Stability problem for non-autonomous systems of non-linear difference equations

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Abstract: - In this paper, a novel approach to the asymptotic stability problem for non-autonomous nonlinear difference equations is presented. We propose a quasi-invariance principle to solve a positive limit set localization problem for such equations. Asymptotic stability proof of the zero solution of non-autonomous difference equation is given by constructing the Lyapunov vector function and comparison system and using the proposed quasi-invariance principle for non-autonomous systems of difference equations. We illustrate the implementation of the proposed approach using the examples of some discrete epidemic models.
Keywords: - Non-autonomous systems, non-linear systems, non-linear difference equations

1 Introduction

Difference equations are widely used in the study of sampled-data control problem for dynamical systems. It is well known that one of the approaches to the stabilization problem solution for sampled-data nonlinear systems consists in an analysis of an approximate discrete-time plant model [1, 2]. Difference equations describe different models in mathematical biology, medicine, economics and other natural sciences as well as technical ones.

An important direction in the qualitative analysis of the difference equations is the study of the stability of their solutions. By now, the theory of stability of linear difference equations has been well developed [3]. But the solution to the stability problem for nonlinear difference equations is far from completeness.

To study the stability problem for nonlinear difference equations the Lyapunov function method is widely used [3, 4]. One of the main extensions of Lyapunov stability theory for autonomous difference systems consists in the application of La-Salle's invariance principle in order to mitigate the conditions of the Lyapunov direct method [4]. The problem on asymptotic behavior of the solutions of non-autonomous difference equations was considered by many authors [3-6]. In [6] an analogue of La-Salle's invariance principle for non-autonomous difference equations was obtained. Note that the Lyapunov direct method was widely developed on the basis of using both the comparison principle and Lyapunov vector function [7-9].

The purpose of this paper is to give new results in the stability theory for non-autonomous systems of nonlinear difference systems. Firstly, we propose a quasi-invariance principle to solve a positive limit set localization problem for non-autonomous systems of difference equations. With respect to the previous result in such direction [10], the proposed approach has the advantage of using a wider class of Lyapunov functions, which allows us to obtain more general results. Secondly, we propose a stability analysis which provides the uniform global asymptotic stability property for the solutions of the non-autonomous systems of nonlinear difference systems using both a Lyapunov vector function method and the theory of limiting equations.

2 Stability analysis of nonlinear non-autonomous difference equations

Consider the nonlinear difference system given by

$$x(n+1) = f(n, x(n)) \tag{1}$$

where x is the $m-$dimensional vector of the real linear space R^m with some norm $\|x\|$, $n \in Z^+$; the function $f: Z^+ \times R^m \to R^m$ is continuous in x for each $n \in Z^+$ and the following equality holds $f(n,0) \equiv 0$.

We will solve the stability problem of the zero solution $x = 0$ of the system (1).

2.1 The quasi-invariance principle for non-autonomous systems of difference equations

In order to prove the quasi-invariance principle for non-autonomous systems of difference equations (1), we introduce the following mathematical notations.

Denote by F the set of the functions $f : Z^+ \times R^m \to R^m$ which are continuous in x. Introduce the following convergence on the set F.

Definition 2.1:1 The sequence $\{f_k \in F\}$ converges to f, if $\forall \varepsilon > 0$, $\forall N \in Z^+$ and for each compact set $D \subset R^m$ there exists $N_0 \in Z^+$ such that for all $k \ge N_0$ the following holds

$$\| f_k(n,x) - f(n,x) \| < \varepsilon \quad \forall (n,x) \in [0,N] \times D$$

This convergence is metrizable if we introduce the following metrics in the space F.

Let $\{D_k\}$ be an aggregate of the embedded compact sets covering the space R^m such that

$$D_1 \subset D_2 \subset \ldots \subset D_k \subset \ldots, \quad \bigcup_{k=1}^{\infty} D_k = R^m$$

The following metrics exists:

$$\rho(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\sup(\| f(n,x) - g(n,x) \|)}{1 + \sup(\| f(n,x) - g(n,x) \|)}$$
$$(n,x) \in [0,k] \times D_k, \quad f, g \in F$$

Assumption 2.1:2 Assume that the right-hand side of (1) satisfies the following two conditions:

a) The function $f(n,x)$ is uniformly bounded on the set $Z^+ \times D$ for each compact set $D \subset R^m$, i.e. the following holds

$$\| f(n,x) \| \le l = l(D) \quad \forall (n,x) \in Z^+ \times D \tag{2}$$

b) The function $f(n,x)$ is uniformly continuous in x on each compact set $D \subset R^m$, i.e. $\forall D \subset R^m$ and $\forall \varepsilon > 0$ there exists $\delta = \delta(\varepsilon, D) > 0$ such that for all $n \in Z^+$ and $x_1, x_2 \in D$: $\| x_2 - x_1 \| < \delta$ the following inequality holds

$$\| f(n,x_2) - f(n,x_1) \| < \varepsilon \tag{3}$$

Lemma 2.1:3 Let Assumption 2.1 hold. Then, the family of translates

$$\{ f_k(n,x) = f(k+n,x), k \in Z^+ \}$$

is contained in some compact set $F_0 \subset F$.

