On the Solution of Multiplicative Inverse Eigenvalue Problem over the Field of Real Numbers

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Abstract: - In this paper, the multiplicative inverse eigenvalue problem is studied. In particular, explicit necessary conditions have been established for the problem to have a solution over the field of real numbers.

Key-Words: - Eigenvalues, inverse eigenvalue problems, matrix completion.

1 Introduction

Let $\mathcal{R}$ be the field of real numbers. Also let $\mathcal{R}[z]$ be the ring of polynomials with coefficients in $\mathcal{R}$.

Let $c(z)$ be a given monic polynomial over $\mathcal{R}[z]$ of degree n. Also let $\mathcal{L}$ be a given class of matrices over $\mathcal{R}$ of size n. Also let $\mathbf{A}$ be a given matrix over $\mathcal{R}$ of size n.

The problem studied in this paper can be stated as follows: Does there exists a matrix $\mathbf{X}$ from the given class $\mathcal{L}$ such that

$$\det[\mathbf{I}z - \mathbf{AX}] = c(z) \quad (1)$$

If so, give conditions for the existence of matrix $\mathbf{X}$ from a given class $\mathcal{L}$. This simple question is known in linear algebra as multiplicative inverse eigenvalue problem. Nevertheless, simply formulated questions do not necessarily have simple answers.

The multiplicative inverse eigenvalue problem has a long history and is probably one of the most prominent open inverse eigenvalue problems . A special case of the multiplicative inverse eigenvalue problem with application to mechanics, is originally considered by Gantmacher and Krein [1]. Friedland in [2] proved that if the matrix $\mathbf{A}$ is complex and all its principal minors are distinct from zero, then the multiplicative inverse eigenvalue problem is solvable over the field of complex numbers, in particular is showed, that there exists a diagonal complex valued matrix $\mathbf{X}$ such that the spectrum of matrix $\mathbf{AX}$ is a given set of complex numbers. The results of [2], was later generalized by Dias da Silva [3] to an arbitrary algebraically closed field. In [4] Rosenthal and Wang were derived necessary and sufficient conditions which guarantee that multiplicative inverse eigenvalue problem has a solution over an algebraically closed field for a generic set of matrices $\mathbf{A}$ and a generic set of polynomials $c(z)$ of degree n. For more complete references and applications of multiplicative inverse eigenvalue problem, we refer to the survey article by Chu [5] and the book by Chu and Golub [6]

In this paper explicit necessary conditions are established for the multiplicative inverse eigenvalue problem to have a solution over the field of real numbers.

2 Basic concepts and preliminary results

This section contains a Lemma that is needed to prove the main result of this paper and some basic notions from matrix theory which used frequently throughout the paper. A matrix $\mathbf{W}(z)$ whose elements are polynomials over $\mathcal{R}[z]$ is called polynomial matrix. A square matrix $\mathbf{V}(z)$ over $\mathcal{R}[z]$ is said to be unimodular if and only if its inverse exists and is also polynomial matrix. A conceptual tool for the study of the structure of polynomial matrices is the following standard form. Every $p \times m$ polynomial matrix $\mathbf{W}(z)$ can be expressed as

$$\mathbf{W}(z) = \mathbf{V}_1(z)\, \mathbf{M}(z)\, \mathbf{V}_2(z) \quad (2)$$

where $\mathbf{V}_1(z)$ and $\mathbf{V}_2(z)$ are unimodular matrices and the matrix $\mathbf{M}(z)$ have the following structur

$$\mathbf{M}(z) = \begin{bmatrix} \mathbf{M}_r(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (3)$$

where

$$\mathbf{M}_r(z) = \text{diag}\,[a_1(z), a_2(z), \ldots a_r(z)] \quad (4)$$

The polynomials $a_i(z)$ for $i=1,2, \ldots, r$ are termed the invariant polynomials of $\mathbf{W}(z)$ and have the following property

$a_i(z)$ divides $a_{i+1}(z)$, for $i=1,2,\ldots, r-1$
(5)

The relationship (2) is known as the Smith – McMillan form of $\mathbf{W}(z)$ over $\mathcal{R}[z]$ and the integer r is the rank of $\mathbf{W}(z)$ [7]. Let $\mathbf{A}$ be a matrix over $\mathcal{R}$ of size n. The monic polynomial c(z) over $\mathcal{R}[z]$ given by

$$\det[\mathbf{I}z - \mathbf{A}] = c(z) \quad (6)$$

is called characteristic polynomial of matrix $\mathbf{A}$ [7] and the roots of characteristic polynomial c(z) are the eigenvalues of the matrix **A.**
The following Lemma is taken from [8] and is needed to prove the main results of this paper.

Lemma 1. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices over $\mathcal{R}$ with dimensions $m \times m$ and $m \times n$ respectively, and let $a_1(z), a_2(z), \ldots a_m(z)$ be the invariant polynomials of $[\mathbf{I}z\text{-}\mathbf{A}, \mathbf{B}]$. Let c(z) be a given monic polynomial over $\mathcal{R}[z]$ of degree $(m+n)$. Then there exist matrices $\mathbf{C}$ and $\mathbf{D}$ over $\mathcal{R}$ with dimensions $n \times m$ and $n \times n$ respectively such that c(z) is the characteristic polynomial of

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

if and only if

$$[\, \Pi_{i=1}^{m} a_i(z)] \text{ divides } c(z) \quad (7)$$

An alternative proof of the Lemma 1 is given by Zaballa in [9].

