Weighted Approximation Properties of Dunkl Analogue of Modified Szász-Mirakjan operators

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Abstract: - Our goal in this paper is to introduce a modified Szász-Mirakjan operators via Dunkl analogue and investigate their approximation properties by using weighted Korovkin theorem. At the end of this study it is concluded that our modified operator can be compared with Dunkl analogue of Szász operators.

Key-Words: - Dunkl operator; Szász-Mirakjan operators; weighted Korovkin theorem; weighted modulus of continuity.

1 Introduction

Positive linear operators are very important in approximation theory. There are a lot of operators that their Korovkin type properties of convergence are studied (for details it can be looked over [1]).
One of the most remarkable operators are Szász operators which were defined by Szász and investigated their approximation results in [7]. These operators are the following.

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \qquad (1)$$

where $x\in[0,\infty)$ and $f\in C[0,\infty)$.
The different generalizations of Szász operators were examined by many mathematicians and their approximation properties were obtained. The papers [3], [9] and [10] can be considered as an examples.
For instance, in 2002, the modified Szász-Mirakjan operators were introduced by Ispir and Atakut in [5] as

$$S_n(f;x)=\frac{1}{e^{a_nx}}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{k!}f\left(\frac{k}{b_n}\right) \qquad (2)$$

here, $x\in[0,\infty), n\in\mathbb{N}$, $\{a_n\}$ and $\{b_n\}$ are the sequences of positive numbers which were given increasing and unbounded that

$$\lim_{n\to\infty}\frac{1}{b_n}=0,\frac{a_n}{b_n}=1+O\left(\frac{1}{b_n}\right) \qquad (3)$$

In 2014, Sucu was reconstructed the Szász operators in [8] by using generalized exponential function defined by Rosenblum in [6]. Now, first let us remind followings.

$$e_\mu(x)=\sum_{k=0}^{\infty}\frac{x^k}{\gamma_\mu(k)} \qquad (4)$$

and

$$\gamma_\mu(2k)=\frac{2^{2k}k!\Gamma\left(k+\mu+{}^1/_2\right)}{\Gamma\left(\mu+{}^1/_2\right)},$$

$$\gamma_\mu(2k+1)=\frac{2^{2k+1}k!\Gamma\left(k+\mu+{}^3/_2\right)}{\Gamma\left(\mu+{}^1/_2\right)},$$

$$k\in\mathbb{N}_0, \mu>-\frac{1}{2}.$$

Noting that for the γ_μ recursion relation is

$$\gamma_\mu(k+1)=(k+1+2\mu\theta_{k+1})\gamma_\mu(k), k\epsilon\mathbb{N}_0 \qquad (5)$$

and here,

$$\theta_k=\begin{cases}0, if\ k\in 2\mathbb{N}\\ 1, if\ k\in 2\mathbb{N}+1\end{cases}. \qquad (6)$$

After bringing back these equations, here is the following operators given by Sucu.

$$S_n^*(f;x)=\frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}f\left(\frac{k+2\mu\theta_k}{n}\right) \qquad (7)$$

$\mu\geq 0, n\in\mathbb{N}, x\geq 0, f\in C[0,\infty)$ whenever the above sum converges.

In the present paper, we first construct a modification of Dunkl analogue of Szász-Mirakjan operators and investigate the weighted approximation properties of these operators. These are constructed as,

$$R_n^*(f;x) = \frac{1}{e_\mu(a_n x)} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} f\left(\frac{k+2\mu\theta_k}{b_n}\right) \quad (8)$$

where $\mu \geq 0, n \in \mathbb{N}, x \geq 0, f \in C[0,\infty)$ and with the equations (4) and (5).

2 Weighted Approximation

As it is known usual Korovkin theorem is used on finite intervals. Therefore, it is required to be used the weighted Korovkin type theorem given by Gadzhiev in [2] to acquire approximation properties of positive linear operators on infinite intervals.

