show. Our focus has been to prove the *Existence* of a solution by following the spirit and letter of the original author, R. Thom. Furthermore, this Swallowtail function has only been used in a reductionist fashion at the level of the synapse, to my knowledge. Finally from the words of a specialist of E.C.T. it's the only catastrophe function that has not found any use!

Drawing on an old philosophical idea equating value with energy (Jean Ville, 1946), we identify the physiological potential as predicted (in the region of the Pons) at the level of the junction between the brain and the spinal cord with the intensity of the output to an external stimulation in a stimuli-response model. Since the novel approach of ECT takes to the letter the introduction from the outset of the observer as participant in the measurement process, we stress here the unique aim of the endeavor: To represent the interaction of the human being with his environment and letting him express it, therefore blending the subject with the object and completing what was missing from Quantum theory. In our polynomial the subjective factor Xo shows up in the numerator and leads to an objective valuation since of a finite degree.

2. The Measurement Process

The basic question we pose is: 'How does the individual process two different inputs in a row, i.e., what, if any, is the scaling function taking the same value either when its argument is z or when we add the respective values of x and y such that $\boldsymbol{z = x + y}$?

$$\boldsymbol{v(z) = v(x + y) = v(x) + v(y)} \tag{1}$$

A special case of Cauchy Equation:

$$\boldsymbol{v(xoy) = v(x) + v(y)} \tag{2}$$

The fundamental equation that mathematicians have tried to solve in different contexts, constituting the basic functional equation for a theory of measurement. This formulation put the mental process into a materialistic frame known as the Grassmanian. The important point is that the addition rule on the argument side could be different from the one on the functional space. If we look for a physical process supporting such a algebraic view, we notice that, for about 50 years now, there exists a physiological theory axiomatizing a potential function as mentioned in the Introduction and processed throughout the brain by a network of about 10^{14} synapses predicting as the only stable functional a ratio scale represented by a fifth degree potential:

$$\boldsymbol{v = \frac{1}{5}x^5 + ax^3 + bx^2 + cx} \tag{3}$$

Corresponding to dimension 3 of the control space (basic rods or cones for visual perception) and dimension 1 (x) of the state space (input stimulus) in the terminology of E.C.T.; the conditions for structural stability, together with homogeneity, give the functional:

$$\boldsymbol{v = \frac{1}{5}x^5 - \frac{2}{3}x_0^2x^3 + x_0^4x} \tag{4}$$

It has been a widespread request to introduce the observer into the observation process in order to do away with the uncertainty principle of quantum theory. In our case this is translated by a self-measurement process giving rise to an emergent unfolding of a specific thermo-dynamical potential known as Free Energy. Under the shape of a function called Swallowtail it is found in Physics in the theory of wave fronts of plasmas and as a singularity of Landau surfaces associated with certain Feynman diagrams. In classical geometric optics it is realized as a singularity of caustics. In regard to embryology, R. Thom propose to consider the extremities of the blastopore furrow as swallowtails:"...anatomy of the nervous system. First, the retraction of swallowtail, defining the motor horns, must occur in the spinal cord at the level of its junction with the brain; might this be the origin of the interlacing of the motor and sensory fibers at the level of the Pons? Second, an interpretation of the neurocoele as the 'support of infinity' of the external world is curiously confirmed in the cephalic extremity of this cavity. It is known to divide into two horns in the brain, the vestiges of the interior cavities of the peduncles of the optic vesicles during eye formation; therefore we can say that the neurocoele ends at its cephalic end in the retinas of the eyes, exactly that nervous zone specialized in a particularly precise simulation of distant phenomena." pp. 199 in Structural Stability& Morphogenesis.

This shows the striking difference between E.C.T. and Fechner's law based solely on empirical data defined only on the positive quadrant of the diagram and for a limited range away from the origin. The closest attempt at representation by the simple square integrable function, without the symmetry toward the origin, is based on the so-called Neural-Quantum hypothesis. As a special case of the functional equation, let us consider:

$$\boldsymbol{f(x + y.f(x)) = f(x).f(y)} \tag{5}$$

Recursive or self-calling.

It is the source of involved work due to its relation to continuous groups (Sophus Lie) and geometric object. Assuming the commutation of the multiplication on the R.H.S. and the convertibility of f(x) we should get, switching the roles of x and y:

$$\boldsymbol{x + yf(x) = y + xf(y) \rightarrow \frac{f(x)-1}{x} = \frac{f(y)-1}{y}} \tag{6}$$

Assuming continuity, the late ratio is a constant

$$C = 1/xo \quad (7)$$

$$f(x) = 1 + x/xo \quad (8)$$

Taking the logarithm of f(x) gives us a special solution of Cauchy Equation:

$$V(x) = log(1 + x/xo) \quad (9)$$

Basis of theorizing in this field from Bernouilli to M. Allais passing by Fechner.

We will compare in the sequel the two functional, our quintic with the Fechner Law, showing their intersection in two different points on the positive quadrant, besides the origin, in order to suggest the small difference in their respective interpretation of the empirical data generated at the time of Fechner.

In the small, the big surprise is that the very same function, the Swallowtail is predicted to occur at the level of the synapse by the same theory (E.C.T.).We quote R. Thom directly:

"I would not have carried these very hypothetical considerations so far if they did not give a good representation of the behavior of nervous activity in the nerve centers, where an excitation (called a stimulus in physiology) remains relatively canalized until it results in a well-defined motor reflex; here the role of diffusion seems to be strictly controlled, if not absent. It is known(the Tonusthal theorem of Uexkull) that, when the associated first reflex of a stimulus is inhibited by artificially preventing the movement, there is a second reflex which, if inhibited, leads to a third reflex, and so forth. This seems to suggest that diffusion of the excitation is, in fact, present, but that, as soon as the excitation find an exit in an effective reflex, all the excitation will be absorbed in the execution of this reflex. This gives a curious analogy with the mysterious phenomenon of the reduction of a wave packet in wave mechanics."Pp.149-150.

Briefly, considering the general 5^{th}degree polynomial, we see on its graph in the positive quadrant that a necessary condition for self-duality is that the local Max and Min, respectively Min and Max to each other, are confounded at the same point. It is an analog to von Neumann's Mini Max concept. With symmetry toward the origin this condition becomes a sufficient one.

This was written over 44 years ago. By now many Quantum Scientists like Henry Stapp firmly believe the following empirical facts related in Jose R. Dos Santos latest book "La clé de Salomon"pp. 452.