Remark 2.1:4 Note that the properties (2) and (3) are the precompactness conditions for the function $f(n,x)$.

Definition 2.2:5 The function $f^* : Z^+ \times R^m \to R^m$ is said to be a limiting to f, if there exists a sequence $n_k \to \infty$ such that the sequence of translates $\{ f_k(n,x) = f(n_k+n,x) \}$ converges to the function f^* in the metrizable space F. Accordingly, the system

$$x(n+1) = f^*(n, x(n)) \tag{4}$$

is said to be a limiting to (1).

Using the result of LaSalle [8] on a topological dynamics for non-autonomous difference equations, the following theorem can be obtained which establishes the relationship between the solutions of (1) and (4).

Theorem 2.1:6 Let for some sequence $n_k \to \infty$ the sequence of translates $\{ f_k(n,x) = f(n_k+n,x) \}$ converge to a limiting function f^* in the space F. Let also the sequence of vectors $\{x_0^{(k)}\}$ ($x_0^{(k)} \in R^m$) converge to some vector $x_0 \in R^m$ if $k \to +\infty$, i.e. $x_0^{(k)} \to x_0$. Then, the sequence of solutions $x_k(n, n_0, x_0^{(k)}) = x(n_k+n, n_0, x_0^{(k)})$ of the systems $x(n+1) = f_k(n, x(n))$ ($f_k(n,x) = f(n_k+n,x)$) converges to the solution $x = x^*(n, n_0, x_0)$ of the limiting system (4). Moreover, this convergence is uniform in $n \in [n_0, n_0+N]$ for each $N \in Z^+$.

Using Theorem 2.1, one can obtain some properties of a positive limit set of a bounded solution of (1).

Definition 2.3:7 Let the solution $x = x(n, n_0, x_0)$ of the system (1) be defined for all $n \ge n_0$. The vector $q \in R^m$ is said to be a positive limit point of that solution, if there exists the sequence $n_k \to \infty$ such that $x(n_k, n_0, x_0) \to q$. The set of all limit points of the solution $x = x(n, n_0, x_0)$ is said to be a positive limit set $\Omega^+(n_0, x_0)$.

Since for each $n_0 \in Z^+$ the translate $f_0(n,x) = f(n_0+n,x)$ is defined on the set $[-n_0, +\infty] \times R^m$, so the definition domain of the limiting function f^* can be extended to the set $Z^- \times R^m$. Therefore, one can define the solutions of the system (4) for all initial points $(n_0, x_0) \in Z \times R^m$. Accordingly, one can define the following function $x^*(n, n_0, x_0)$, $x^* : Z \times Z \times R^m \to R^m$.

Definition 2.4:8 The set $H \subset R^m$ is said to be quasi-invariant, if, for each $x_0 \in H$ there exist both the limiting system (4) and its solution $x = x^*(n)$, $x^*(0) = x_0$ such that $x^*(n) \in H\ \forall n \in Z$.

Theorem 2.2 [8]:9 Let the solution $x = x(n, n_0, x_0)$ of the system (1) be bounded for all $n \in Z^+$. Then, the positive limit set $\Omega^+(n_0, x_0)$ is bounded and quasi-invariant. Moreover, the solution $x(n, n_0, x_0)$ asymptotically tends to $\Omega^+(n_0, x_0)$ as $n \to \infty$.

2.2 V.M. Alekseev's formula of nonlinear variation of parameters for a nonlinear difference system

Consider the nonlinear difference system

$$v(n+1) = g(n, v(n)) + Q(n, v(n)) \tag{5}$$

where the function $g : Z^+ \times R^k \to R^k$ is continuously differentiable in $v \in R^k$ for each $n \in Z^+$, the function $Q(n, v)$ is continuous in $v \in R^k$ for each $n \in Z^+$.

Let $w = w(n, n_0, w_0)$ be a solution of the unperturbed system

$$w(n+1) = g(n, w(n)) \tag{6}$$

Define the following matrix [4, 9]

$$\Phi(n, n_0, w_0) = \frac{\partial w(n, n_0, w_0)}{\partial w_0} \tag{7}$$

Note that the matrix (7) is a fundamental one of the linear variational system

$$\begin{aligned} &\varphi(n+1) = H(n)\varphi(n) \\ &H(n) = \frac{\partial g}{\partial w}(n, w)\Big|_{w = w(n, n_0, w_0)} \end{aligned} \tag{8}$$

In other words, the matrix (7) satisfies the following equation

$$\Phi(n+1) = H(n)\Phi(n), \quad \Phi(n_0) = I \tag{9}$$

where I is the identity matrix.

