## 2 Brief overview of SA methods

The model is represented by a mapping $f$ (a deterministic or stochastic function) which relates the inputs domain to the output space:

$$Y = f\left(X_1, X_2, ..., X_M\right) \quad (1)$$

where $Y$ is a scalar. The input factors ($X_1$, $X_2$, . . ., $X_M$) are supposed to be random variables described by identified probability density functions which reflect the uncertain knowledge of the system under analysis [13].

Sensitivity analysis relates to the problem of investigating the contribution of the uncertainty in the input factors to the uncertainty in the model output $Y$ [14]. Through SA, it is possible to decompose the model output uncertainty back to the input sources of uncertainty. The quantification of the signification of input uncertainties is useful for the identification of factors which must be measured with precision to reach the given accurateness of the model output.

As far as the SA methodologies are concerned, the distinction can be made between qualitative and quantitative methods [13]. Qualitative methods are oriented to the identification of significant and insignificant factors using screening, but these provide only soft observations of the relative difference of importance [15]. Quantitative technologies can be designed and adapted so to provide information on the uncertainty quantity explained by each factor [16]. Mostly, a variance is considered [17] as the degree of uncertainty; nevertheless, also other degrees of sensitivity between statistic interferences are possible [7, 18].

Another classification of available SA methodologies makes distinction between local and global methods [13, 16].

In local approaches (also known as one-at-a-time, OAT, methods), the influence of a single factor is studied supposing that all the other factors are fixed on nominal values, see e. g. [19]. The main shortcoming of this approach is the impossibility to find the interaction between the factors, because it manifests itself, when the inputs chase simultaneously. The local sensitivity index can be obtained very intuitively by computation of derivatives, see e.g. [20]. The influence of the input factor on the model output $Y$ is computed, for the deterministic value of input $X_i^*$ as [13]:

$$Y'_{X_i} = \left.\frac{\partial Y}{\partial X_i}\right|_{X_i = X_i^*} \quad (2)$$

The inclusion of stochastic uncertainty of the input factor can be reached by normalizing the derivatives by the factors' standard deviations.

$$Y^{\sigma}_{X_i} = \left.\frac{\sigma X_i}{\sigma Y}\frac{\partial Y}{\partial X_i}\right|_{X_i = X_i^*} \quad (3)$$

The relations (2) and (3) provide applicable information only if the model is linear or if the range of uncertainty of the input factors is small. The interactions among factors cannot be detected.

The majority of published sensitivity analyses are either local or OAT analyses, relying on unjustified assumptions of model linearity and additivity [21]. GSA which would obviate these shortcomings, are applied by a minority of researchers [21].

Generally, GSA allow the use of model independent methods as they do not require assumptions of linearity or additivity [2, 3]. GSA can be deterministic [22], stochastic [2, 3] or based on fuzzy sets [23]. The stochastic GSA is often noticed particularly in connection with the analysis of variance (ANOVA), where the studied variance in a particular variable is partitioned into components attributable to different sources of variation [24]. More details of GSA can be found in the reviews about sensitivity analysis [21, 25-28].

In GSA, variance-based methods are commonly used [2, 4, 5] for quantifying the sensitivity of the output to the inputs in terms of a reduction in the variance of model output. Non-variance approaches to GSA are applied by a minority of researchers, and therefore it is great challenge for further research work. There exist many papers in which the objective classification of the SA method is not possible, because the term "sensitivity analysis" is generally used in the context of uncertainty analysis [21].

The paper presented deals with the study into the qualities of fairly new type of quantitative global "non-variance" GSA, called MSA in the paper, as an alternative method to be applied to classical ANOVA techniques.

## 3 Decomposition-based GSA

The comparison of the Sobol method (based on the measurement of distance between $Y$ and mean of $Y$) with SA applying contrasting functions (distance between $Y$ and median of $Y$) [7, 18] is presented in the following chapters.

