Stability Analysis of Stochastic Model for Stock Market Prices

IYAI DAVIES[1], AMADI INNOCENT UCHENNA[2] AND NDU ROSELINE .I[3]

[1] & [3] Department of Mathematics, Rivers State University, Port Harcourt, Rivers State , Nigeria.
[2]Department of Mathematics/Statistics, Port Harcourt Polytechnic, Rivers State, Nigeria.

davies.iyai@ust.edu.ng, amadiuchennainno@gmail.com, favourose@yahoo.com

Abstract: - In this paper, the unstable natures of stock market forces were analysed using new differential equation model that could impact the expected returns of investors in stock exchange market with a stochastic volatility in the equation. The new results were obtained by developing stochastic vector differential equation, exploring the properties of the fundamental matrix solution (a function of the drift) of this equation and by placing continuity condition on the stochastic part (a function of the volatility). Example is given to illustrate the effectiveness of the model and simulation results presented graphically using MATLAB.

Key-Words: - Stability, fundamental matrix, stochastic volatilities, Black Scholes, stock exchange market

1 Introduction

As seen in [1] and [2] , a differential equation is an important tool for harnessing different components into a single system and analysing the inter-relationships that exists between these components, these component might remain independent of each other. Differential equations are one of the most frequently used tools for mathematical modelling in engineering and sciences. Generally, dynamics of changing process can be modelled into ordinary differential equation or partial differential equation, depending on the nature of the problems.

If some randomness is allowed into differential equations for example; environmental effects, then a more realistic mathematical model of the problem or situation known as stochastic differential equation can be obtained. The random quantities or variables in the equation can be parameterized into discrete or continuous variables; the parameterized collection of random variables is known as stochastic processes. The concept of stochastic process was first introduced in 1902 by Gibbs. J. W who studied the integral of Hamilton-Jacobi differential equation for conversation system in statistical mechanics with random initial state see [3]. Stochastic processes can serve as mathematical models of systems and phenomena for predicting real-life behaviour of any random dynamic problem in physical, biological and social sciences [4] where the dynamical behaviour of system processes is difficult to be describe by ordinary mathematical model, because, their analysis is based on stochastic point of view instead of deterministic.

The unstable property and other considerable factors such as liquidity on stock return, expected return, cumulative return, abnormal return, value and size effects, stock volatility, trading frequencies, price imbalance and other market conditions has led fund managers to raise various investment styles to catch the attention of investors in the stock market [5]. These factors and other market anomalies attracted the attention of many independent researches for example [6], [7], [8], and led to the development of Black Scholes model [9] in the 1970s. The development of the Black Scholes model changed the way financial market was being analysed through earlier observations and also impacted the world of pricing derivatives using stocks as the underlying asset see [10], [11], and [12]. Though the Black Scholes is widely used in market predictions, its main short coming is its assumption of constant volatility; several evidences have shown that this assumption is not suitable for real data. There are however several models like the jump diffusion [13], Markov switching that does not have close form solution [14], Heston processes [15] and so on whose analysis of the financial market data have shown that volatility is a stochastic process.

The aim of this paper is first to establish a dynamic stochastic model in terms of vector valued differential equations, analyse the unstable nature of the stock price changes in the stock market by exploring the properties of the fundamental matrix solution as well as using practical stock exchange market data to illustrate the effectiveness of the theoretical results.
The rest of the paper is organized as follows; Section 2 contains the mathematical notations, preliminaries, definitions and model formulation. In Section 3, stability theorems for the system which is the main result of this paper is stated and proved. Finally, Section 4 contains numerical examples of the theoretical results prior to the discussion and conclusion in Section 5.