Theorem 2.3:10 Let $v = v(n, n_0, v_0)$ and $w = w(n, n_0, v_0)$ be the solutions of the systems (5) and (6) respectively, defined for all $n \ge n_0$. Then, for these solutions one can easily find the following relationship

$$\begin{aligned} &v(n, n_0, v_0) = w(n, n_0, v_0) + \\ &+ \sum_{j=n_0}^{n-1} \int_0^1 \Phi(n, j+1, g(j, v[j]) + sQ(j, v[j]))ds\, Q(j, v[j]) \end{aligned} \tag{10}$$

where $v[j] = v(j, n_0, v_0)$.

Proof: For each $j = n_0, n_0+1, ..., n-1$ one can find the following

$$\begin{aligned} &w(n, j+1, v[j+1]) - w(n, j, v[j]) = \\ &= w(n, j+1, v[j+1]) - w(n, j+1, w[j+1]) \end{aligned} \tag{11}$$

where $w[j+1] = w(j+1, j, v[j])$.

Applying the Mean Value Theorem, from (11) one can obtain the following relationship

$$\begin{aligned} &w(n, j+1, v[j+1]) - w(n, j, v[j]) = \\ &= \int_0^1 \Phi(n, j+1, sv[j+1] + \\ &+(1-s)w[j+1])(v[j+1] - w[j+1])ds = \\ &= \int_0^1 \Phi(n, j+1, g(j, v[j]) + \\ &+ sQ(j, v[j]))ds\, Q(j, v[j]) \end{aligned} \tag{12}$$

Summarizing the equalities (12) for j from n_0 to n, we have

$$\begin{aligned} &w(n, n_0, v[n]) - w(n, n_0, v[n_0]) = \\ &= \sum_{j=n_0}^{n} \int_0^1 \Phi(n, j+1, g(j, v[j]) + \\ &+ sQ(j, v[j]))ds\, Q(j, v[j]) \end{aligned} \tag{13}$$

Since $w(n, n, v[n]) = v[n] = v(n, n_0, v_0)$ and $w(n, n_0, v[n_0]) = w(n, n_0, v_0)$, from (13) we obtain the formula (10). This completes the proof.

Remark 2.2:11 The relationship (10) represents the V.M. Alekseev formula of variation of constants for nonlinear difference equations. Note that in the other form the discrete version of V.M. Alekseev's formula was obtained in [11, 12]. We will use the formula (10) in combination with a comparison method in order to get the new solutions of the asymptotic stability problem for non-autonomous systems of nonlinear difference equations.

2.3. Main result

In this subsection, new theorems on the limiting behavior and asymptotic stability property for the solutions of system (1) are obtained.

Assumption 2.2: 12 Assume that one can find a Lyapunov vector function candidate $V = V(n, x)$,

$V : \mathrm{Z}^+ \times \mathrm{R}^m \to \mathrm{R}^k$, such that it is continuous in x for each $n \in \mathrm{Z}^+$ and the following equation holds

$$V(n+1, x(n+1)) = \\ = g(n, V(n, x(n))) + Q(n, x(n), V(n, x(n))) \qquad (14)$$

where the functions $g : \mathrm{Z}^+ \times \mathrm{R}^k \to \mathrm{R}^k$ and $Q : \mathrm{Z}^+ \times \mathrm{R}^m \times \mathrm{R}^k \to \mathrm{R}^k$ satisfy the following conditions:
1) The function $g = g(n, w)$ is quasi-monotonically nondecreasing and continuously differentiable in $w \in \mathrm{R}^k$
2) The functions $g = g(n, w)$ and $Q = Q(n, x, w)$ satisfy the precompactness criteria such as (2) and (3).
3) The inequality $Q(n, x, w) \leq 0$ holds for all $(n, x, w) \in \mathrm{Z}^+ \times \mathrm{R}^m \times \mathrm{R}^k$.

Using Assumption 2.2 one can easily obtain that $V(n, x)$ is a comparison vector function and (6) is a comparison system [7].

Lemma 2.2: 13 Let Assumption 2.2 hold. Let also $w(n) = w(n, n_0, V_0)$ ($V(n_0, x_0) = V_0$) be a solution of (6) defined in the interval $[n_0, N]$. Then, for all $n \in [n_0, N]$ we have

$$V(n, x(n, n_0, x_0)) \leq w(n, n_0, V_0) \qquad (15)$$

where $x(n) = x(n, n_0, x_0)$ is a solution of the system (1).