### 3.1 Sensitivity measure - method of Sobol’

The one of traditional form of GSA is the Sobol’ method [4]. The basic idea of Sobol’ method [4] is to decompose the function ($X_1$, $X_2$,..., $X_M$) into terms of increasing dimensionality,

$$f(X_1, X_2,..., X_M) = f_0 + \sum_{i=1}^{M} f_i(X_i) + \\ + \sum_{1\le i\le j\le M} f_{ij}(X_i, X_j) + ... \\ + f_{1,2,...,M}(X_1, X_2,..., X_M) \tag{4}$$

If the input factors are statistically independent, then there exists a unique decomposition of (4) such that all the terms are mutually orthogonal. The decomposition (5) is not a series expansion, because it has $2^M$ (finite number) of members. The variance of the output variable $Y$ can be decomposed into:

$$V(Y) = \sum_{i=1}^{M} V_i + \sum_{1\le i\le j\le M} V_{ij} + ... + V_{1,2,...,M} \tag{5}$$

where $V_j$, $V_{jl}$,..., $V_{1,2,...M}$ denote the variance of $f_i$, $f_{ij}$, ..., $f_{1,2,...,M}$, respectively. In this approach the Sobol first-order sensitivity index for factor $X_i$ is given by:

$$S_i = \frac{V(E(Y|X_i))}{V(Y)} \tag{6}$$

In [4], Sobol proposed an alternate definition $S_i$=corr$^2$($Y$,$E$($Y$|$X_i$)) based on the evaluation of the correlation between output random variable $Y$ and the conditional random arithmetical mean $E(Y|X_i)$. The sum of all $S_i$ is equal to 1 for additive models, and less than 1 for non-additive models. The difference $1-\sum_i S_i$ is an indicator of the presence of interactions in the model.

The second and third orders sensitivity indices can be write as

$$S_{ij} = \frac{V(E(Y|X_i, X_j))}{V(Y)} \tag{7}$$

Sensitivity index $S_{ij}$ expresses the influence of doubles on the monitored output.

$$S_{ijk} = \frac{V(E(Y|X_i, X_j, X_k))}{V(Y)} + \\ - S_{ij} - S_{ik} - S_{jk} - S_k - S_j - S_i \tag{8}$$

The other higher orders sensitivity indices can be written and calculated analogously.

The Sobol indices have been widely used in many contexts. Application studies generally show a common drawback, Sobol indices are based on mean value as on the central parameter. The mean value, however, is, in general, not necessarily appropriate for each statistical purpose. It seems very intuitive that the sensitivity analysis based on another central parameter can show different results.

### 3.2 Sensitivity measure - method of contrast

A more general sensitivity measurement is based on contrasts [7]. The choice of the contrast (loss) function can determine global sensitivity indices of different types [7]. In the present paper, the SSA will be compared to the sensitivity analysis based on the measurement of distance between $Y$ and median of $Y$. Median as the alternative central parameter could involve very different variables than mean value.

The contrast (1oss) function $\psi$ associated with median can be written with parameter $\mu$ as

$$\psi(\mu) = 0.5E|Y - \mu| \tag{9}$$

and the estimator of median $\mu^*$ is given by $\mu^*$=Argmin $\psi(\mu)$. The first order median contrast index $M_i$ can be written as

$$M_i = 1 - \frac{E\left(\min_{\mu} E(\psi(\mu)|X_i)\right)}{\min_{\mu} \psi(\mu)} \tag{10}$$

The second order sensitivity index can be written as

$$M_{ij} = 1 - \frac{E\left(\min_{\mu} E(\psi(Y,\mu)|X_i, X_j)\right)}{\min_{\mu} \psi(\mu)} - M_i - M_j \tag{11}$$

The higher order quantile contrast indices can be written and calculated analogously. The sum of all indices is equal to one.

$$\sum_i M_i + \sum_i \sum_{j>i} M_{ij} + \sum_i \sum_{j>i} \sum_{k>j} M_{ijk} + ... + M_{123...M} = 1 \tag{12}$$

The decomposition (12) is similar to the Sobol decomposition described in detail and discussed, e.g., in [3], nevertheless, the significance of indices can be different.