2 Problem Formulation

Here, we consider the stochastic volatility model with volatility of returns following a stochastic process of the form

$$dS_t = \mu_t S_t dt + \sigma_t S_t dw_t \quad (1)$$

where S_t the price process, μ_t is the drift coefficient of the stock price, σ_t is a positive volatility process and w_t is a Weiner process (see [16] and references therein), and develop a vector valued stochastic volatility model, where the securities invested on some bond processes with observed volatility for real existing price processes have some reversion drifts with the leverage effect, stock returns and implied volatility have negative correlation and are given by the equations,

$$\left.\begin{array}{c} dS_1(t) = \tau\mu_1 S_1(t)dt + \mu_2 S_2(t)dt + \cdots + \\ \mu_n S_n(t)dt + S_1(t)\big(\sigma_{11}dw_1(t) + \sigma_{12}dw_2(t)\big) \\ dS_2(t) = \mu_1 S_1(t)dt + \tau\mu_2 S_2(t)dt + \cdots + \\ \mu_n S_n(t)dt + S_2(t)\big(\sigma_{21}dw_1(t) + \sigma_{22}dw_2(t)\big) \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ dS_n(t) = \mu_1 S_1(t)dt + \mu_2 S_2(t)dt + \cdots + \\ \tau\mu_n S_n(t)dt + S_n(t)\big(\sigma_{n1}dw_1(t) + \sigma_{n2}dw_2(t)\big) \end{array}\right\} (2)$$

Here, it is assumed that the processes $S_1, \cdots, S_n$ are correlated and $\sigma_{ji} = \sigma_{(j-(j-1))(i+1)} \neq 0$; $j = 2, \cdots, n; i = 1, \cdots, n$.

To develop a vector equation we write the equations (2) in matrix form by letting $x = (S_1, \cdots, S_n)^T$, where T denotes the matrix transpose

$$A(t) = \begin{pmatrix} \tau\mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \dots & \vdots \\ \mu_1 & \mu_2 & \dots & \tau\mu_n \end{pmatrix},$$

$$B_i(\cdot) = \begin{pmatrix} \sigma_{1i} & 0 & \cdots & 0 \\ 0 & \sigma_{2i} & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_{ni} \end{pmatrix}$$

Let R be the real line, then a generalized vector valued SDE for equation (2) can be written in the form

$$dx(t) = A(t)x(t)dt + \sum_{i=1}^{n} B_i\left(t, x(t)\right)dw_i(t), \quad (3)$$

$$x(0) = x_0$$

2.1 Preliminaries

Where $A(t) \in R^{n\times n}$, $B_i(\cdot) \in R^{n\times n}$ are $n \times n$ matrices, $x \in R^n$ is the price volatility process, $w_i(t) \in R^n$ is a Weiner process, $dw_i(t)$ denotes the differential form of the Weiner process with $dw_i(t) = \xi_t dt$, and ξ_t is the correlation coefficient (white noise process).

To compute the future stock price from the initial value for a time period t of the equation (3), we introduce the fundamental matrix $X(t) \in R^{\mathrm{n}\times n}$ (see [17]) a function of the expected rate of correlated stock price returns processes in the trading periods in order to hedge any risk associated with the implied volatility. It is assumed that $X(t)$ is a continuously differentiable function with

$$X(t) = \begin{cases} 0, t < 0 \\ I, t = 0 \ (I = identity) \end{cases}$$

Then the stock strike price $x(t)$ of (3) with time t can be computed as an integral equation by the variation of parameter method ([17]) as

$$x(t) = X(t)x_0 + \int_0^t X(s)X^{-1}(s)\sum_{i=1}^{n} B_i\big(s, x(s)\big)dw_i(s) \quad (4)$$

The stochastic integral in equation (4) can be calculated using the Ito's integral for stochastic processes with Brownian motion (see [18]) by assuming that B_i can only change at discrete time points $t_i (i = 1, \cdots, N-1)$, where $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N < T$, $t \in [0, T]$. However, the method to be applied will not require the use of the Ito's integral for stochastic process evaluation since the introduced fundamental matrix is a continuous function of the stock price return, we only need to assume that the Weiner process w_i are correlated with correlation coefficient ξ_t.