We first bring to mind the definitions and theorem relative to weighted approximation.

$$B_\rho(\mathbb{R}^+) := \{f : |f(x)| \leq M_f \rho(x)\} \quad (9)$$

$$C_\rho(\mathbb{R}^+) := \{f : f \in B_\rho(\mathbb{R}^+) \cap C[0,\infty)\} \quad (10)$$

$$C_\rho^k(\mathbb{R}^+) := \left\{f : f \in C_\rho(\mathbb{R}^+) and \ \lim_{x\to\infty} \frac{f(x)}{\rho(x)} = k\,,\right\} \quad (11)$$

here k is a constant, $\rho(x) = 1 + x^2$ is a weighted function and M_f, depends only on f, is a constant. C_ρ and B_ρ are the normed spaces endowed with the norm

$$\|f\|_\rho = sup_{x\geq 0} |f(x)|/\rho(x).$$

Theorem 2.1 [2] Let $\{T_n\}$ be the sequence of linear positive operators which are mappings from C_ρ into B_ρ satisfying the conditions

$$\lim_{n\to\infty} \|T_n(t^\gamma, x) - x^\gamma\|_\rho = 0, \gamma = 0,1,2$$

then, for any function $f \in C_\rho^k$

$$\lim_{n\to\infty} \|T_n f - f\|_\rho = 0,$$

and there exists a function $f^* \in C_\rho \backslash C_\rho^k$ such that

$$\lim_{n\to\infty} \|T_n f^* - f^*\|_\rho \geq 1.$$

Lemma 2.2 *The positive linear operators R_n^* which were defined by (8) have the following properties.*

$$R_n^*(1;x) = 1 \quad (12)$$

$$R_n^*(t;x) = \frac{a_n x}{b_n} \quad (13)$$

$$R_n^*(t^2;x) = \frac{a_n^2 x^2}{b_n^2} + \left\{\frac{a_n}{b_n}\frac{1}{b_n} + 2\mu \frac{a_n}{b_n}\frac{1}{b_n}\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right\} x \quad (14)$$

$$R_n^*(t^3;x) = \frac{a_n^3 x^3}{b_n^3} + \left\{\frac{3a_n^2}{b_n^2} - 2\mu \frac{a_n^2}{b_n^2}\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right\} \frac{x^2}{b_n} + \left\{\frac{a_n}{b_n} + \frac{4\mu^2 a_n}{b_n} + \frac{4\mu a_n}{b_n}\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right\} \frac{x}{b_n^2} \quad (15)$$

$$\begin{aligned} R_n^*(t^4;x) = &\frac{a_n^4 x^4}{b_n^4} + \left\{\frac{6a_n^3}{b_n^3} + 4\mu \frac{a_n^3}{b_n^3}\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right\} \frac{x^3}{b_n} \\ &+ \left\{\frac{7a_n^2}{b_n^2} + 4\mu^2 \frac{a_n^2}{b_n^2} - 8\mu \frac{a_n^2}{b_n^2}\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right\} \frac{x^2}{b_n^2} \\ &+ \left\{(1+12\mu^2)\frac{a_n}{b_n} + 2\mu(3+4\mu^2)\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right\} \frac{x}{b_n^3} \end{aligned} \quad (16)$$

Proof. We can easily obtain (12) and (13) from (4) and (5).

Now let us prove (14).

$$R_n^*(t^2;x) = \frac{1}{e_\mu(a_n x)} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} \left(\frac{k+2\mu\theta_k}{b_n}\right)^2$$

Using (5) and from (12) and (13) we can write

$$\begin{aligned} R_n^*(t^2;x) &= \frac{a_n x}{b_n}\left\{\frac{1}{e_\mu(a_n x)} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} \frac{k+2\mu\theta_k}{b_n}\right\} \\ &+ \frac{a_n x}{b_n^2}\frac{1}{e_\mu(a_n x)} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} \left(1 + 2\mu(-1)^k\right) \\ &= \frac{a_n^2 x^2}{b_n^2} + \frac{a_n x}{b_n^2} + \frac{a_n x}{b_n^2} 2\mu \frac{e_\mu(-a_n x)}{e_\mu(a_n x)} \end{aligned}$$

and the proof is completed.

$$R_n^*(t^3;x) = \frac{1}{e_\mu(a_n x)} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} \left(\frac{k+2\mu\theta_k}{b_n}\right)^3.$$