# 4 GSA test using Ishigami function

The nonlinear and non-monotonic Ishigami function [8] is a widely used test tool in the study of sensitivity analysis techniques [9-12]. The applications SA methods described in the Chapter 2 are studied with the purpose of illustrating the properties of the new indicators $M_i$, $M_{ij}$, $M_{ijk}$.

## 4.1 Ishigami function

The mathematical expression of the Ishigami function is

$$Y = \sin x_1 + a \sin^2 x_2 + b x_3^4 \sin x_1 \tag{13}$$

where the three inputs $X_i$ ($i = 1, 2, 3$) are independent and follow the uniform distribution $U(-\pi, \pi)$. The step by step description is used for parameters $a$, $b$. The change of parameters $a$ and $b$ proceeded numerically through the intervals $a \in [0, 10]$, $b \in [0, 1]$ with steps 0.2 for $a$, and 0.02 for $b$.

The Latin Hypercube Sampling (LHS) method [29, 30] was applied to generate input random variables. Twenty one thousand LHS runs were used for the evaluation of $E(Y|X_i)$ in (6), and other twenty one LHS runs were used for the evaluation of $V(E(Y|X_i))$. The variance $V(Y)$ was evaluated with forty two thousand LHS runs. It can be noticed that SSA of (13) has the analytical solution, too [9], which is in very exact agreement with numerical results presented here. Other indices described in Chapter 2 were evaluated analogously with the above described number of LHS runs. Each index defined in 2.1 and 2.2 is computed using double loop, thus the total numerically demanding character of computation of one index is $22000^2$ LHS runs.

The general agreement occurs in case of sensitivity indices which are zero value; these are $M_3=S_3=0$ and $M_{23}=S_{23}=0$. The examples in which the general agreement does not occur are presented in Figs 1 to 10 where parametric analyses of dependence of sensitivity indices on parameters $a$, $b$ are presented. The courses of $M_1$ and $S_1$ are similar in shapes but they are not identical, the maximum agreement (minimum difference) is abs($M_1$-$S_1$)=0.028 in the point $b$=0, $a$=10. The courses $M_2$ and $S_2$ are similar in shapes, and there exist pairs of parameters $a$, $b$, where $M_2$ and $S_2$ are identical. The courses $M_{12}$ and $S_{12}$ are identical (zero) approximately for pro $b$>0.25, see Fig. 3 and Fig. 6. The agreement of indices $M_{13}$ and $S_{13}$ is illustrated in Fig. 11.

The courses of $M_{123}$ and $S_{123}$ are not similar in shape, but there exist parameters $a$, $b$, where $M_{123}=S_{123}=0$. In general, it holds that the higher is the order of indices, the greater differences are between MSA and SSA.

In general, the GSA as also the MSA, are model-free settings, because neither of them requires assumptions of additiveness or linearity [3]. Therefore, both of them can be applied with effectiveness to the reliability analysis based on many types of stochastic models [31-36]. The disadvantage of GSA is the fact that it usually leads to computationally demanding estimations, nevertheless the potential for research into the GSA method for the future is evident. It can be noted that CPU time consumption can be effectively reduced by using GSA-oriented types of meta-models [33-36].

# 5 Conclusion

It has been found by the Ishigami function sensitivity analyses that MSA provide relevant information which cannot be revealed by the SSA. First order sensitivity indices are similar in shape, or identical. Higher differences between results of MSA and SSA are identified for higher order sensitivity indices.

It is important to take into consideration that MSA has not any analogy with variance decomposition. The sensitivity measured by the MSA measures absolute distances from the median, whereas the SSA measures square power of the distance from mean value. It can be noticed that the absolute distance can be measured also between another point than median. MSA can be generalized for any upper or lower quantile.