2.1.1 Definitions

We now give some definitions upon which the study hinges.

Definition 1: Asymptotic Stability
The initial stock price $x_0(t)$ of system (3) is stable if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that every future stock price $x(t)$ of (3) with
$\|x(0) - x_0(0)\| < \delta(\varepsilon)$ exist and satisfies
$\|x(t) - x_0(t)\| < \varepsilon$ on R. The initial stock price $x_0(t)$ of the system (3) is called asymptotically stable if it is stable and there exist a constant $\eta > 0$ such that $x(t) - x_0(t) \to 0$ as $t \to \infty$ whenever $\|x(t) - x_0(t)\| \le \eta$ see [19].

3 Problem Solution

We now give the theorem and proof as the main result of this paper.

Theorem 1: Let $X(t)$ be the fundamental matrix of equation (3). Assume that there are constants $M > 0, K > 0$ such that

$$\int_0^t \| X(t)X(s)^{-1} \| \, ds \le K, \;\; t \ge 0 \qquad (5)$$

with $\| X(t) \| \le Me^{-K^{-1}t}$, $t \ge 0$. Furthermore, if

$$\| \sum_{i=1}^{n} B_i\left(t, x(t)\right)\xi_t \| \le \rho \| x \|, \;\; \forall t \ge 0 \quad (6)$$

with ρ satisfying $0 \le \rho < K^{-1}$. Then, the zero stock price of (3) is asymptotically stable.

Proof: Let $X(t)$ be the fundamental matrix of (3) with $X(0) = x_0$. Then since $X(t)X^{-1}(s) = Y(t)Y^{-1}(s)$ for any other fundamental matrix $Y(t)$ of (3), then there exists a constant $M > 0$ such that $\| X(t) \| \le Me^{-tk^{-1}}$, where $k > 0$ and $t \ge 0$. Thus $x(t) \to 0$ as $t \to \infty$.

If $x(t)$ is a local solution of (3) defined to the right of $t = 0$, then $x(t)$ satisfies the system

$$dx(t) = A(t)x(t)dt + \sum_{i=1}^{n} B_i\left(t, x(t)\right)dw_i(t).$$

By the variation of constants formula for this system we have that

$$x(t) = X(t)x_0 + \int_0^t X(t)X^{-1}(s)\sum_{i=1}^{n} B_i\big(s, x(s)\big)\xi_t ds \quad (7)$$

Letting $L > 0$ be such that $\|X(t)\| \le L$ for $t \ge 0$, we obtain

$$\| x(t) \| \le L \| x_0 \| + \rho K \max_{0 \le s \le t}\|x(s)\|$$

which implies

$$\max_{0 \le s \le t}\|x(s)\| \le (1 - \rho K)^{-1} L \| x_0 \|$$

It follows that $\|x(s)\| \le (1 - \rho K)^{-1} L \| x_0 \|$ as long as $x(t)$ is defined. This implies that $x(t)$ is continuable to $+\infty$ (see [19]) and that the zero or initial stock price is stable. We now show that $x(t) \to 0$ as $t \to \infty$. To this end, let $c = \limsup_{t\to\infty}\|x(t)\|$ and pick d such that $\rho K < d < 1$. If $c > 0$, then, since $d^{-1}c > c$, there exists $t_0 \ge 0$ such that $\|x(t)\| \ge d^{-1}c$ for every $t \ge t_0$. Thus, equation (7) implies