Note that the comparison system (6) satisfies the precompactness criteria. Therefore, one can find the family of limiting comparison systems

$$w(n+1) = g^*(n, w(n)), \qquad g^* \in F_g \qquad (16)$$

Using the properties of the function $g = g(n, x)$, we get that all the solutions $w = w(n, n_0, w_0)$ of the system (6) are differentiable in $w_0 \in \mathrm{R}^k$. Moreover, since the function $w(n, n_0, w_0)$ is nondecreasing in w_0, one can easily obtain that the matrix

$$\Phi(n, n_0, w_0) = \frac{\partial w(n, n_0, w_0)}{\partial w_0}$$

is semi-definite positive and normalized, i.e. $\Phi(n_0, n_0, w_0) \geq 0$ and $\Phi(n_0, n_0, w_0) = I$. Besides, $\Phi(n_0, n_0, w_0)$ is the fundamental matrix for the linear variational system (8).

Assumption 2.3: 14 Suppose that for each compact set $D \subset \mathrm{R}^k$ there exist positive reals $M(D)$ and $m(D)$ such that for all $(n, n_0, w_0) \in \mathrm{Z}^+ \times \mathrm{Z}^+ \times D$ the matrix $\Phi(n, n_0, w_0)$ satisfies the following conditions

$$\|\Phi(n, n_0, w_0)\| \leq M(D)$$
$$\det \Phi(n, n_0, w_0) \geq m(D)$$

Theorem 2.4: 15 Let Assumptions 2.2 and 2.3 hold. Let also the solutions $x(n, n_0, x_0)$ and $w(n) = w(n, n_0, V_0)$ ($V_0 = V(n_0, x_0)$) of the systems (1) and (6) respectively, be bounded for all $n \geq n_0$. Then, for each positive limit point $q \in \Omega^+(n_0, x_0)$ there exists the aggregate of the limiting functions (f^*, V^*, g^*, Q^*) such that for the solution $x = x^*(n, q)$ of the system (4), satisfying the initial condition $x^*(0, q) = q$, the following holds

$$x^*(n, q) \in \Omega^+(n_0, x_0)$$
$$x^*(n, q) \in \{V^*(n, x) = w^*(n)\} \cap \{Q^*(n, x, w^*(n)) = 0\}$$
$$\forall n \in \mathrm{Z}$$

where $w^*(n)$ is the solution of the limiting comparison system (16) such that $w^*(0) = V^*(0, q)$.

Remark 2.3: 16 Theorem 2.4 presents the new result of a positive limit set localization problem for non-autonomous difference equations by using vector Lyapunov functions and comparison principle. This result is an analogue of the well-known La-Salle's invariance principle for autonomous systems [4]. The main difference between our result and the known one [6] is that we apply a wider class of vector Lyapunov functions which can depend on time, as well as a comparison method which is a development of the direct Lyapunov method.

Theorem 2.5: 17 Let the Lyapunov vector function candidate $V = V(n, x)$ exist such that it satisfies the precompactness criteria and the following holds

1) The function $\bar{V} = \max(V_1, V_2, ..., V_k)$ is positive definite, i.e. there exists a function $a \in K$ such that $\bar{V}(n, x) \geq a(\|x\|)$;
2) The equality (14) holds;
3) The zero solution $w = 0$ of comparison system (6) is stable;
4) For each limiting aggregate (f^*, V^*, W^*, Q^*) and each bounded solution $w = w^*(n) \neq 0$ of the limiting comparison system (16) there are no solutions of (4) which stay forever in the set

$$\{V^*(n, x) = w^*(n)\} \cap \{Q^*(n, x, w^*(n)) = 0\}$$

Then, the zero solution $x = 0$ of the system (1) is asymptotically stable.

Theorem 2.6: 18 Let the Lyapunov vector function candidate $V = V(n, x)$ exist such that it satisfies the precompactness criteria and the following holds

1) The function $\bar{V}(n,x) = \max(V_1, V_2, ..., V_k)$ is positive definite and radially unbounded, i.e. there exists a function $a \in K_\infty$ such that $\bar{V}(n,x) \geq a(\|x\|)$;
2) The functions $V_1(n,x)$, $V_2(n,x)$, ... , $V_k(n,x)$ converge to zero uniformly in n as $\mathrm{P}\, x\, \mathrm{P} \to 0$;
3) The equality (14) holds, where $Q = Q(n,x)$;
4) The zero solution $w = 0$ of comparison system (6) is uniformly globally stable;
5) For any limiting pair (f^*, Q^*) there are no solutions of (4) which stay forever in the set

$$\{Q^*(n,x) = 0\}$$

except for the zero solution $x = 0$.