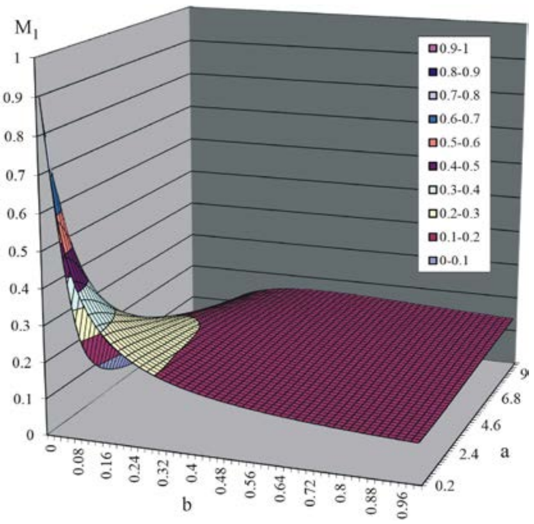


Fig.1: First order sensitivity index $M_1$

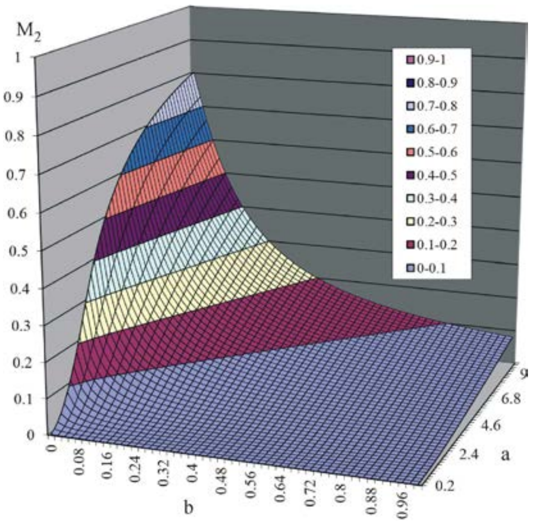


Fig.2: First order sensitivity index $M_2$

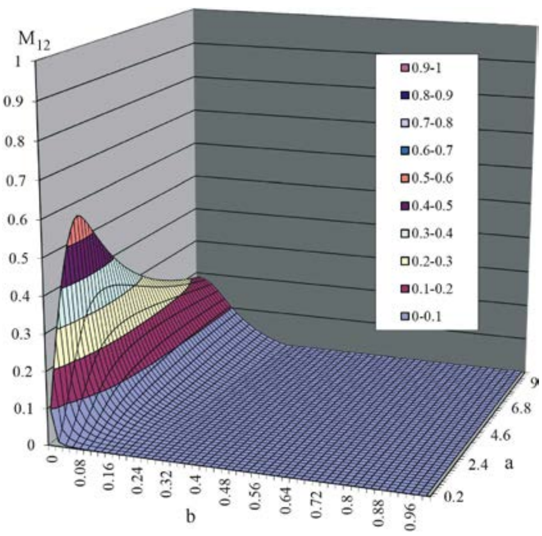


Fig.3: Second order sensitivity index $M_{12}$

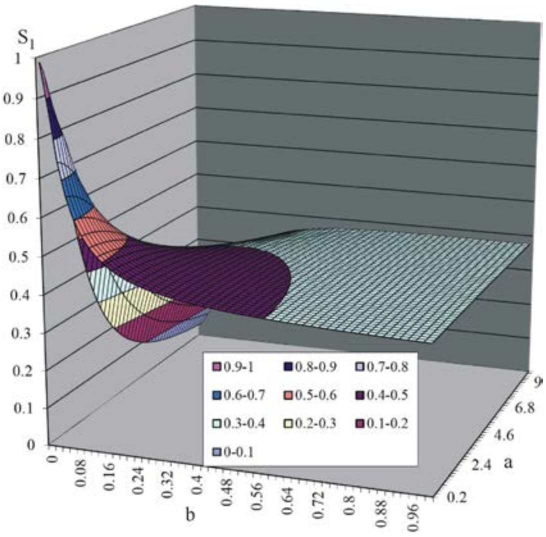


Fig.4: First order sensitivity index $S_1$

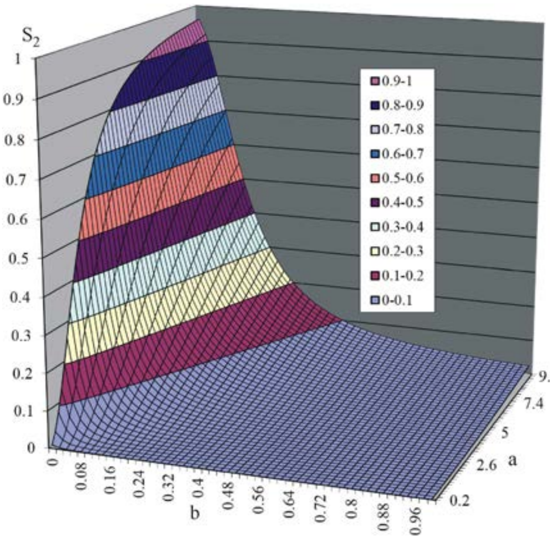


Fig.5: First order sensitivity index $S_2$

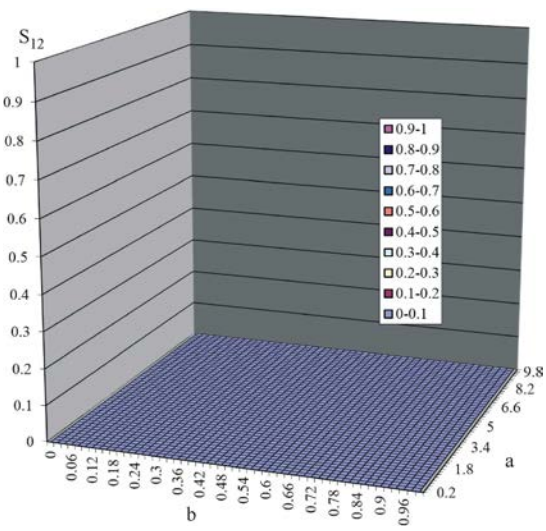


Fig.6: Second order sensitivity index $S_{12}$

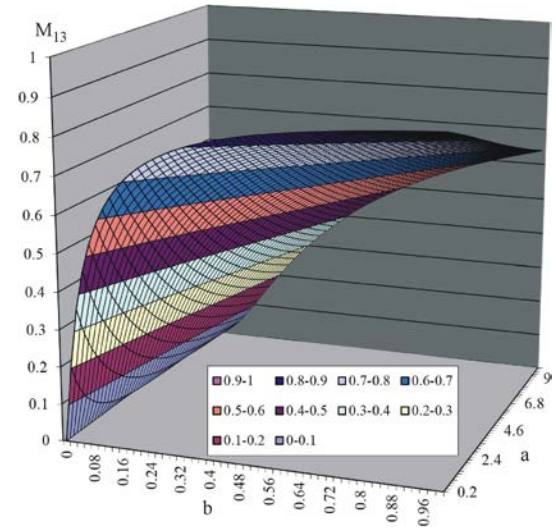


Fig.7: Second order sensitivity index $M_{13}$

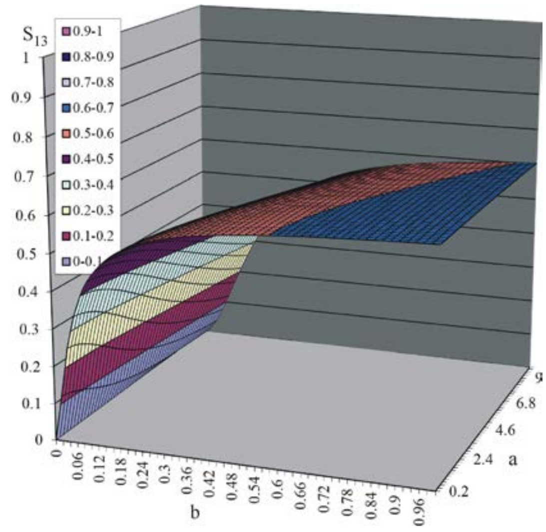