$$\begin{aligned}
\|x(t)\| &= \left\| X(t)x_0 + X(t)\int_0^t X^{-1}(s)\sum_{i=1}^{n} B_i\big(s, x(s)\big)\xi_t ds \right\| \\
&= \left\| X(t)x_0 + X(t)\int_0^t X^{-1}(s)\sum_{i=1}^{n} B_i\big(s, x(s)\big)\xi_t ds \right. \\
&\quad \left. + X(t)\int_0^{t_0} X^{-1}(s)\sum_{i=1}^{n} B_i\big(s, x(s)\big)\xi_t ds \right\| \\
&\le \|X(t)\|\|x_0\| \\
&\quad + \left\| X(t)\int_0^{t_0} X^{-1}(s)\sum_{i=1}^{n} B_i\big(s, x(s)\big)\xi_t \right\| ds \\
&\quad + \int_{t_0}^{t} \|X(t)X^{-1}(s)\| \left\| \sum_{i=1}^{n} B_i\big(s, x(s)\big)\xi_t \right\| ds \\
&\le \|X(t)\|\|x_0\| \\
&\quad + \|X(t)\|\int_0^{t_0} \left\| X^{-1}(s)\sum_{i=1}^{n} B_i\big(s, x(s)\big)\xi_t \right\| ds \\
&\quad + \rho K d^{-1} c
\end{aligned}$$

Taking the limsup above as $t \to \infty$, we obtain $c \le \rho K d^{-1} c$, that is, a contradiction. Thus, $c = 0$ and the proof is complete.

4 Example

Here, we give a two dimensional example of the n-dimensional model developed in Section 3. Consider the two dimensional volatility model given by

$$\left.\begin{aligned} dS_1(t) &= \frac{-1}{0.25}\mu_1 S_1(t)dt + \mu_2 S_2(t)dt + \\ &\quad S_1(t)\big(\sigma_{11}dw_1(t) + \sigma_{12}dw_2(t)\big) \\ dS_2(t) &= \mu_1 S_1(t)dt - \frac{1}{0.25}\mu_2 S_2(t)dt + \\ &\quad S_2(t)\big(\sigma_{21}dw_1(t) + \sigma_{22}dw_2(t)\big) \end{aligned}\right\} \quad (8)$$

This can be written in matrix form with

$$x = (S_1,\ S_2)^T,\ A(t) = \begin{pmatrix} \frac{-\mu_1}{0.25} & \mu_2 \\ \mu_1 & \frac{-\mu_2}{0.25} \end{pmatrix},$$

$$B_i(\cdot) = \begin{pmatrix} \sigma_{1i} & 0 \\ 0 & \sigma_{2i} \end{pmatrix},\ i = 1, 2$$

where the processes $S_1(t),\ S_2(t)$ are correlated with $\sigma_{21} = \sigma_{12} \neq 0$ and

$$B_1(t) = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{21} \end{pmatrix},\ B_2(t) = \begin{pmatrix} \sigma_{12} & 0 \\ 0 & \sigma_{22} \end{pmatrix}.$$

So that,

$$dx(t) = A(t)x(t)dt + \sum_{i=1}^{2} B_i\big(t, x(t)\big)dw_i(t) \quad (9)$$

$$x(0) = x_0.$$

4.1. Data Analysis and Result

To illustrate the unstable nature of the above model, we use eighteen years (1997-2014) closing stock price data from the Nigerian exchange market (NSE) extracted from [20]. The data on Table 1 shows a ten year stock price data traded on the floor of the NSE, it shows the initial stock prices, drift, volatility and trading days for 1998-2005, 2009, 2010, and 2013 which is 252, that of 2006, 2007 and 2012 is 251, while 1997, 2008 is 253 and 2014 is 96. This values where calculated using stock returns over a period of one year each by allocating each year its own unit period as given by the number of days for the stock year ([20])