Then, the zero solution $x = 0$ of the system (1) is uniformly globally asymptotically stable.

Remark 2.4: 19 The conditions of Theorems 2.5 and 2.6 make it possible to extend the classes of the comparison systems and the Lyapunov vector functions used to study the asymptotic stability property of non-autonomous difference systems. Therefore, Theorems 2.5 and 2.6 represent the development of the classical comparison method for asymptotic stability analysis of non-autonomous difference equations. On the other hand, Theorems 2.5 and 2.6 are the development of the authors's previous results [10] obtained for non-autonomous systems of differential equations to non-autonomous systems of difference equations.

3 Stability analysis of the second-order epidemic model

3.1 Stability analysis of the second-order epidemic model

Consider a discrete epidemic model for the spread of the disease (gonorrhea or chlamydia), which consists of two heterosexual populations P_1 and P_2 where infected members of one population can infect the healthy members of the other population [8, 13, 14]. Assume that the recovery of the infected individuals is possible without the immunity. Assume also that the population sizes are constant. A non-autonomous discrete model of the disease course is given by

$$\begin{cases} x_1(n+1) = \dfrac{\alpha_{12}\Delta t M}{W} x_2(n)(1-x_1(n)) + \\ +(1-\gamma_1\Delta t)x_1(n) \\ x_2(n+1) = \dfrac{\alpha_{21}\Delta t W}{M} x_1(n)(1-x_2(n)) + \\ +(1-\gamma_2\Delta t)x_2(n) \end{cases} \tag{17}$$

where $n \in \mathrm{Z}^+$, x_1 and x_2 are the fractions of the infected members of the populations P_1 and P_2 respectively; $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$; W and N are the sizes of the populations P_1 and P_2 respectively; α_{jk} ($j,k = 1,2$, $j \neq k$) and γ_i ($i = 1,2$) are the coefficients which characterize the process of the infection spread (contact and recovery rates), for which the following inequalities hold $0 \leq \alpha_{jk}\Delta t \leq W/M$ and $0 \leq \gamma_i \Delta t \leq 1$.

Assume that the contact and recovery rates can vary with the season during the year, i.e. $\alpha_{jk} = \alpha_{jk}(n)$ and $\gamma_i = \gamma_i(n)$ where $j,k = 1,2$, $j \neq k$ and $i = 1,2$.

Note that the set $\Gamma = \{0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 1\}$ is invariant with respect to the solutions $x(n, n_0, x_0) \in \Gamma$ of the equation (17) for all initial points $x(n_0) = x_0$, where $(n_0, x_0) \in \mathrm{Z}^+ \times \Gamma$ and $\forall n \geq n_0$.

The system (17) satisfies the precompactness criteria (2) and (3). Therefore, the following limiting system can be obtained

$$\begin{cases} x_1(n+1) = \dfrac{\alpha^*_{12}(n)\Delta t M}{W} x_2(n)(1-x_1(n)) + \\ +(1-\gamma^*_1(n)\Delta t)x_1(n) \\ x_2(n+1) = \dfrac{\alpha^*_{21}(n)\Delta t W}{M} x_1(n)(1-x_2(n)) + \\ +(1-\gamma^*_2(n)\Delta t)x_2(n) \end{cases} \tag{18}$$

where $\alpha^*_{ij}(n) = \lim_{k\to+\infty}\alpha_{ij}(n_k + n)$ and $\gamma^*_i(n) = \lim_{k\to+\infty}\gamma_i(n_k + n)$, $i, j = 1,2$, $i \neq j$ are limiting functions.

Choose the Lyapunov vector function candidate such as

$$V = (V_1, V_2)^T, \quad V_1 = x_1, \quad V_2 = x_2 \tag{19}$$

One can easily obtain the comparison system

$$\begin{cases} w_1(n+1) = \dfrac{\alpha_{12}(n)\Delta t M}{W} w_2(n) + \\ +(1-\gamma_1(n)\Delta t)w_1(n) \\ w_2(n+1) = \dfrac{\alpha_{21}(n)\Delta t W}{M} w_1(n) + \\ +(1-\gamma_2(n)\Delta t)w_2(n) \end{cases} \tag{20}$$

The vector $Q = (Q_1, Q_2)^T$ is given by

$$Q_1 = \frac{\alpha_{12}(n)\Delta t M}{W} x_1 x_2$$

$$Q_2 = \frac{\alpha_{21}(n)\Delta t W}{M} x_1 x_2$$

The limiting functions Q^*_1 and Q^*_2 are defined as follows

$$Q_1^* = \frac{\alpha_{12}^*(n)\Delta t M}{W} x_1 x_2$$
$$Q_2^* = \frac{\alpha_{21}^*(n)\Delta t W}{M} x_1 x_2$$