Fig.9: Second order sensitivity index $S_{13}$

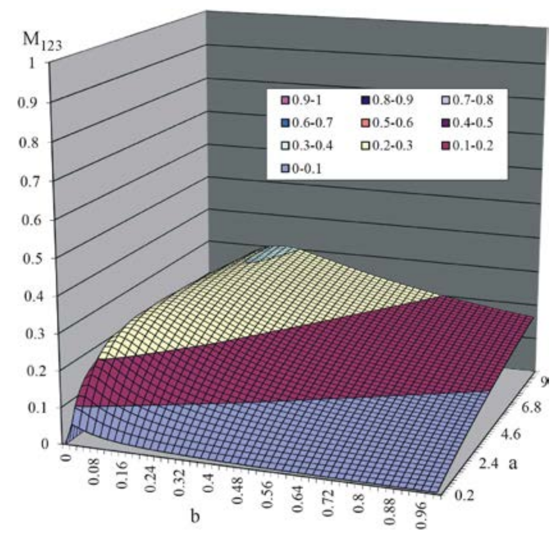


Fig.8: Third order sensitivity index $M_{123}$

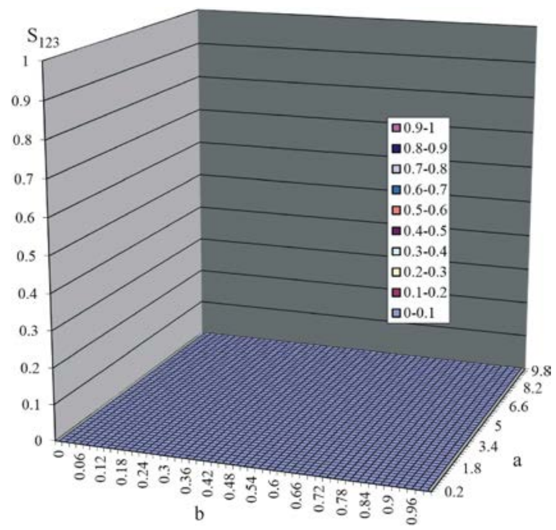


Fig.10: Third order sensitivity index $S_{123}$

MSA belongs into the group of global sensitivity analysis, because it is based on decomposition, and thus it is possible to compute all first and higher order indices (describe all interaction effects) the sum of which must be equal to one. The next research can concern the output sensitivity with respect to each factor individually and the total factor sensitivity inclusive of interactions. The research into the total effect can link up with extensive knowledge