Table 1: Values of initial stock prices, drift, volatility and trading days

Year	Initial stock price (S_0)	Drift μ	Volatility σ	Trading days n
1997	17.50	-0.4908	0.593452	253
1998	66.25	1.7862	0.906322	252
1999	24.80	1.8429	10.615442	252
2000	475.00	1.2322	1.162434	252
2001	28.19	-2.1212	1.147489	252
2002	18.63	0.1112	0.982237	252
2003	17.60	0.1483	0.747571	252
2004	45.40	1.0283	0.412699	252
2005	36.18	0.1204	0.784289	252
2006	40.91	0.0684	0.290624	251
2007	25.61	-0.3906	0.406075	251
2008	23.72	-0.0283	0.369683	253
2009	12.85	-0.3803	0.75789	252
2010	17.10	0.3567	0.42141	252
2011	16.75	0.0156	0.295142	252
2012	16.29	0.0407	0.396402	251
2013	20.08	0.2211	0.20114	252
2014	39.59	0.7039	0.44278	96

To obtain the values of the volatilities in the $B_i, i = 1, 2$ matrix in equation (9), we subdivide the volatility σ and trading days n columns in Table 1 into three groups as shown in Table 2, and compute the mean of each group to the get the $\sigma_{ji}, j = i = 1, 2$ as follows

Table 2: The mean values of volatility

Year	Volatility $\sigma 1$	Trading days $n1$	$\sigma 1 n1$
1997	0.593452	253	150.143356
1998	0.906322	252	228.393144
1999	10.615442	252	2675.091384
2000	1.162434	252	292.933366
2001	1.147489	252	289.167228
2002	0.982237	252	247.523724
Total		1513	3883.252204
Year	Volatility $\sigma 2$	Trading days $n2$	$\sigma 2 n2$
2003	0.747571	252	188.387892
2004	0.412699	252	104.000148
2005	0.784289	252	197.640828
2006	0.290624	251	72.946624
2007	0.406075	251	101.924825
2008	0.369683	253	93.529788
		1511	758.430105
Year	Volatility $\sigma 3$	Trading days $n3$	$\sigma 3 n3$
2009	0.75789	252	190.98828
2010	0.42141	252	106.19532
2011	0.295142	252	74.375784
2012	0.396402	251	99.496902
2013	0.20114	252	50.68728
2014	0.44278	96	42.50688
Total		1355	564.250446

$$\sigma_{11} = \frac{\sum ó1n1}{\sum n1} = \frac{3883.252204}{1513} = 2.5666,$$

$$\sigma_{21} = \sigma_{12} = \frac{\sum \sigma 2n2}{\sum n2} = \frac{758.430105}{1511} = 0.5019,$$

$$\sigma_{22} = \frac{\sum \sigma 3n3}{\sum n3} = \frac{564.250446}{1355} = 0.4164,$$

$$B_1 = \begin{pmatrix} 2.5666 & 0 \\ 0 & 0.5019 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0.5019 & 0 \\ 0 & 0.4164 \end{pmatrix},$$

$$\sum B_i = B_1 + B_2 = \begin{pmatrix} 3.0685 & 0 \\ 0 & 0.9183 \end{pmatrix} \qquad (10)$$

To obtain the values of drift in the A matrix of equation (9), we subdivide the drift μ and trading days n columns in Table 1 into two groups as shown in Table 3, and compute the mean of each group to the get the $\mu_i,\ i = 1, 2$ values as follows

Let

$$\mu_1 = \frac{\sum f\mu_i}{\sum f_i} = \frac{921.1992}{2269} = 0.4060,$$

$$\mu_2 = \frac{\sum f\mu_j}{\sum f_j} = \frac{38.4548}{2110} = 0.0182,$$

Having computed the values of the drift, we now have the matrix A as

$$A(t) = \begin{pmatrix} -1.6240 & 0.0182 \\ 0.4060 & -0.0728 \end{pmatrix}$$

The eigenvalues of matrix A are $\lambda = -0.0681, -1.6288$ and the fundamental matrix solution of equation (9) can be found following the methods in [17] as

$$X(t) = \begin{pmatrix} 0.0182e^{-0.0681t} & 0.0182e^{-1.6288\,t} \\ 1.5559e^{-0.0681t} & -0.0048e^{-1.6288\,t} \end{pmatrix}$$