The zero solution $w_1 = w_2 = 0$ of the comparison system (20) is uniformly stable if for each $k_0 \in Z^+$ and each $k \geq k_0$ the following holds

$$\left\| \prod_{j=k_0}^{k} A(j) \right\| \leq M = const \qquad (21)$$

where the matrix $A(j) \in R^{2\times 2}$ is defined as follows

$$A(j) = \begin{pmatrix} 1-\gamma_1(j) & \dfrac{\alpha_{12}(j)\Delta t M}{W} \\ \dfrac{\alpha_{21}(j)\Delta t W}{M} & 1-\gamma_2(j) \end{pmatrix}$$

Since the comparison system (20) is linear so the uniform stability of its zero solution $w_1 = w_2 = 0$ is global.

The set $\{Q_1^* = Q_2^* = 0\}$ doesn't contain the solutions of (18) except for $x_1 = x_2 = 0$. Using Theorem 2.6 one can obtain the uniform global asymptotic stability property for the zero solution $x_1 = x_2 = 0$ of (17) if the inequality (21) holds. Then, there is no epidemic and the disease dies out.

In particular, the inequality (21) is true if there exists $\gamma = const > 0$ such that for all $j \in Z^+$ the following holds

$$\frac{\alpha_{21}(j)}{\gamma_2(j)} \leq \gamma \leq \frac{\gamma_1(j)}{\alpha_{12}(j)} \qquad (22)$$

Note that in autonomous case the inequality (22) coincides with the condition proposed by [14], which consists in the requirement that the basic reproductive rate is not more than one, i.e.

$$R = \frac{\alpha_{12}\alpha_{21}}{\gamma_1\gamma_2} \leq 1$$

3.2 Stability analysis of the third-order epidemic model

Consider a third-order discrete epidemic model that describes the course of the disease in some system with three subpopulations. Assume that the infected members of the first and second populations can infect each other. Assume also that the infected members of a third population can infect the members of all three populations. Let x_i be an infected part of the population P_i, $i = 1,2,3$. Then $(1-x_i)$ is a healthy part that perceives an infection.

The nonlinear discrete model of the course of the disease is given by

$$\begin{cases} x_1(n+1) = (a_{12}(n)x_2(n) + a_{13}(n)x_3(n))(1-x_1(n)) + \\ +a_{11}(n)x_1(n) \\ x_2(n+1) = (a_{21}(n)x_1(n) + a_{23}(n)x_3(n))(1-x_2(n)) + \\ +a_{22}(n)x_2(n) \\ x_3(n+1) = (a_{34}(n)x_3(n) + a_{31}(n)x_1(n) + \\ +a_{32}(n)x_2(n))(1-x_3(n)) + a_{33}(n)x_3(n) \end{cases} \qquad (23)$$

where $\varepsilon \leq a_{ij} \leq 1-\varepsilon$, $i = 1,2,3$, $j = 1,2,3,4$, $x_i(n) \in \Gamma = \{x: \ 0 \leq x_i \leq 1, \ i = 1,2,3\}$, $\forall n \geq n_0$, $x(n_0) \in \Gamma$

$$\begin{cases} a_{12}(n) + a_{13}(n) \leq 1 \\ a_{21}(n) + a_{23}(n) \leq 1 \\ a_{31}(n) + a_{32}(n) + a_{34}(n) \leq 1 \end{cases}$$

The equations limiting to (23) are given by

$$\begin{cases} x_1(n+1) = (a_{12}^*(n)x_2(n) + a_{13}^*(n)x_3(n))(1-x_1(n)) + \\ +a_{11}^*(n)x_1(n) \\ x_2(n+1) = (a_{21}^*(n)x_1(n) + a_{23}^*(n)x_3(n))(1-x_2(n)) + \\ +a_{22}^*(n)x_2(n) \\ x_3(n+1) = (a_{34}^*(n)x_3(n) + a_{31}^*(n)x_1(n) + \\ +a_{32}^*(n)x_2(n))(1-x_3(n)) + a_{33}^*(n)x_3(n) \end{cases} \qquad (24)$$

where the functions $a_{ij}^*(n)$ are limiting ones for a_{ij} correspondingly, i.e. there exists the sequence $n_k \to +\infty$ such that

$$a_{ij}^*(n) = \lim_{n_k \to +\infty} a_{ij}(n+n_k)$$

Consider the Lyapunov vector function candidate $V(x) = (V_1, V_2, V_3)^T$ such that $V_1 = x_1$, $V_2 = x_2$ and $V_3 = x_3$.