By using (5) and from (12), (13) and (14)

$$R_n^*(t^3;x) = \frac{a_n x}{b_n^3}\frac{1}{e_\mu(a_n x)} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} \times (k+1+2\mu\theta_{k+1})^2$$

$$= \frac{a_n x}{b_n^3} \frac{1}{e_\mu(a_n x)} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} (k + 2\mu\theta_k)^2$$

$$+ \frac{2a_n x}{b_n^3 e_\mu(a_n x)} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} (k + 2\mu\theta_k) \times \left(1 + 2\mu(-1)^k\right)$$

$$+ \frac{a_n x}{b_n^3 e_\mu(a_n x)} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} \left(1 + 2\mu(-1)^k\right)^2$$

and we get the (15). With the same methods of proofs of the others we acquire (16) by using (5) and from (12), (13), (14) and (15). So,

$$R_n^*(t^4; x) = \frac{1}{e_\mu(a_n x)} \sum_{k=1}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} \left(\frac{k + 2\mu\theta_k}{b_n}\right)^4$$

$$= \frac{a_n x}{b_n^4 e_\mu(a_n x)} \sum_{k=1}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} \left(k + 2\mu\theta_k + 1 + 2\mu(-1)^k\right)^3$$

and the equation is proved.

Theorem 2.2 *Assume that* $\{R_n^*\}$ *are the sequence of linear positive operators and they are defined as in (8). Then for each function* $f \in C_\rho^k$,

$$\lim_{n\to\infty} \|R_n^*(f; x) - f(x)\|_\rho = 0. \qquad (17)$$

Proof.

i. From (12) it is easy to say that

$$\lim_{n\to\infty} \|R_n^*(1; x) - 1\|_\rho = 0.$$

ii.

$$\sup_{x\in[0,\infty)} \frac{|R_n^*(t; x) - x|}{1 + x^2} = \left(\frac{a_n}{b_n} - 1\right) \times \sup_{x\in[0,\infty)} \frac{x}{1 + x^2} = O\left(\frac{1}{b_n}\right) \sup_{x\in[0,\infty)} \frac{x}{1 + x^2}$$

and because of the (3) the proof is completed.

iii. Using the properties of supremum and from (14) we obtain the following.

$$\sup_{x\in[0,\infty)} \frac{|R_n^*(t^2; x) - x^2|}{1 + x^2} \le \left(\frac{a_n^2}{b_n^2} - 1\right) \sup_{x\in[0,\infty)} \frac{x^2}{1 + x^2} + \frac{a_n}{b_n^2}(1 + 2\mu) \sup_{x\in[0,\infty)} \frac{x}{1 + x^2}$$

then we get the proof from (3).

It is conspicuous that we get the proof of Theorem 2.2 by using above proofs and from Theorem 2.1 Consequently, Theorem 2.2 is obtained.

3 Order of Approximation

In this section we obtain the order of approximation of functions f belongs to the space C_ρ^k by the operators R_n^* on $[0, \infty)$. Taking into consideration that the usual modulus of continuity $\omega(\delta)$ does not tend to 0, as $\delta \to 0$, on infinite interval, İspir and Atakut in [5] defined a weighted modulus of continuity $\Omega(f; \delta)$ of the functions f, coorespondingly in [4]. It tends to zero, as $\delta \to 0$, on infinite interval and defined as

$$\Omega(f; \delta) = \sup_{|h|\le\delta, x\in[0,\infty)} \frac{|f(x + h) - f(x)|}{(1 + h^2)(1 + x^2)}, f \in C_\rho^k \qquad (18)$$

Lemma 3.1 [4] *Let* $f \in C_\rho^k$

i. *$\Omega(f; \delta)$ is a monotonically increasing function of $\delta \ge 0$.*

ii. *For every $f \in C_\rho^k$, $lim_{\delta\to 0}\, \Omega(f; \delta) = 0$.*

iii. *For each positive value of λ*

$$\Omega(f; \lambda\delta) \le 2(1 + \lambda)(1 + \delta^2)\Omega(f; \delta). \quad (19)$$

Using the definition of $\Omega(f; \delta)$ and the inequality (19), we have

$$|f(t) - f(x)| \le 2\left(1 + \frac{|t - x|}{\delta}\right)(1 + \delta^2)\Omega(f; \delta) \times (1 + x^2)(1 + (t - x)^2), f \in C_\rho^k, x, t \in [0, \infty). \qquad (20)$$

Theorem 3.2 *Assume that $f \in C_\rho^k$. Then ,*

$$\sup_{x\ge 0} \frac{|R_n^*(f; x) - f(x)|}{(1 + x^2)^3} \le K_\mu \left(1 + \frac{1}{b_n}\right) \Omega\left(f; \frac{1}{\sqrt{b_n}}\right) \qquad (21)$$

is obtained for a sufficiently large n plus K_μ is a constant independent of a_n and b_n.