on total sensitivity indices realized within the framework of SSA [3]. The failure probability is the most important contribution to the reliability analysis. The above presented MSA can be modified for identification of crucial input quantities, which influence the failure probability to maximum, or, on the contrary, they are peripheral.

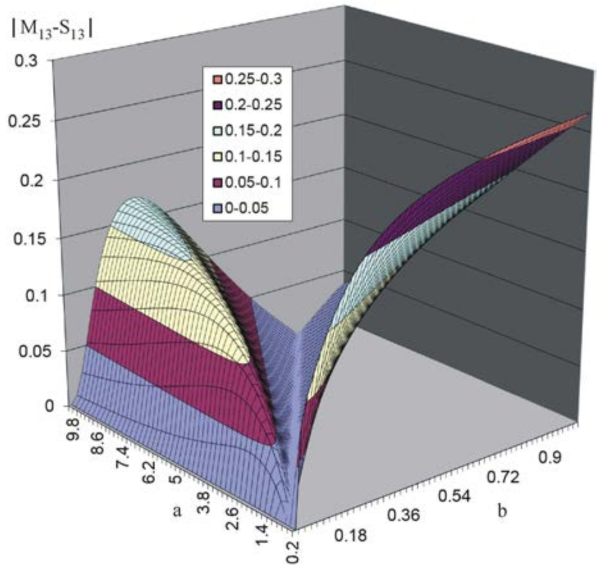


Fig.11: Difference between indexes $M_{13}$ and $S_{13}$

It shows that MSA and SSA are only part of a higher whole of the Goal Oriented Sensitivity Analysis, which merits much further work to become a practical and useful tool for the future.

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