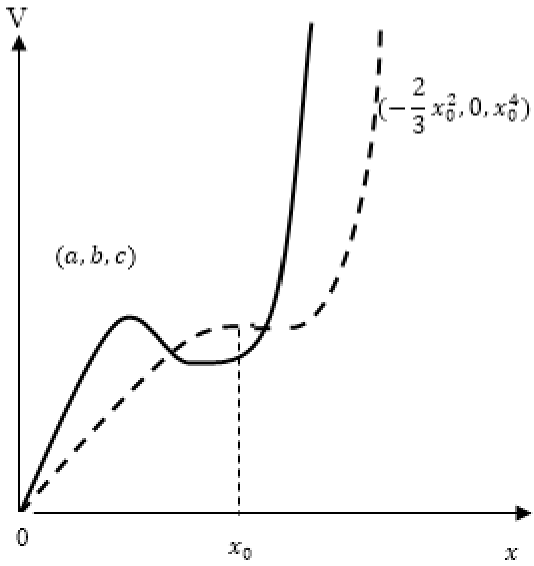

Fig.0 : the general quintic polynomial

"En général, ces sauts quantiques ne sont possibles que dans des espaces d'une largeur équivalente à sept atomes, mais, dans des cas rares, ils peuvent se produire dans des largeurs pouvant aller jusqu'a cent quatre-vingts atomes au maximum. Et bien, il se trouve, par coïncidence, ou peut-être pas, que la largeur de la fente synaptique est justement de cent quatre-vingts atomes. Or, comme les électrons sont constamment en mouvement, ils peuvent tenter des milliards de fois de traverser la membrane synaptique pendant le millième de seconde que met une synapse électriquement polarisée à s'activer, ce qui porte a 50% leur taux de réussite dans le tunnel quantique pour cette largeur. En étudiant attentivement la structure d'une synapse, on s'est aperçu que son architecture, encore une fois par une étrange coïncidence, est parfaite pour exploiter un effet de tunnel quantique. Lorsqu'une impulsion arrive à la synapse, la fente devient électriquement polarisée et c'est ce puissant champ électrique qui permet l'effet de tunnel quantique. C'est pourquoi on peut supposer que la fonction d'onde s'effondre dans les synapses lorsqu'une pensée se produit, et c'est de ce phénomène qu'émerge la conscience."

3. A Formalist Approach

Taking the derivative of our potential gives us our expression of a force field as follows:

$$v' = x^4 - 2x_0^2x^2 + x_0^4 = (x^2 - x_0^2)^2 \quad (10)$$

This algebraic picture is seen to be compatible with a Machian view of the force as the result of inertia there inducing inertia here; or in other words, depicting value as the expression of the subject's attractiveness in the presence of the object of his(her) attention.

The black box of the brain is subject to determinism due to its interaction with its physical milieu, but the point of reference to which the input is compared, representing an aggregate of the psychic phenomenon and parameterizing completely the control space of dimension 3, is an unfathomable island of free will or volition.

In computer terminology, the likely picture is one of software built in the hardware. A relevant theory of the brain as a holographic process has been advanced by K.Pribram and D.Bohm, compatible with the singularity concept, an essential ingredient to E.C.T. The second essential ingredient being the concept of analytical continuation. They are essential in the sense of allowing the representation to go from the local to the global, hence to be complete.

To that effect, I will only mention the crucial new feature of self-duality valid with our functional because the control space (regular physical one) has dimension 3, compatible with the tunneling effect in quantum theory and the collapse of the wave function in the Von Neumann measurement process. Also compatible with it is the hidden parameter theory revived by D. Bohm.

Before leaving the domain of physics, we should mention the representation of free energy in thermo-dynamical scaling where the exponent 5 of the state variable around the critical point has been identified, corresponding to dimension 3 of the parameter space. This again is an added feature of E.C.T.: Connecting its ultimate purpose of geometrizing thermodynamics with the well-known scaling hypothesis in critical phenomenon. Finally, a succinct picture of the brain, functioning like a superconductor at 0°K (Kelvin), provides an additional impetus to the overall conjecture that such a theory may have closed the gap between the physical and the mathematical continua.

Last but not least the symmetrical shape put our formula within the scope of the theorem (Poincare – Dulac) on normal forms deployment around the critical point in case of resonance, completing the picture of “Morphic Resonance” à la R. Sheldrake and the linkage of thermodynamics with the quantum level Hence the importance of verifying the empirical grounding of our quintic, a task accomplished by Pr M. Allais and Pr K. Arrow both separately. This shows the ultimate appeal of such a functional, which could be guessed at, due to its simplicity, ever since the XIX century.

Although this polynomial checks the most well known criteria for utility functions, like the S.Ross' one, and the cardinal impossibility result corresponding to Arrow famous Theorem both presented at Oslo in 1982 these are seen as sufficient conditions to qualify such a value function. The brain, with its hundred trillions synapses, could well serve the purpose as an integrator or inductor of our free energy by resonance, since the same shape is preserved from an infinitesimal scale to a much higher one.

4. An Impossibility Result

4.1 Formulation:

Let the seti = 1,2,….,n represent the different individuals parameterizing their utility functions as follows:

$$v_i = 1/5\, x^5 - 2/3\, a_i\, x^3 + a_i^2\, x, \tag{11}$$

$a_i \geq 0$

We are looking for a_0 whose utility is a weighted sum of the n individuals:

$$1/5\ x^3 - 2/3\ a_0\ x^3 + a_0^2\ x$$
$$= \sum_i a_i (\, 1/5\ x^3 - 2/3\ a_i\ x^3 + a_i^2\ x) \tag{12}$$

With

$$\sum_i a_i = 1, \qquad a_i \geq 0$$

Identifying the coefficients of the powers of x, left and right:

$$a_0 = \sum_i a_i a_i \text{ and } a_0^2 = \sum_i a_i\ a_i^2 \tag{13}$$

Therefore, a condition of compatibility:

$$(\sum_i a_i a_i)^2 - \sum_i a_i\ a_i^2 = 0 \tag{14}$$

Developing and simplifying

$$\sum_i a_i (a_i - 1) a_i^2 + 2 \sum_{i<j} a_i a_j a_i a_j = 0 \tag{15}$$

Which is recognized to be:

$$\sum_{i<j}^{n} a_i a_j (a_i - a_j)^2 = 0 \tag{16}$$

Assuming that all individuals have the same weight $(a_i = 1/n, i = 1, \ldots, n)$:

$$na_0 = \sum_i a_i \text{ and } na_0^2 = \sum_i a_i^2 \tag{17}$$

Compatibility requires:

$$\sum_{i<l}^{n} (a_i - a_j)^2 = 0$$

$$(n-1) a_n^2 - 2a_n (\sum_i^{n-1} a_i) + (n-1) \sum_i^{n-1} a_i^2 - 2 \sum_{i<j}^{n-1} a_i a_j = 0 \tag{18}$$

The discriminant

$$D'_n = (\sum_i^{n-1} a_i)^2 - (n-1)^2 (\sum_i^{n-1} a_i^2) + 2(n-1)(\sum_{i<j}^{n-1} a_i a_j)$$
$$= \sum a_i^2 + 2\sum a_i a_j - (n-1)^2 (\sum_i a_i^2)$$
$$+ (2n-2)\sum_{i<j} a_i a_j$$
$$= (2n-n^2)(\sum a_i^2) + 2n\sum_{i<j} a_i a_j$$
$$= n[(2-n)\sum a_i + 2\sum_{i<j} a_i a_j]$$
$$= n[(\sum a_i)^2 (1-n)\sum a_i^2] = n\sum_{i<j}^{n-1}(a_i - a_j)^2 \quad (19)$$

4.2 Proposition:

For n individuals whose cardinal indices are all real, we cannot have a real representative of their aggregated preferences over the whole range of variation of the variable x, unless all n individuals are identical.

4.3 Proposition:

Even when there is such a representative (for some indices ∈ C), the set of such indices, including the representative, is defined up to scalar addition and scalar multiplication (affine transformation).

4.4 Proof:

Set $a_i' = \lambda a_i + \mu$

λ, μ ∈C.

with $na_0 = \Sigma a_i$

$$na_0^2 = \Sigma a_1^2 \quad (20)$$

Then $a_0' = \lambda(\Sigma a_i / n) + \mu = \lambda a_0 + \mu$ (21)

Now

$$a_0'^2 = \Sigma a_i'^2 / n = \lambda^2 \Sigma a_i^2 / n + 2\lambda\mu \Sigma a_i / n + n.\mu^2 / n$$
$$= \lambda^2 a_0^2 + 2\lambda\mu a_0 + \mu^2 = (\lambda a_0 + \mu)^2 = (a_0')^2. \quad (22)$$

The expansion of (1) shows that the root a_n will be the average of the previous a_i's as soon as $D_n' = 0$. In order for (1) to be verified, it is readily seen that one or more of the squares under the summations would be negative.