One can easily obtain the following

$$V(x(n+1)) = A(n)V(x(n)) - Q(n,x)$$

where the matrix $A(n)$ and the vector $Q(n,x) = (Q_1, Q_2, Q_3)^T$ are such as

$$A(n) = \| a_{jk}(n) \|$$
$$Q_1 = (a_{12}(n)x_2(n) + a_{13}(n)x_3(n))x_1(n)$$
$$Q_2 = (a_{21}(n)x_1(n) + a_{23}(n)x_3(n))x_2(n)$$
$$Q_3 = (a_{31}(n)x_1(n) + a_{32}(n)x_2(n) + a_{34}(n)x_3(n))x_3(n)$$

The comparison system is then given by

$$w(n+1) = A(n)w(n) \tag{25}$$

The zero solution $w_1 = w_2 = w_3 = 0$ of (25) is uniformly stable if the following holds

$$\left\| \prod_{j=k_0}^{k} A(j) \right\| \le M = const \tag{26}$$
$$\forall\, k_0 \in \mathrm{Z}^+, \quad \forall k \ge k_0$$

One can easily see that the set $\{Q^*(n, x_1, x_2, x_3) = 0\}$ does not contain the solutions of (24) except the zero solution $x_1 = x_2 = x_3 = 0$. From (26), using Theorem 2.6, we get the global asymptotic stability of the equilibrium point $x_1 = x_2 = x_3 = 0$ of the system (23).

Now, consider the Lyapunov scalar function $V = V(n,x)$ such that $V = fx_1 + gx_2 + hx_3$, where f, g and h are some positive reals.

One can obtain that the sufficient conditions of the uniform stability of the equilibrium state position $x_1 = x_2 = x_3 = 0$ of the system (23) are such as follows

$$\begin{cases} fa_{11}(n) + ga_{21}(n) + ha_{31}(n) \le f\,\mu(n) \\ fa_{12}(n) + ga_{22}(n) + ha_{32}(n) \le g\,\mu(n) \\ fa_{13}(n) + ga_{23}(n) + ha_{33}(n) + ha_{34}(n) \le h\mu(n) \end{cases} \tag{27}$$

where the function $\mu(n)$ satisfies the following inequality

$$\prod_{j=n_0}^{n} \mu(j) \le m_0 = const, \forall\, n > n_0, n_0 \in \mathrm{Z}^+ \tag{28}$$

The set $\{Q_1^* = Q_2^* = Q_3^* = 0\}$ doesn't contain the solutions of (24) except for the zero solution $x_1 = x_2 = x_3 = 0$.

Therefore, using Theorem 2.6 we obtain that the zero solution $x_1 = x_2 = x_3 = 0$ of (23) is globally uniformly asymptotically stable if there exist the positive constants f, g and h and the function $\mu : \mathrm{Z}^+ \to \mathrm{R}^+$ such that the inequalities (27) and (28) hold.

Analyze the conditions (27). Introducing the parameters $u = f/g > 0$ and $v = h/g > 0$ we obtain that the system of inequalities (27) can be written as follows

$$\begin{cases} \mu(n) > a_{11}(n) \\ \mu(n) > a_{22}(n) \\ \mu(n) > a_{33}(n) + a_{34}(n) \\ v \le \dfrac{\mu(n) - a_{11}(n)}{a_{31}(n)} u - \dfrac{a_{21}(n)}{a_{31}(n)} \\ v \le \dfrac{\mu(n) - a_{22}(n)}{a_{32}(n)} - \dfrac{a_{12}(n)}{a_{32}(n)} u \\ v \le -\dfrac{a_{13}(n)}{a_{33}(n) + a_{34}(n) - \mu(n)} u - \\ \quad - \dfrac{a_{23}(n)}{a_{33}(n) + a_{34}(n) - \mu(n)} \end{cases} \tag{29}$$

From the last three inequalities in (29), we obtain some domain (see Figure 1) on the plane (u,v).

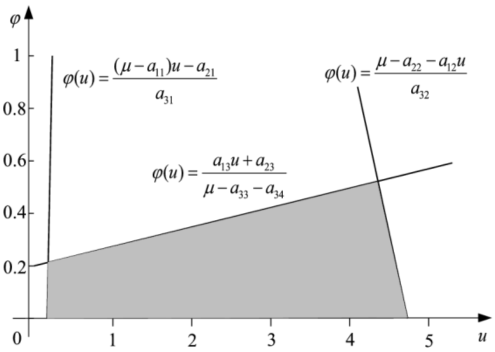

Fig. 1. Graphical interpretation of the conditions (29).

Let find the intersection points of the half-planes in Figure 1.