To show the stability of equation (9), we use the conditions (i) and (ii) of Theorem 1 as follows. By condition (i) of Theorem 1, we get

$$\int_0^t \left\| \begin{pmatrix} 0.0182e^{-14.6843t} & 0.0182e^{-0.6140\,t} \\ 1.5559e^{-14.6843t} & -0.0048e^{-0.6140\,t} \end{pmatrix} \times \begin{pmatrix} 0.0115e^{-14.6843s} & 0.0437e^{-14.6843\,s} \\ 89.2267e^{-0.6140s} & -1.0438e^{-0.6140\,s} \end{pmatrix} \right\| ds,$$

Table 3: Mean values of drift

Year	Drift $\mu 1$	Days $n1$	$\mu 1 n1$
1997	-0.4908	253	-124.1724
1998	1.7862	252	450.1224
1999	1.8429	252	464.4108
2000	1.2322	252	310.5144
2001	-2.1212	252	-534.5424
2002	0.1112	252	28.0224
2003	0.1483	252	37.3716
2004	1.0283	252	259.1316
2005	0.1204	252	30.3408
Total		2269	921.1992

Year	Drift $\mu 2$	Days $n2$	$\mu 2 n2$
2006	0.0684	251	17.1684
2007	-0.3906	251	-98.0406
2008	-0.0283	253	-7.1599
2009	-0.3803	252	-95.8356
2010	0.3567	252	89.884
2011	0.0156	252	3.9312
2012	0.0407	251	10.2157
2013	0.2211	252	55.7172
2014	0.7039	96	62.5744
Total		2110	38.4548

It follows by the definition of the Euclidean norm, that

$$\int_0^t \| X(t)X(s)^{-1} \| \, ds = 1.6768 \leq K$$

To obtain condition (ii) of Theorem 1, we know from the calculations of the B_i matrices in equation (10) that

$$\left\| \sum B_i \left(t, x(t)\right) \xi_t \right\| = 3.2030 \leq \rho$$

It is now clear that not all the conditions on Theorem 1 are satisfied; that is $0 \leq \rho \nless k^{-1}$ i.e. $0 \leq 3.2030 \nless 1.6768$ and the system (8) is not asymptotically stable.

5 Discussion

5.1. Effect of Parameters on Return Distribution

There are statistical evidence of stock market data and many economic arguments of stock market which suggests that, the volatility of the stock is a time dependent quantity, which exhibit various random features and that stock return is not normally distributed; they have higher peaks and fatter tails than a normal distribution. This departure from normality is the main shortcoming of the Black-Scholes model. In this section, we analyse the effect of coefficient of correlation (ξ_t and σ on the stability of system (3) considered

5.2. Effect of ξ_t on Return Distribution

The coefficient of correlation ξ_t denotes the sources of randomness for the underlying Weiner process and the volatility. It captures the leverage effect, affecting the size of the tails; the skewness of the return distribution. That is, if $\xi_t < 0$ (ρ will be less than zero), then volatility increases and asset price return decreases, thus, leading to the spread of left tail and squeeze in right tail of the distribution, thus, creating a fat left-tailed distribution. If $\xi_t > 0$, volatility increases with increase in asset price return. This causes the right tail to spread with a squeeze in the left tail of the distribution, thus, creating a fat-tailed distribution. If $\xi_t = 0$, the skewness is close to zero , these effects of ξ_t on the skewness of the distribution are illustrated in

Figures 1-3 using varying values of ρ.