Bat this is so only if the difference under such square is a pure imaginary number. Because of the symmetric permutation of the indices, an argument could be made that each such couple (I;j) should correspond to a complex conjugate pair. Since the a_i's are defined up to affine transformation, we need consider only the pure imaginary part.

Let us particularize for **n=3** or pose what we call the inverse problem. How to decompose a would-be representative Ω in three "constituents?"

(1') compatibility gives:

$$a_3^2 - a_3(a_1 + a_2) + a_1^2 + a_2^2 - a_1 a_2 = 0. \quad (23)$$

(2')
$$\Delta = (a_1 + a_2)^2 - 4(a_1^2 + a_2^2 - a_1 a_2) = -3(a_1 - a_2)^2. \quad (24)$$

In order to have a_3 solution∈**R,** Δ must be ≥0, i.e. $a_1 - a_2 = 2j\beta$, hence

$\mathrm{Re}a_1 = \mathrm{Re}a_2 = 0$ where $j = \sqrt{-1}$. β∈R.

Now, since a_3 and $\sum \boldsymbol{a_i}$ ∈ **R**, we must have:

$\mathrm{I}a_2 = -\mathrm{I}a_1 = \beta$ hence a_1 and a_2 are complex conjugate, then $\Delta = -3(2j\beta)^2 = 12\beta^2 \succ 0.$ (25)

We will retain the only positive root:

$$a_3 = +\beta\sqrt{3}. \quad (26)$$

Hence the average will be**:**

$$(\Sigma a_i)/3 = \Omega = (\beta\sqrt{3})/3 = \beta/\sqrt{3} \quad (27)$$

For n=4, we have a binomial in a_4:

$$3a_4^3 - 2a_4(a_1 + a_2 + a_3) + (a_1^2 + a_2^2 + a_3^2)$$
$$- 2\Sigma a_i a_j = 0$$

$$\Delta' = -8(1') = 0 \Rightarrow a_4 = (\textstyle\sum_1^a a_i)/3 = \Omega \quad (28)$$

as expected.

If we were to give a_3 a value between 0 and$\boldsymbol{\sqrt{3}}$, then (1') will be negative and **Δ'** positive, implying two different real roots for a_4.

Therefore, a decomposition over **n=4** implies an extra degree of freedom for the choice of a_3 and hence is not unique.

We see then that given **Ω** ∈ **R**, we can decompose it uniquely in exactly three constituents:

$$+ j\Omega\sqrt{3} = a_1$$
$$- j\Omega\sqrt{3} = a_2$$
$$3\Omega = a_3$$

This latter decomposition, together with the impossibility result, gives us a cardinal analogy to the ordinal result of arrow's impossibility theorem.

Our interpretation of impossibility is very different from the above.

It is that the rule of addition is not the usual arithmetic summation defined on the real numbers.

5. Comparison With The State Of The Art

We focus our attention at the positive quadrant where we take the Taylor expansion of

$$\boldsymbol{log}\left(\mathbf{1}+\left|\frac{x}{x_0}\right|\right)=\frac{x}{x_0}-\frac{x^2}{2{x_0}^2}+\frac{x^3}{3{x_0}^3}-\frac{x^4}{4{x_0}^4}+\frac{x^5}{5{x_0}^5}\quad \dots$$

(29)

Approximation valid for small $IxI < IxoI$. We seek its intersection with our quintic:

$\frac{x^5}{5}-\frac{2x^3{x_0}^2}{3}+x_0^4x$ Setting $\boldsymbol{xo=1}$ for sake of simplicity, we get:

$\frac{x^4}{4}-x^3+\frac{x^2}{2}=\mathbf{0}$ Whose first solution $x' < 1$ is $x' = \mathbf{2}-\sqrt{\mathbf{2}} = \mathbf{0.6}$ so the two curves, starting tangential from their common origin zero, have one dominating the other until $x' = \mathbf{0.6}$ then invert their dominance until their next intersection

$x'' = \mathbf{2}+\sqrt{\mathbf{2}} = \mathbf{3.4}$

Beyond 1, limit of validity of the Taylor expansion. It is pertinent at this point to recall that Fechner was influenced in picking the logarithmic by the choice of Bernouilli!

This could be compared to the early attempts of Cramer or the so-called the Engineering transformation curve. This latter is the subject of representation by A.C.M.S. (Arrow, Chenery, Minhas and Solow) in 1961.

Although the empirical attempts to find a fit stopped at the level of the cubic, even if they had guessed the fit with the symmetric quintic, a result which was possible a century earlier, there would have been no explanation by any measurement process nor behavioral interpretation neither to any link to the early attempts of the logarithmic shape of Psycho-Physics. Only the presence of E.C.T. could provide the underlying structure leading to the emergence of the quintic polynomial as an observable response to a stimulus to our senses, as sought after a century earlier by Weber and Fechner themselves. This constitutes a right turn of things since Psycho-Physics finally returned to D. Bernouilli the gift they borrowed from him.

As for the impossibility result, if we surrender to the current interpretation then it could be compared to Einstein proving the impossibility of summing the velocities via the arithmetic addition, without finding the exact algebra and the use of Minkowsky framework. Which would mean no Special Theory of Relativity and no Einstein!

6. The Missing Link

Going back to the last attempt by Pr Maurice Allais to axiomatize the utility function, I noticed 2 new axioms that Pr Allais added, after reviewing his 1952 experiments and analyzing the diverse reactions for over a generation, more precisely in “The expected utility hypotheses and the Allais paradox”,1979:

Axiom (VI): Axiom of invariance and homogeneity of the index of psychological value and

Axiom (VII): Axiom of cardinal isovariation.

After analyzing two cases, the log linear and the non log linear approximation one, he finds an excellent fit with a behavior verifying his axioms, up to approximation to the errors due to psychological introspection. About the same time, he was made aware of my first paper on the subject” The essential Tension”, which he approved wholeheartedly.

To make a long story short, there are two ingredients worth noticing:

First: the new conceptualization of ECT brings an essential feature under the name of critical point or Catastrophe point, fitting the bill for a reference point, but the novelty being that it shows on the numerator side, instead of the denominator one, retained since the time of Weber-Fechner.

Second: the basic problematic, also from the time of Weber-Fechner, as to the search for an origin and scale of the logarithmic shape inherited from the time of Bernouilli, precisely represented by his two new axioms, was addressed with success by ECT, by the hypothesis of “diffeomorphism”. This latter is a “smooth” and “reversible” feature of the representation. Hence the relevance of ECT to the problem at hand .Now, the final punch line to relate it to our problem resides in the methodology brought about by ECT. Indeed, one recalls from classical Analysis, two basic results:

1. Any function could be approximated to any degree (of approximation) by a polynomial.
2. The Taylor series expansion cannot be sure to converge. And even when it does, it’s not sure the convergence will be to the original function which led to the local Taylor expansion.

ECT formalism had, a decade earlier, solved this problem in an original fashion.