• One can find the minimum value of u on the half-plane defined by the fourth relationship in (29) using the following equality

$$\frac{\mu(n) - a_{11}(n)}{a_{31}(n)} u = \frac{a_{21}(n)}{a_{31}(n)}$$

• There exists the intersection point $u_1^*(n)$ of the boundaries of the half-planes defined by the fourth relationship and the sixth one in (29) if the following holds

$$u_1^*(n) = \frac{a_{23}a_{31} + a_{21}(\mu - a_{34} - a_{33})}{(\mu - a_{34} - a_{33})(\mu - a_{11}) - a_{13}a_{31}}$$

$$(\mu - a_{11})(\mu - a_{34} - a_{33}) > a_{13}a_{31}$$

• There exists the intersection point $u_2^*(n)$ of the boundaries of the half-planes defined by the fifth relationship and the sixth one in (29), which is defined by

$$u_2^*(n) = \frac{\mu^2 - (a_{34} + a_{33} + a_{22})\mu + a_{22}a_{34} - a_{23}a_{32} + a_{22}a_{33}}{a_{13}a_{32} + a_{12}(\mu - a_{34} - a_{32})}$$

• One can find the minimum value of u on the half-plane defined by the fifth relationship in (29) using the following equality

$$\frac{\mu(n) - a_{22}(n)}{a_{32}(n)} = \frac{a_{12}(n)}{a_{32}(n)}u$$

If $\forall n \in Z^+$ the following inequalities hold

$$0 < \gamma^* \le \frac{a_{21}(n)}{\mu(n) - a_{11}(n)} \le \alpha^* \le \frac{\mu(n) - a_{22}(n)}{a_{12}(n)}$$
$$\frac{(\mu(n) - a_{11}(n))\alpha^*}{a_{31}(n)} - \frac{a_{21}(n)}{a_{31}(n)} \ge \beta^*$$
$$\frac{a_{13}(n)\alpha^* + a_{23}(n)}{\mu(n) - a_{33}(n) - a_{34}(n)} \ge \beta^*$$
$$\frac{\mu(n) - a_{22}(n) - a_{12}(n)\alpha^*}{a_{32}(n)} \ge \beta^*$$

then there exists the point (α^*, β^*) which belongs to the domain (29), where $\alpha^* = const > 0$ and $\beta^* = const > 0$.

Therefore, one can obtain the following estimations for the function $\mu(n)$: (28) and

$$\begin{gathered}
\mu(n) \ge a_{22}(n) + \alpha^* a_{12}(n) \\
\mu(n) \ge a_{11}(n) + \frac{1}{\alpha^*} a_{21}(n) \\
\mu(n) \ge a_{11}(n) + \frac{1}{\alpha^*}(a_{21}(n) + \beta^* a_{31}(n)) \\
\mu(n) \ge a_{22}(n) + a_{12}(n)\alpha^* + \beta^* a_{32}(n) \\
\mu(n) \le a_{11}(n) + \frac{\gamma^*}{a_{21}(n)} \\
\mu(n) \le a_{33}(n) + a_{34}(n) + \frac{1}{\beta^*}(a_{13}(n)\alpha^* + a_{23}(n))
\end{gathered} \tag{30}$$

The compatibility of the estimates (30) is expressed as follows. Let there exist the positive constants $\alpha^* > 0$, $\beta^* > 0$, $\gamma^* > 0$ such that the following inequality holds

$$\begin{gathered}
\max\{a_{11}(n) + \frac{1}{\alpha^*}[a_{21}(n) + \beta^* a_{31}(n)], \\
a_{22}(n) + a_{12}(n)\alpha^* + \beta^* a_{32}(n)\} \le \\
\le \min\{a_{11}(n) + \frac{\gamma^*}{a_{21}(n)}, \\
a_{33}(n) + a_{34}(n) + \frac{1}{\beta^*}(a_{13}(n)\alpha^* + a_{23}(n))\}
\end{gathered}$$

Then, the zero state $x_1 = x_2 = x_3 = 0$ of (23) is uniformly globally asymptotically stable.

4 Conclusion

We have addressed the global asymptotic stability problem for non-autonomous systems of nonlinear difference equations. A quasi-invariance principle for such systems has been proposed and new theorem of La-Salle's type on the limit behavior of the solutions has been proved. By employing both the Lyapunov vector function method and the quasi-invariance principle, a novel solution to the global asymptotic stability problem has been obtained. It is well-known that the basic condition of the classical comparison theorem for the asymptotic stability property consists in the requirement of this property for the zero solution of the comparison system. In this paper the aforementioned condition is weakened to the requirement of the stability property for the zero solution of the comparison system. In order to show the effectiveness of the proposed approach we provide the examples on stability analysis of some non-autonomous discrete epidemic models.

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