2$

Substituting $\lambda = -\,\omega^2$ in (17) gives

$$y = \frac{2\,\mathrm{E}}{i\omega} \sin(i\omega\,[x + \phi]). \qquad (18)$$

Since

$$\sin(i\,\theta) = \frac{e^{-\theta} - e^{\theta}}{2i}, \qquad (19)$$

we have

$$\sin(i\omega[x + \phi]) = \frac{e^{-\omega\,[x+\phi]} - e^{\omega\,[x+\phi]}}{2i}, \qquad (20)$$

so that

$$y = \mathrm{E}\frac{e^{\omega\,[x+\phi]} - e^{-\omega\,[x+\phi]}}{\omega}. \qquad (21)$$

4.1.1 Compairing (21) with (6)

Consider

$$\mathrm{E}\frac{e^{\omega\,[x+\phi]} - e^{-\omega\,[x+\phi]}}{\omega} = C_5 x + C_6, \qquad (9)$$

for $\lambda = 0$. But the latter implies $\omega = 0$. Evaluating the expression on the left of (22) at $\omega = 0$ requires limits. That is,

$$\lim_{\omega \to 0} \mathrm{E}\frac{e^{\omega\,[x+\phi]} - e^{-\omega\,[x+\phi]}}{\omega} = C_5 x + C_6, \qquad (23)$$

which yields

$$2E(x+\phi) = C_5 x + C_6. \qquad (24)$$

This is not a contradiction. It is a consolidation of constants, meaning $C_5 = 2E$ and $C_6 = 2E\phi$.

4.2 The case $\lambda = \omega^2$

Substituting $\lambda = \omega^2$ in (17) gives

$$y = 2E\,\frac{\sin(\omega\,[x+\phi])}{\omega}. \qquad (25)$$

This can be expanded, so that

$$y = \left(2E\,\frac{\sin(\omega\phi)}{\omega}\right)\cos(\omega x) + [2E\cos(\omega\phi)]\sin(\omega x), \qquad (26)$$

which is similar to Euler's result in (5). Similar, but not the same.

4.2.1 Compairing (25) with (6)

Consider

$$2E\,\frac{\sin(\omega\,[x+\phi])}{\omega} = C_5 x + C_6, \qquad (27)$$

for $\lambda = 0$. Again, evaluating the expression on the left of (27) at $\omega = 0$ requires limits. That is,

$$\lim_{\omega\to 0} 2E\,\frac{\sin(\omega\,[x+\phi])}{\omega} = C_5 x + C_6, \qquad (28)$$

so that

$$2E(x+\phi) = C_5 x + C_6. \qquad (29)$$

This is a duplication of the results observed in Sub-subsection 4.1.1.

4 Conclusion

We have demonstrated that Euler's solutions for ODEs of the type I (1) have errors, and provided an alternative solution, presented in (17).

These errors are very hard to detect in most instances of applications. An area wherein it is easy to detect is Optics. See [11] and [12].

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