Inspired by his thorough correspondence with Waddington and Zeeman about physiology, (chreods…) R.Thom embarked on a research work leading to the emergence of potential functions in finite number, the famous seven catastrophes, corresponding to a combination of the dimensions of two spaces, the control and state spaces. It was the first time the control space was introduced from the outset in conjunction with the state space and I identified it with the physical space corresponding to the brain wiring. Hence I focused on the dimension 3 in order to have a chance to have a measurement by the human brain corresponding to an observable result. The basic procedure of ECT states that, starting from the "germ", highest term of the Taylor expansion at which one wishes to stop the approximation, a diffeomorphic change of coordinates leads to a UNIQUE representation with "determinacy", i.e. exact polynomial. And this is possible only by the introduction of the catastrophe point which insures both the "structural stability" and the "analytical continuation" i.e. the passage from the local to the global. It was the striking answer to the problem dormant since the beginning.

How could this connect with the logarithmic shape? Taking the Taylor expansion of the neural quantal function:

Log(1+x)= x $-x^2/2 + x^3/3 - x^4/4 + x^5/5$ -....

Retaining the "germ" $x^5/5$, corresponding to the dimension 3 by definition,since the germ has to have two degrees higher than the dimension of the control space, ECT leads to a function, called "self-intersection curve", defined by the cancellation of its first and second derivatives at the critical(catastrophic)point. This was the perfect topological translation of a non-decreasing function with its derivative reaching its strict minimum (zero) at the reference point, parameterizing the control space, by definition. Symmetry towards the origin completes the representation from – to + infinity.

With one stroke, we have an exact polynomial representing an observable phenomenon at the level of the brain. This was exactly what Pr Allais was predicting since before WWII, coming from Psycho-Physics. By the same token, it was immune to all the criticism that ECT had been exposed to, since they dealt with the applications in the social sciences, but never to the rigor of its mathematics. Of course, R. Thom had to pioneer new terminology, since there were many other definitions of stability. His theory was dubbed semi-open. He used Gateaux rather than Frechet Derivative and other concepts I will not encumber the reader with.

The identification of the same quintic, the Swallowtail, as emerging at the level of the synapse opens new vistas to the functioning of the brain, and more importantly the basis of the case of resonance. In the meantime, the overall picture unlocks the door to an alternative to Weber-Fechner's Just Noticeable Difference as defined in P.Buser last book to be the central problem of neuro-physiology.It also provides a new insight to the perennial Mind-Body problem.

It's remarkable that R.Thom himself thought that Consciousness emerges from the linking together of the right and left hemispheres of the brain and P. Buser still labelled the brain response as a "subjective" outcome.

7. Extensions and Conclusions

My attempt should be seen in light of L. Brentano ultimate aim to criticize the objective theories of Value (Smith,Ricardo…Marx) after synthesizing the history of such studies since Aristotle. In a nutshell, this alternative to the logarithmic function put to rest the dichotomy entertained since WW2 between the "descriptive" approach and the normative, "prescriptive" one, since the latter was built on the explicit premise that the former "doesn't exist, and could not, even in Plato' heavens". This is typically illustrated by the prominence of the Prospect Theory seeking to explain people behavior thanks to hypothetical supper additive weights "in spite of the value function" having given up on the usefulness of any such function. The same highest order (the third) differential invariant, the negative Schwarz' derivative, has been known as "extrinsic risk aversion" since J-M. Grandmont in his analysis of business cycles. It was well known to characterize the frontier of chaos in dynamical systems. Moreover, an alternative theory of general equilibrium in economy has been devised since the 70s' of last century by Y. Balasko, using specifically the mathematics of ECT for the society as a whole, without the need for concavity. This should put to rest any discomfort that these new concepts are stand alone and on the contrary this shows that they bring a solid grounding to the new burgeoning macro-economics of the last generation.

Having an expression with a finite degree, our polynomial brings a unit of measurement, common to the consumer and producer, therefore letting the exchange process take place by value equivalence, not permitted under any concave representation. Even if one could handle only pathological cases, one would need still a rod of rationality to which one could compare his observations. See my paper presented in London july 2017: "A synthetic approach to value as a standard of reference".

The same Mathematics of ECT brings about another curve, again from cognitive introspective mode, tackling the 2-dimensional input space, known as the Umbilic Hyperbolic, again fitting the empirical data with 2 corresponding reference points, to replace the well-known hypothetical Cobb-Douglass curve in classical Economics.

Because the most salient points of E.C.T. stress that there is no need to the (1) Law of excluded middle, (2) Law of contradiction.

We have shown that it fulfilled Einstein intuition: “God does not play dice”. Indeed this stems directly from the message of ECT: “And the word was made flesh” which underlies the semantic trend of Thom’s endeavor. Specifically the paradox of the double slit experiment from Quantum Mechanics is dealt with at the synaptic junction, whose function in ECT terminology is precisely “to slice”. We have thus shown how people “speak their mind up”. No wonder then that the linguistic aspect, proper to humans, has been taken up by Jean Petitot for more than 40 years now. Moreover, the force unfolded, being the perfect square of the difference of two squares the one of the state variable with the one of the control variable has all the ingredients to fulfill another dream of Einstein: joining the basic forces of nature in one unified view, one where no scalars nor constants neither fractions interfere, in an expression of the force directly proportional to the distance of the two prime entities, coming from a deterministic formalization of Thermodynamics. Indeed, the independence of E.C.T. from its substratum, together with the resonance phenomenon, unused until now even in Physics, and the logic we uncovered grounded on the dimension 3, could provide an answer to the “curse of dimensionality” nagging all recent attempts for unification.

ANNEX: About Risk

1. Introduction

The Pratt’s approach (1964) has been a seminal work on which various theoretical and empirical concepts for insurance and finance have been based. However, it reached very soon its limits for generalization, in spite of the consensus as to the importance of its methodology.

We show in the following pages that the reasons for the deadlock could rigorously be removed by simply expanding the Taylor approximation two more levels .This will result in identifying the negative sign of a third order differential invariant, the Schwarz’ derivative, as a practical quantification of the concept of systematic risk. The resulting solution set for the utility functions retained actually enjoys a stronger characteristic involving the derivative of a second order differential invariant, namely $(\frac{U''}{U'})'<0$.

This fully confirms the specific intuition of Pr Arrow as mentioned in the Pratt’s paper, concerning an alternative definition for the absolute risk aversion measure, together with its decreasing behavior. The same characteristic insures not only the convergence of the relative risk measure but also its increasing asymptotic behavior for a distinguished class of polynomials. The premium emanating from this higher level, combined with the reference points, as defined by the critical points of the utility functions, gives a descriptive account of the gambler’s behavior independent from probability considerations.

The evidence for the existence of a solution subset monotonically non-decreasing comes from the proof of the Singer’s conjecture (1978).It embeds all the polynomial functions coming from the so -called Elementary Catastrophe Theory under the same roof, namely the negative Schwarz’ derivative. This explains and generalizes the main characteristics of an initial solution, the symmetric quintic with its linguistic meaning (Non satiation axiom). This transfer of technology from the domain of hard sciences was suggestive to us because of the old idea equating value to energy, together with the existence of the stimuli response procedure, as traditional in this field of inquiry as Ramsey’s approach.

The emerging picture points toward the need for the generalization of the conventional view of preferences representation over a scale defined by two points. Indeed, the latter is seen as a special case of the Mobius transformation toward which the Schwarz’ derivative is invariant. Its importance lies in the infrastructure building the foundations of discontinuous groups for linear substitutions in the theory of Fuchsian functions. This richer structure seems to be a necessity, not only on theoretical grounds, but more so for a realistic appraisal of the applications in the physical world, involving people and economic goods with more than one attribute.