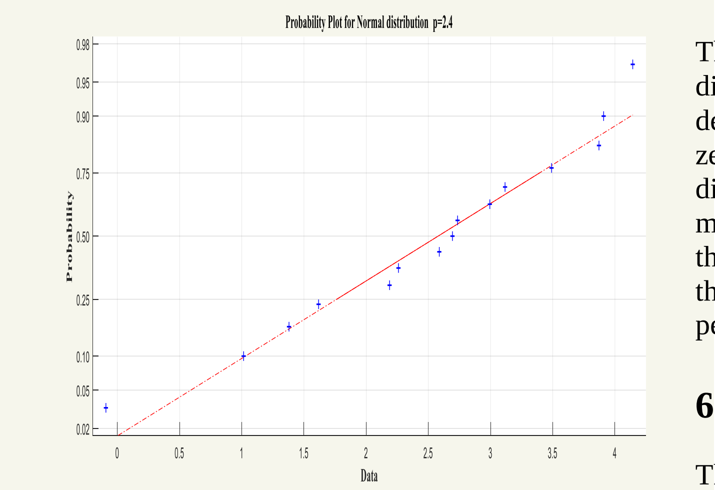

Fig. **1.** Probability plot when $\rho > 0$

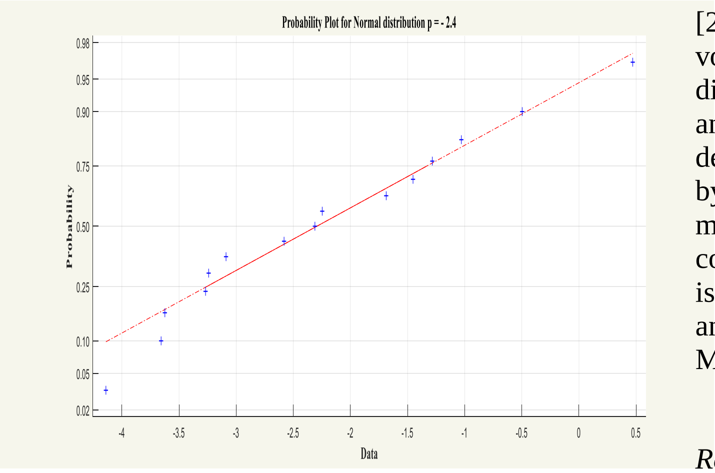

Fig. **2.** Probability plot when $\rho < 0$

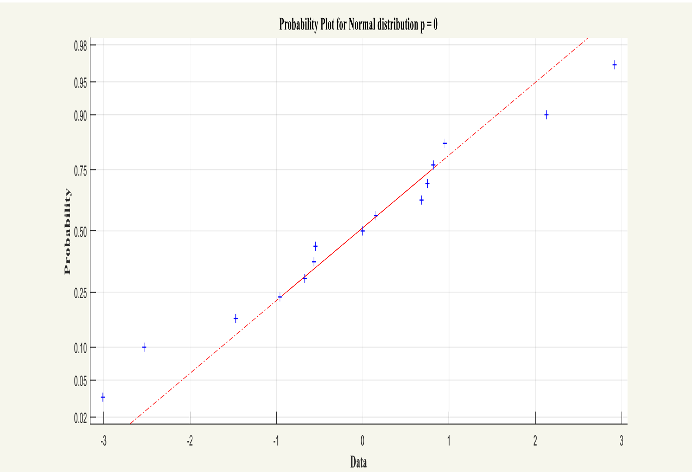

Fig. **3.** Probability plot when $\rho = 0$

5.2. Effect of σ on Return Distribution

The effect of σ is mainly on the peak of the distribution. When $\sigma = 0$, the volatility becomes deterministic because the processes $\sum B_i(\cdot)$ will be zero and ρ also zero. This will lead to a normal distribution of stock returns as in the Black Scholes model. Theoretically, increase in σ also increases the peak, creating fatty tails on both sides implying that, increase in market volatility σ gave higher peak of the distribution and vice versa.

6 Conclusion

The statistical analysis as described in [20] shows that volatility of stock is a time dependent quantity and also exhibits various random features. The randomness of stochastic volatility is addressed in [21] by assuming that both the stock price and the volatility are stochastic processes affected by the different sources of risk. In this work stability analysis on stock market forces were obtained by developing stochastic vector differential equation; by exploring the properties of the fundamental matrix solution of this equation and by placing continuity condition on the stochastic part. Example is given to illustrate the effectiveness of the model and simulation results presented graphically using MATLAB.