It would be useful to give a lemma for the proof of this theorem.

Lemma 3.3 *Let the R_n^* operators defined in (8). Then, for these operators the second and the fourth moments are the followings.*

$$i.\ R_n^*((t-x)^2;x) = O\left(\frac{1}{b_n}\right) \times [x^2 + x(1+2\mu)] \quad (22)$$

$$ii.\ R_n^*((t-x)^4;x) = \left(\frac{1}{b_n}\right) \times [x^4 + (8\mu^3 + 12\mu^2 + 24\mu + 3)(x^3 + x^2 + x)] \quad (23)$$

Proof. To prove this Lemma it is used (12), (13), (14), (15) and (16).

i. $R_n^*((t-x)^2;x) = R_n^*(t^2 - 2tx + x^2;x)$

$$R_n^*((t-x)^2;x) = \left(\frac{a_n}{b_n} - 1\right)^2 x^2 + \frac{x}{b_n}\left\{\frac{a_n}{b_n} + 2\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right\} \le \frac{1}{b_n}\left(x^2 + x(1+2\mu)\right)$$

ii. $R_n^*((t-x)^4;x)$

$$= R_n^*(t^4 - 4t^3x + 6t^2x^2 - 4tx^3 + x^4;x)$$
$$= x^4\left[\frac{a_n^4}{b_n^4} - 4\frac{a_n^3}{b_n^3} + 6\frac{a_n^2}{b_n^2} - 4\frac{a_n}{b_n} + 1\right]$$
$$+ \frac{x^3}{b_n}\left[6\frac{a_n^3}{b_n^3} + 4\frac{\mu a_n^3}{b_n^3}\frac{e_\mu(-a_n x)}{e_\mu(a_n x)} - 12\frac{a_n^2}{b_n^2} + 8\frac{\mu a_n^2}{b_n^2}\frac{e_\mu(-a_n x)}{e_\mu(a_n x)} + 6\frac{a_n}{b_n} + 12\frac{\mu a_n}{b_n}\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right]$$
$$+ \frac{x^2}{b_n^2}\left[7\frac{a_n^2}{b_n^2} + 4\frac{\mu^2 a_n^2}{b_n^2} - 8\frac{\mu a_n^2}{b_n^2}\frac{e_\mu(-a_n x)}{e_\mu(a_n x)} - 4\frac{a_n}{b_n} - 16\frac{\mu^2 a_n}{b_n} - 16\frac{\mu a_n}{b_n}\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right]$$
$$+ \frac{x}{b_n^3}\left[(1+12\mu^2)\frac{a_n}{b_n} + 2\mu(3+4\mu^2)\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right]$$
$$\le \frac{1}{b_n}\{x^4 + (8\mu^3 + 12\mu^2 + 24\mu + 3)x^3 + (8\mu^3 + 12\mu^2 + 24\mu + 3)x^2 + (8\mu^3 + 12\mu^2 + 24\mu + 3)x\}$$
$$= \frac{1}{b_n}x^4 + \frac{1}{b_n}(8\mu^3 + 12\mu^2 + 24\mu + 3) \times (x^3 + x^2 + x)$$

then the proofs are completed.

Proof of Theorem 3.2

Using (20),

$$|R_n^*(f;x) - f(x)| \le 2(1+\delta^2)(1+x^2)\Omega(f;\delta) \times \Big\{1 + R_n^*((t-x)^2;x) + \frac{1}{\delta}[R_n^*((t-x)^2;x)]^{1/2} + \frac{1}{\delta}\Big[[R_n^*((t-x)^2;x)]^{1/2} \times [R_n^*((t-x)^4;x)]^{1/2}\Big]\Big\}$$

By using (22) and (23) and choosing $\delta_n = \frac{1}{\sqrt{b_n}}$ for sufficiently large n's, we obtain the proof of the theorem.

Remark. It is remarkable to say that by choosing $\mu = 0$ and $a_n = b_n = n$ our operators turn into the operators given by Szász in [7]. If it is chosen only $a_n = b_n = n$ then it will be the operators defined by Sucu in [8]. On the other hand, by selecting $\mu = 0$ then the operators turn into the operator given by Ispir and Atakut in [5].

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