2. Pratt’s framework

We start with the general formulation of the risk premium as pioneered in Pratt’s paper:

By reducing our consideration to an actuarially neutral case $E(\tilde{z}) = 0$, and expanding U around x, in order to assess π (x,z) as the variance σ_z^2 goes to zero in a first approximation:

$$U[x + E(\tilde{z}) - \pi(x,z)] = E[U(x+\tilde{z})] \tag{1}$$

$$U(x-\pi) = u(x) - \pi u'(x) + \pi^2/2\, u''(x) + \dots \tag{2}$$

$$E[U(x+\tilde{z})] = E[u(x) + \tilde{z}\, u'(x) + (\tilde{z}^2)/2u''(x) + \dots] = U(x) + \sigma_z^2(u''(x))/2 + \dots \tag{3}$$

Or

$\frac{\pi^2}{2}u''(x) - \pi u'(x) - \frac{\sigma^2}{2}u''(x) = 0$ Considered as a second degree equation in π, we get the solutions:

$$\pi_{1,2} = \frac{\left(u' \pm \sqrt{\Delta}\right)}{u''} \text{ with } \Delta = u'^2 + \sigma^2 u''^2 \tag{4}$$

With $\sigma^2 \Rightarrow 0$ the first root goes to zero:

$$\pi_1 = (1-\sqrt{1+\sigma^2 u''^2/u'^2}).u'/u'' = -\sigma^2\ u''/2u' \tag{5}$$

The second root is the relevant one:

$$\pi_2 = 2u'/u'' + (\sigma^2\ u'')/2u' \tag{6}$$

Pushing the approximation to order three

$$U(x-\pi) = u(x) - \pi u'(x) + \pi^2/2!\,U''(x) - \pi^3/3!\,u'''(x) + \ldots \tag{7}$$

$$E[U(x+\tilde{z})] = E[u(x) + \tilde{z}u'(x) + \tilde{z}^2/2!\,u''(x) + \tilde{z}^3/3!\,u'''(x) + \ldots] = U(x) + \sigma^2 u''/2\,(x) + \mu^3\ u'''/3!\ (x) \tag{8}$$

Equating we get:

$$(\pi^3/3!)u''' - (\pi^2/2!)u'' + \pi u' + (\sigma^2/2!)\,u'' - \mu^3(u'''/3!) = 0 \tag{9}$$

with $\sigma^2, \mu^3 \Rightarrow 0$, π *factors out*:

$$(*)\ \pi^2 - \pi^3(u''/u''') + 6(u'/u''') = 0 \tag{10}$$

$$\Delta = 9u''^2/u'''^2 - 24\ u'/u''' = 9(u''^2/u'''^2)(1 - 8\,u'\,u'''/3u''^2) \tag{11}$$

In order to have a premium =/= 0 we must have $\Delta > 0$:
With roots:

$$\pi_{1,2} = 3u''/2u'''\,(1 \pm \sqrt{1-(8u'u'''/3u''^2)}) \tag{12}$$

or assuming $u'\,u''' << u''^2$

$$\pi_1 = 3u''/2u'''\ (1-1+(4u'u''')/3u''^2) = 2\,u'/u'' \tag{13}$$

$$\pi_2 = 3u''/2u'''\ (1+1-4\ (u'\,u'''/3u''^2) = 3u''/u''' - 2\,u'/u'' \tag{14}$$

Therefore this higher order brings a new premium. We will show consequently that the negativity of Schwarz' derivative of the utility function is the main characteristic of this premium. Also it is the relevant one at this level since we are looking for the root with the maximum modulus. We prove this assertion immediately:

Proposition: (Saade,1984) Whatever the signs of U',U',U"' and the premiums, we always have the following:

(I) $|\pi_1| < |\pi_2|$

(II) $Su_{\pi_2} < 0$

Where | | is absolute value and SU_{π_2} the Schwarz' derivative associated with the root π_2

From equation (*) and

$$SU = (U'''/U') - (3/2)(U''/U')^2 \tag{15}$$

Proof: Let U'>0.

$U'' < 0 < U''': \pi_1 < 0; \pi_2 < 0$

since $\Delta > 0$ we have:

$$(3U''/U''') < 8(U'/U'') < 4(U'/U'') < 2(U'/U'') \tag{16}$$

So

$$3(U''/U''') < 2(U'/U'') \Rightarrow 3(U''/U')^2 > 2(U'''/U') \tag{17}$$

$i.e. SU < 0$

Also

$$\pi_2 - \pi_1 = 3(U''/U''') - 4(U'/U'') < 0 \Rightarrow |\pi_2| > |\pi_1| \tag{18}$$

$U'' > 0 > U''': \pi_1 > 0 > \pi_2$

with $|\pi_2| > \pi_1 > 0$

From π_2:

$$3(U''/U''') < 2\,(U'/U'') \Rightarrow 3U''^2 > 2U'\,U''' \quad (\text{or } SU < 0) \tag{19}$$

$U'', U''' > 0:\ \pi_2 > \pi_1 > 0$ and SU < 0 trivially

$U'', U''' < 0: \pi_2 > 0 > \pi_1$ with $\pi_2 > |\pi_1| > 0$

and $3U''^2 > 2U'\,U'''$ so $SU < 0$ again

Remark I: For $U' < 0$ the permutation $(a \Leftrightarrow b\,; c \Leftrightarrow d)$, leads to the same conclusion

Remark II: It is easily seen that $\Delta > 0 \rightarrow SU < 0$

Therefore $SU < 0$ is a minimally necessary condition. Although there is a variety of functions verifying this property, like the exponential, the arctangent, or the sin, by far the most studied and well understood are the polynomial ones. In our case, we will see in a subsequent section, where we will show the complementarity of the present study to the classical comparative statics, the unique importance of this distinguished class to the definition of risk. To further stress the role played by the critical points and their stabilizing effects we tackle next a concept coming from the exact sciences which has proven to be unifying for non-linear Dynamics.

3. Singer's conjecture

In the domain of map iteration, May (1976) formulated a hypothesis of the concavity of the function being iterated in order to insure stable convergence. His concern, based on population dynamics, was to generalize the results already known for the quadratic mapping. Since this was a problem which already had been the concern of scholarly pursuit for generations, most notoriously by Cayley in the 19th century, Singer, drawing on the masterwork done by Julia (1917) and Fatou, substituted the negativity of the Schwarz' derivative to the concavity assumption. Taken up by Eckmann and Collet in their book on map iteration (1980) as a minimally necessary assumption, it became the standard in the field of non linear dynamics .The Singer's conjecture generalizes to every polynomial whose first derivative has real roots of any order of multiplicity the negativity of the Schwarz' derivative, already proven for real simple distinct roots.

For the proof we will use a different formulation of the Schwarz' derivative, namely:

$$SU = (U''/U')' - 1/2(U''/U')^2 \tag{20}$$

since we have the identity:

$$(U''/U')' = (U'''/U') - (U''/U')^2 \tag{21}$$

Let us pose $U' = r'^{l} s'^{m} t'^{n}$, where r', s', t' are all linear expressions of the same argument than U' and represent the derivatives of quadratic forms in x. l,m,n are all integer exponents of positive values.

$$U'' = lr'^{l-1} r'' s'^{m} t'^{n} + mr'^{l} s'^{m-1} s'' t'n$$
$$+ nr'^{l} s'^{m} t'^{n-1} t'' \tag{22}$$

so

$$(U''/U') = (lr''/r') + (ms''/s') + (nt''/t')$$
$$= (l/x + a) + (m/x + b) + (n/x + c) \tag{23}$$

And their derivatives are all negative, therefore (U''/U')' is negative, no matter the number of factors entering in the product defining U' as long as it is finite. Since we can also let l,m,n,…. Take all the positive integer values imaginable, we have therefore constructed all possible polynomials with real roots for their derivative and proven the proposition. Q.E.D.