References:

[1] I. Davies, Euclidean Null Controllability of Infinite Neutral Differential Systems, *Anziam J.*, vol. 48, no. 2, pp. 285–293, 2006.

[2] I. Davies and O. L. C. Haas, Robust Guaranteed Cost Control for a Nonlinear Neutral System with Infinite Delay, in *European Control Conference*, 2015, no. 1, pp. 1255–1260.

[3] X. Mao, A Note on the LaSalle-Type Theorems for Stochastic Differential Delay Equations, *J. Math. Anal. Appl.*, vol. 268, pp. 125–142, 2002.

[4] X. Mao, Robustness of Stochastic Differential Delay Equations with Markovian Switching, *SACTA*, vol. 3, no. 1, pp. 48–61, 2000.

[5] K. Chang, G. Y. Wang, and C. Lu, The Effect of Liquidity on Stock Returns : A Style Portfolio Approach, *WSEAS Transactions on Mathematics*, vol. 12, no. 2,

pp. 170–179, 2013.

[6] L. Bachelier, *Theories de la speculative paris: Gauthier-Villar. Transplanted in Cooler*. 1964.

[7] M. F. M. Osborn, Browmian motion in the stock market operations research, *Ind. Manag. Rev.*, vol. 6, p. 13 – 32., 1965.

[8] P. Samuelson, Economics theorem and mathematics – An appraisal cowls foundation paper 61, *Repr. Am. Econ. Rev.*, vol. 42, pp. 56 – 69, 1952.

[9] F. Black and M. Scholes, Valuation of options and corporate liabilities, *J. Polit. Econ.*, vol. 81, pp. 637–654, 1973.

[10] M. K. Mondal, A. Alim, F. Rahman, and H. A. Biswas, Mathematical Analysis of Financial Model on Market Price with Stochastic Volatility, *J. Math. Financ.*, vol. 7, pp. 351–365, 2017.

[11] H. A. Biswas, On the Evolution of AIDS / HIV Treatment : An Optimal Control Approach, *Curr. HIV Res.*, vol. 12, pp. 1–12, 2014.

[12] J. Hull and C. John, *Options, Futures, and other Derivatives*, 7th ed. Pearson/Prentice Hall, Upper Saddle River, 2008.

[13] P. Wilmott, *Paul Wilmott on Quantitative Finance*, 2nd ed. John Wiley and Sons, Inc., 2006.

[14] A. Neisy, Least -Squares Method for Estimating Diffusion Coefficient, *IUST Int. J. Eng. Sci. Math. Ind. Eng. Spec. Issue*, vol. 19, no. 1–2, pp. 17–19, 2008.

[15] S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Rev. Financ. Stud.*, vol. 6, no. 2, pp. 327–343, 1993.

[16] A. Neisy and M. Peymany, Financial Modeling by Ordinary and Stochastic Differential Equations, *World Appl. Sci. J.*, vol. 13, no. 11, pp. 2288–2295, 2011.

[17] I. Davies and O. C. L. Haas, Null controllability of neutral system with infinite delays, *Eur. J. Control*, vol. 26, pp. 28–34, 2015.

[18] F. Herzog, Stochastic Differential Equations. pp. 1–64, 2013.

[19] A. G. Kartsatos, *Advanced Ordinary Differential Equations*. Tampa, Florida: Mariner Publishing Company, Inc, 1980.

[20] M. E. Adeosun, S. O. Edeke, and O. O. Ugbebor, Stochastic analysis of stock market price models: A case study of the Nigeria stock exchange (NSE), *WSEAS Trans. Math.*, vol. 14, pp. 353–363, 2015.

[21] C. Hull and A. White, An analysis of the bias in option pricing caused by a stochastic volatility, *Adv. Futur. Options Res.*, vol. 3, pp. 27–61, 1988.