Since $(U''/U')' < 0 \rightarrow SU < 0$, we have therefore proven the equivalence between the Arrow's risk measure and this functional form of the new solution set.

The importance of Singer's statement is primordial to our purpose since we are dealing with non-decreasing functions. Indeed, in this case, every root of U' is also root of U'', therefore is at least double root. Polynomials with negative Schwarz' derivative have many interesting properties none the least the fact that they are invariant up to Mobius transformation $(aU + b)/(cU + d)$ thereby generalizing the usual linear transformation invariance retained for utility functions in general. This latter is seen as a special case of the subset: ad-bc=1, with c=0. Another pertinent property is the preservation of the cross-ratio between 4 points. This leads directly to complete determinacy via three points.

4. Compatibility with comparative statics

The aim here is to show that replacing two differential equations (brought by setting the local and global measures equal to constants), incompatible by definition, by one differential inequality is perfectly logical if we want to define a curve over the whole field. Our way to proceed is to apply the method of variation of the constants, bringing to bear the Sturm theorem reconciling the two levels.

4.1 In the Small

we have seen that, at first approximation as the variance is small, we reached the following expression for the premium:

$$\pi_2 = 2(U'/U'') + \sigma^2 U''/2U' \tag{24}$$

In the framework of Mean Variance (Tsiang) usually retained, we see that U''/U' is the coefficient of the variance up to multiplication by a positive constant. Since it is also decreasing, as seen in the proof of Singer's conjecture, Pr Arrow's alternative definition of the local absolute risk measure is perfectly appropriate. We will see that for the asymptotic behavior of the traditional relative measure, this same property is all we need to have a stable convergence. A further remark for this subsection is that the principal term, missing from previous considerations, is $2U'/U''$, precisely representing systematic risk, after uncertainty is resolved. In order to have a positive premium to account for aversion to risk we should, at first approximation, have $U'' > 0$, assuming of course $U' > 0$. Assuming for a moment that $U'' = kU'$ therefore $U''' = k^2U'$ we, by substitution for $U'': U''' = k^2U'$ whose solution is

$$U' = A\cosh kx + B\sinh kx. \tag{25}$$

We retain only the first term for monotonically increasing solutions. The result $U = (A/k)\sinh kx$ is a generalization of the exponential function usually assumed at this local level.

It is clear that the choice of the new premium hinges on the criterion of Maximum modulus among the different possible roots. It has further the virtue of embodying in its expression the qualitative feature of the negative Schwarz derivative. It is easily seen as the solution emanating from this new level of the Taylor series. However, the negativity of the Schwarz' derivative should not be seen as dependent on the choice of the appropriate root. Indeed, as stated earlier, the positivity of the discriminant as a condition for the existence of solutions for the binomial is sufficient to guarantee the Schwarz' derivative property. Moreover,

even when the variance is small but not identically zero, it is shown that the neglected root is a higher order equivalent to the classical statics, fully compatible with the same property $(SU < 0)$, independently from the sign of U'''. we assume $U' > 0$ hereafter:
The product of the three roots is

$$\mu^2 - 3(U''/U''')\sigma^2 \tag{26}$$

The main part of the product of the roots assumed so far is (the variance being small):

$$2(U'/U'')[3(U''/U''') - 2(U'/U'')] \tag{27}$$

with the pertinent assumption of $U'U''' << U''^2$ It is clear that, for the variance small enough, the skewness is negligible. It is also a fact that when the second term in the double product is small enough, we recover the traditional premium

$$\cong (-3\sigma^2 \; u''/u''')/(6u'/u''') \tag{28}$$

$$\pi_0 = (-3\sigma^2 \; U''/U''')/(6u'/u'''-4u'^2/u''^2) \tag{29}$$

Let $U''' < 0$. Then SU<0 trivially and we will have risk aversion if $U'' < 0$

{Let $U''' > 0$ now. Then $(U'/U''') > (U'/U'')^2$, $SU < 0$ and again we reach the same conclusion. It is remarkable in this case to notice that even if the variance is not small, the effect of a positive skewness is to increase risk aversion.

4.2 In the Large

When we take the derivative of the relative measure of risk we get:

$$(-xU''/U')' = -(U''/U') + x(-U'''/U' + U''^2/U'^2)$$
$$= x(U''/U')^2 - U''/U' - xU'''/U' \tag{30}$$

Or in terms of decreasing powers of U''/U': a binomial with discriminant $\Delta = 1 + 4^2 xU'''/U'$
Since the usually assumed condition $U''' > 0$ leads directly to the positivity of the discriminant we get two roots, only one of them convergent. Indeed we see immediately that U''=0 always gives a Decreasing Relative Risk (DRR) since in this case (relative Risk Aversion=RRA)' has the opposite sign of X i.e. negative by assumption: $-X(U'''/U')$. therefore U''=0 is between the two roots, hence one root corresponds to $U'' > 0$ the other to $U'' < 0$. We can now read directly the variations of the traditional RRA with the alternative Arrow ARA(Absolute Risk Aversion) and its limits of decreasing behavior on the following diagram: X>0

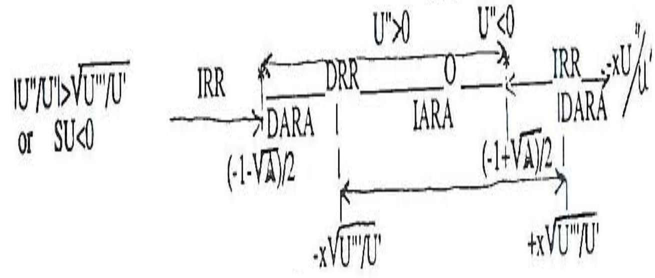

Fig.1: Variation of the different risk measures.

Nota: IRR= Increasing Relative Risk aversion
DRR= Decreasing Relative Risk
IARA= Increasing Absolute Risk Aversion
DARA= Decreasing Absolute Risk Aversion
We verify, by inspecting directly the expression of the solution as a function of the discriminant, that the only convergent solution corresponds to DARA and to $U'' > 0$. The diagram assumes not only $X > 0$ but also X increasing to infinity. Should we change the direction of convergence of X toward zero, the arrows change directions and the roots' stability as well. For $X < 0$ the reader could verify easily that the same root is still the convergent one although now the sign of U'' has changed relatively to the roots. The immediate implication of the change of the sign (U'') corresponding to the stable root with the change of sign (X) seems to indicate that a stable solution around $X = 0$ entails an expression of the second derivative negatively proportional to X. It is pertinent at this point to remark that the case $X < 0$ corresponds to debt or insurance since it relates to a negative amount, and therefore that the framework extends the classical analysis to the whole domain of variation.
We will now consider the general solution (RRA)'=0 giving, by the same token, the classes of functions either with constant RRA or their asymptotic equivalents when X tends to its boundaries of variation. From the formal expression of $(RRA)' = 0$, we get:

$$X[(U''/U')^2 - (U'''/U')] = U''/U' \tag{31}$$

$$\text{OR } X(-U''/U')' = U''/U' \tag{32}$$

$$(U''/U') = k/X$$

hence

$$U = (L/k+1)X^{K+1} + C \tag{33}$$

With k,L,C parameters.
What we need to show now is the compatibility between this direct formulation of the solution and the

previous one stemming from the analysis based on the two roots.
We see first that k=constant is equivalent to a constant discriminant. Since

$$K = XU''/U' = (1 + \text{or} - \sqrt{\Delta})/2 \qquad (34)$$

Let us set $U' = V$ and $\Delta = 1 + 4a,\ a = (x^2 U'''/U')$.
we now have $XV' - kV = 0$ to be compatible with
$X^2 V'' - aV = 0$
By differentiating (1) and multiplying by X we get:

$$X^2 V'' + XV'(1-k) = 0 \qquad (35)$$

substituting kV for XV' from (1) we have
$X^2 v'' + k(1-k)V = 0$ identical to (2) by setting $a = k(k-1)$. We see then that the same Δ gives two corresponding k (conjugate) related by:
$K(k-1) = k'(k'-1)$ or $k + k' = 1$
More importantly perhaps is the question whether we can have a classification of the solutions of (2) as the risk measure U''/U' i.e 'a' changes.
The answer is provided by the famous theorem of Sturm from differential geometry that we state without proof:
Given two differential equations

$$\text{I} \quad V'' - a(x)V = 0 \qquad (36)$$

$$\text{And} \quad \text{II} \quad W'' - b(x)W = 0 \qquad (37)$$

Or any interval $(X0, X1)$ where a>b with

$$V(X0) = W(X0) = 0 \qquad (38)$$

the solutions are such that $W(x) > V(x)$ on the same interval and as long as they are non negative.
Nota: our focus here is on the representation of one individual's preferences and not on the risk comparison between individuals.
The same theorem applies to the local level U'''-k^2U'=0 when k varies. It is becoming clear why a utility curve for an individual cannot classify him, as far as risk attitude is concerned, v/s another individual over the whole domain of definition, unless these critical points are rejected to infinity. Otherwise we have to accommodate a switch in the classification by intervals. The exponential function fulfills the first alternative, the polynomial the second one.

4.3 The Increasing Relative Risk Hypothesis

We have seen in the first part that the alternative (Arrow) DARA condition was compatible with the existence of a premium at the level two of the Taylor approximation. We have further seen that the same DARA was compatible with the stable root (convergence) for (RRA)'=0. It is legitimate to wonder whether the previous condition induces any information on the variation of RRA near the root. One can appreciate the difficulty from the diagram where the root could be approached from either side (IRR) or (DRR).
Indeed setting the condition of positivity for the discriminant:

$$U''^2/U' - U''' > 5U'''/3 \qquad (39)$$

By local consideration even at infinity (X⇒+∞) we have the following equivalence:

$$X(U''/U') = k, X^2(U'''/U') = a = k(k-1) \qquad (40)$$

So $XU'''/U'' = k - 1$

$$\Delta\pi > 0 \Leftrightarrow X[U''^2/U' - U'''] > 5XU'''/3$$
$$= 5(k-1)\,U'''/3$$

In order to have (RRA')>0 all we need is

$$5(k-1)\,U''/3 > U'' \qquad (41)$$

Assuming U''>0, it is sufficient to have: $k \geq 1 + 3/5 = 8/5$ Q.E.D
In the case of the retained solution set i.e. starting from $X^3/3$ the hypothesis is confirmed.

A Special example
The simplest shape after the cubic, fitting qualitatively, is the quintic (fig.0):

$$U' = (x^2 - x_0^2),\ U'' = 4x(x^2 - x_0^2), \qquad (42)$$

$$U''' = 4(3x^2 - x_0^2). \qquad (43)$$

$$\text{With first premium} = (x^2 - x_0^2)/2x$$

$$\text{And second one} = 3x(x^2 - x_0^2)/(3x^2 - x_0) - (x^2 - x_0^2)/2x$$

$$= (x^2 - x_0^2)(3x^2 + x_0^2)/2x\,(3x^2 - x_0^2)$$

We can recall here from St Petersburg that the gambler's behavior has been dubbed as irrational or impulsive in the region after the plateau where $U''' > 0$ .. More importantly one observes the same changes in concavity, on the positive quadrant starting from zero and higher amounts. The usual description is to call such psychological process an overestimation of small values (Risk taker) and underestimation of higher ones (Prospect Theory).
We can see that in our model the theory fit the behavioral interpretation precisely due to the introduction of the threshold Xo. Now for small values relative to Xo the theoretical possibility that one takes an insurance premium (and avoid the lottery) could be explained by the fact that the "amount is not worth the gamble". That is a region where both U'' and U''' are negative. This gives us a first level premium negative and a second level one positive. Keeping in mind that the second one, in absolute value, is always bigger than the first, and will be the dominant one, we can still detect a secondary effect due to the smaller premium. Closer to Xo, when U'''

becomes positive, while U" is still negative, the second premium becomes negative. The gambler is more inclined to seek the lottery.
At higher amounts $(X > Xo)$, all the derivatives of U as well as the two premiums are positive. However, psychologically, the opposite phenomenon is at work due precisely to the relative position of the stakes v/s the threshold Xo. The decision maker will be a risk avoider because he prefers to assure himself a comfortable certain equivalent comparatively to Xo.
On the negative quadrant, where the insurance reflex is relevant, near the origin and by symmetry, we have $U''>0$ but $U'''<0$, therefore the first premium is positive and the second one negative. He will still risk losing small amounts, but will reverse himself when the amount reaches the root of U'''. The behavior is symmetrical to the positive quadrant case… Indeed we now have a situation with a inflated fear of a loss for which he could be held accountable, thereby triggering his risk aversion reflex. Both premium are now positive since U" still positive and U''' has become so… But the more we approach-Xo the less risk averse the behavior, turning into a risk seeking region below $-Xo(U''<0, U'''>0)$ and increasingly so, precisely because once the debt is beyond the accountancy threshold (-Xo) the phenomenon of irresponsibility becomes present (ruin).
This interpretation helps to explain for high Xo, given the same stake X, the phenomenon of self-insurance. Ironically, similar behavior, but for completely opposite reason (poor enough to afford insurance), finds its explanation here also. From the insurer's point of view, the rationale to enter the contract agreement comes from the empirical observation of the high frequency of occurrence for small losses and the small one (rare events) for the big losses.
In the case where $U'=0\,(X=+\text{ or }-Xo), U''=0$ also and the only premium left is equal to μ. Consequently, the skewness becomes the unique relevant factor, independently from the sign of U''' or the value of the variance. Also, when $U''=0$ without having $U'=0$, like at the origin, the premium, again independent from the variance, is a function of the skewness and the ratio U'/U'''.
Finally the question of deductibles as well as the problem of portfolio selection would find a satisfactory answer if the RRA was high enough, like greater or equal to three. This follows from some theoretical studies coupled with empirical computation (J.M. Grandmont) and goes to show the adequacy of the degree of the representation chosen. Behaviorally speaking, however, it is clear why a higher deductible induces the imposition of a lower premium by the insurer. We see here the practical importance of knowing the reference point Xo.

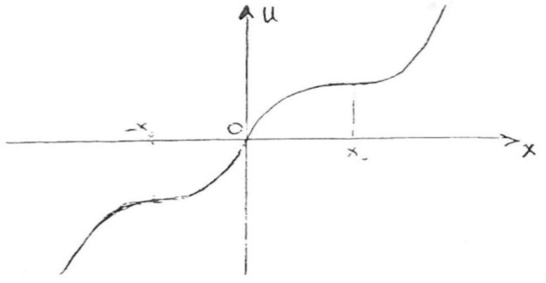

Fig.2: The importance of the reference point X_0.

4.4 The mean risk approach to St Petersburg

Following up on the mean variance approach of Harry Markowitz (1959), Paul Weirich proposes a variant to the expected utility decision rule. His version evaluates an option (O) by separating the utility of the causal consequences from the utility of the risk involved. He then considers a series of elementary gambles. The St Petersburg being the limit of this series. The mth gamble involves the coin to be tossed exactly m times. If the first heads comes on the nth toss, the gambles 2*n dollars. If heads never comes up, the gamble pays nothing. He then formulates a set of seemingly plausible assumptions toward risk in order to insure that the two partial series, corresponding to the consequences and to the risk respectively, have their principal terms converge so fast to each other that the two infinities reach a finite difference.
The chief premise of the approach is that the small chances for large prizes create big risks. In other terms aversion to risk puts a limit on the attractiveness of gambles. More Pictural, there is some number of birds in hand worth more than any number of birds in the bush.
This last image seems to precisely correspond to what the classical literature refers to as the systematic risk that we have formally quantified above.
"Suppose that all firms tend to be profitable or unprofitable together, due, for example to shifts in foreign demand. The investors would like to find insurance against a generally unfavorable development, but they cannot find it by any amount of diversification. There may indeed be individuals or organizations who would be willing at a price to pay compensation for the occurrence of the unfavorable event, but the stock market does not provide any opportunity for a mutually advantageous insurance transaction to occur". (Arrow, 1971, p. 139).
Indeed two important consequences of the intuitively appealing premise is that aversion to risk increases with the dispersion of the outcomes, and that the

greater the stakes, the greater the rate of increase in risk for subsequent increases in the stakes.

Although the mean risk approach does not assume expected utility, it does not precludes it either as P. Weirich point out in a note:

"One might conjecture that the sum on the right hand side of the equation is equal to the expected value of some utility encompassing both the utility of the consequences of O and the utility of a risk involved in O, perhaps the expected value of the utility of O itself, i.e., P(Sn)U(O,Sn), with Sn representing mutually exclusive and collectively exhaustive states of the world independent of the option O. If this conjecture were correct, our mean risk method of evaluation would be compatible with an expected utility method of evaluation. I will not explore this issue here, however".

We will show presently that the main tenets of mean risk method of evaluation could be accounted for in our classical framework.

Since our second level premium is greater, in absolute value, than the one at the first level, and the inclusion of the variance at the second approximation would oblige us to take into account the skewness and solve a third degree equation, for sake of comparison, we will focus on the first level premium with only an extra term corresponding to the variance.

For the stakes big enough, X is large relatively To Xo, hence U">0 and each term of the premium is positive. Therefore aversion to risk increases with the dispersion. Also the principal term is always increasing in X: $\left(U'/U"\right)' > 0$, as proven earlier.

Finally, the main point of convergence between the two methods is the unboundedness of the utility function. However, one has to stress that Pratt's approach is rigorously valid only as the variance tends to zero, i.e. near and at certainty. Its limitation comes from the fact that it could only handle a limited number of moments of the distribution. But the main point here is to see the compatibility of the two methods as far as the attitudes to risk are concerned. More important from our perspective is the fact that we have been able to account for the gambler's behavior regardless of the probability distributions. This seems to fit, together with the introduction of the threshold, the original approach of D. Bernoulli as well as the subsequent derivation of the probabilities from the assessment of the utilities first, assuming that the expected value hypothesis holds. The work of Friedman and Savage (1948) comes immediately to mind. In the same vein, Weibull (1982) shows the existence of the functional form, called concatenation, for the preference function, identical to the relation holding for observable phenomena in physics, in an article starting from the expected utility hypothesis and proving a dual theorem to Von Neumann's.

It might come as a surprise that our result does not use the specifies of the probability concept but rather isolates a phenomenon appearing with everyday pervasive presence of risk. This is made possible precisely because the Pratt's approach leads to the complete resolution of uncertainty. This ubiquity of risk is the raison d'etre of the futures contracts as an extension of specialized risk shifting on debt instruments in Neil S. Weiner (Stock index Futures 1984):"Although our economics textbooks have remarkably little to say about the matter, nothing is more obvious than the university of risks in the economic system… in a capitalist society, the success of new businesses and the movements of the stock market cannot be foreseen; and above all, technological progress and the development of new knowledge are by their very nature leaps into unknown". (Arrow, 1971 p. 135).

5. conclusion

The closest resemblance in this field to the concepts we have been putting together comes from Richard Jeffrey's "The Logic of Decision". Although he dismisses the St Petersburg game as a big lie, he clearly sees the need to define a scale with three points instead of two. His utilities, when referring to the lottery are unbounded. More importantly, he is dealing with a world of propositions giving a linguistic flavor to his endeavor. The method he adopts is not causal but constitute a variant from Ramsey's procedure to elicit probabilities from preferences' profiles: "Here, the elementary logical operations on propositions (denial, conjunction, disjunction) do the work which is done by the operations of forming gambles in the "classical" theory of Ramsey and Savage…but here (see chapter 6), the preference ranking of propositions determines the utility function only up to a fractional linear transformation with positive determinant… The classical case is obtained here if the preference ranking is of the sort that can only be represented by a utility function that is unbounded both above and below; and it is shown (chapter 10) that the present theory is immune to the St Petersburg paradox, so that one can reasonably be a Bayesian in the present sense and still have an unbounded utility function. "(Richard C.Jeffrey in the preface of The Logic of Decision'1983)

We could also see the relevance of Arrow-Pratt notion of certain equivalent in H. Stoll formula for parity relationship between put and call where the risk premium was missing initially.

Also the connection with Newton method in numerical analysis, where Schwarz derivative was found instrumental, brings to light the equivalence to Policy Iteration Procedure (PIP) of Dynamic Programming. A final quote should go to Tsiang: "The influence of skewness is positive. That is to say, a positive skewness of the distribution is a desirable feature, and other things being equal, a greater skewness would increase the expected utility .This is not a result peculiar to the assumption of a negative exponential utility function, but may be shown to be a general pattern of behavior towards uncertainly on the part of all risk avert individuals with decreasing or constant absolute risk aversion with respect to increases in wealth…"
"Thus if we regard the phenomenon of increasing absolute risk-aversion as absurd, we must acknowledge that a normal risk-avert individual would have a preference for skewness, in addition to an aversion to dispersion (variance) of the probability distribution of returns. It is interesting to note that Harry Markowitz once remarked that "the third moment of the probability distribution of returns from the portfolio may be connected with a propensity to gamble". Nevertheless, as we have shown above, skewness preference is certainly not necessarily a mark of an inveterate gambler, but a common trait of a risk-aversion. I cannot, therefore, go along with Markowitz in taking the view that since gambling is to be avoided, the third moment need not be considered in portfolio analysis"…
"Anyway, skewness preference must be fairly prevalent pattern of investor's behavior, for modern financial institutions provide a number of devices for investors to increase the positive skewness of the returns of their investments, for example, the organziation of limited liability joint stock companies, prearranged stop-loss sales on the stock and commodity markets, puts and calls in stocks, etc., which otherwise would perhaps not have been developed". (Tsiang, 1972, pp. 358 – 360) .

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