3 Main results

Associated with any inverse eigenvalue problem [6] is the theory of solvability. A major effort in solvability has been to determine necessary and sufficient conditions which guarantee that the inverse eigenvalue problem has a solution. In this section, explicit necessary conditions are established for the existence of solution of multiplicative inverse eigenvalue problem, over the field of real numbers.

Theorem 1. Let $\mathbf{A}$ be a given matrix over $\mathcal{R}$ of size n. Also let rank$[\mathbf{A}]=r$. Let c(z) be a given monic polynomial over $\mathcal{R}[z]$ of degree n. Then there exists a matrix $\mathbf{X}$ from a given class $\mathcal{L}$ such that c(z) is the characteristic polynomial of the matrix $\mathbf{AX}$, only if

(a) (z^{n-r}) divides c(z)

Proof: Let rank$[\mathbf{A}]=r$. Then there exist real nonsingular matrices $\mathbf{P}$ and $\mathbf{Q}$ of size n respectively such that

$$\mathbf{A} = \mathbf{P}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix}\mathbf{Q} \quad (8)$$

we have that

$$[\mathbf{I}z - \mathbf{AX}] =$$

$$= \left[\mathbf{I}z - \mathbf{P}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix}\mathbf{QX}\right] = \mathbf{P}[\mathbf{I}z -$$

$$-\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix}\mathbf{QXP}]\mathbf{P}^{-1} \quad (9)$$

Let

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix}\mathbf{QXP}] = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \quad (10)$$

where $\mathbf{U}_1$ and $\mathbf{U}_2$ are matrices over $\mathcal{R}$ with dimensions $r \times (n\text{-}r)$ and $r \times r$ respectively. Substituting (10) into (9) we have that

$$[\mathbf{I}z - \mathbf{AX}] = \left[\mathbf{I}z - \mathbf{P}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix}\mathbf{QX}\right] =$$

$$= \mathbf{P}[\begin{bmatrix} \mathbf{I}_{n-r} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix}z - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix}]\mathbf{P}^{-1} \quad (11)$$

Suppose that there exists a matrix $\mathbf{X}$ from a given class $\mathcal{L}$ such that c(z) is the characteristic polynomial of the matrix $\mathbf{AX}$. From relationship (11) and since the matrix $\mathbf{P}$ is non-singular, we have that the matrices

$$\mathbf{AX} \text{ and } \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \quad (12)$$

are similar and thus have the same characteristic polynomial c(z). From the above it follows that

$$\det\left[\begin{bmatrix} \mathbf{I}_{n-r} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_r \end{bmatrix} z - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix}\right] = \mathrm{c}(z) \qquad (13)$$

Since the invariant polynomials of $[\mathbf{I}_{n-r}\, z,\ \mathbf{0}]$ are $a_i(z) = z$ for $i=1,2, \ldots, (n-r)$, from relationship (13) and Lemma 1 we have that

$$[\ \Pi_{i=1}^{n-r} a_i(z)] \text{ divides } c(z) \qquad (14)$$

or equivalently

$$(z^{n-r}) \text{ divides } c(z) \qquad (15)$$

Condition (a) of Theorem 1 follows from (15) and the proof is complete.
A more fundamental question [6] related to the solvability of multiplicative inverse eigenvalue problem is whether the matrix **AX** can have arbitrarily prescribed eigenvalues by adjusting the matrix **X** from a given class $\mathcal{L}$. A partial answer to this question gives the following Theorem.

Theorem 2. Let **A** be a given matrix over $\mathcal{R}$ of size *n*. Also let c(*z*) be an arbitrary monic polynomial over $\mathcal{R}[z]$ of degree *n*. Also let the class $\mathcal{L}$ contains singular and non-singular matrices. Then there exist a matrix **X** from a given class $\mathcal{L}$ such that c(*z*) is the characteristic polynomial of the matrix **AX**, only if

(a) The matrix **A** is non-singular.

Proof: Let rank[**A**]=*r* and let there exists a matrix **X** from a given class $\mathcal{L}$ such that the arbitrary monic polynomial c(*z*) $\mathcal{R}[z]$ of degree *n*, is the characteristic polynomial of the matrix **AX**. Then from Theorem 1 we have that

$$(z^{n-r}) \text{ divides } c(z) \qquad (16)$$

Since by assumption c(*z*) is arbitrary monic polynomial over $\mathcal{R}[z]$ of degree *n*, we assume without any loss of generality that the polynomial c(*z*) has no roots at the point *z*=0. Since (16) hold, the polynomial (z^{n-r}) divides c(*z*) which by assumption has no roots at the point *z*=0, if and only if

$$(z^{n-r}) = 1 \qquad (17)$$

From (17) it follows that

$$n = r \qquad (18)$$

From relationship (18) we have that the matrix **A** is nonsingular. This is condition (a) of Theorem 2 and the proof is complete.
Remark: Besides the explicit necessary conditions which are established in this paper, as far as we know, there are no published explicit necessary conditions for the solvability multiplicative inverse eigenvalue problem in its full generality. This demonstrates the originality of the contribution of Theorem 1 and Theorem 2 of this paper with respect to existing results.

4 Conclusion

The multiplicative inverse eigenvalue problem is hard and challenging open algebraic problem. In this paper explicit necessary conditions are established for the multiplicative inverse eigenvalue problem to have a real solution. We believe that our results are useful for further understanding of this challenging algebraic problem.

In our point of view our results are also useful in studying some unsolved and probably most prominent problems of linear systems theory that can be converted to the solution of multiplicative inverse eigenvalue problem over the field of real numbers. Typical examples are the arbitrary pole placement problem by constant output feedback and dynamics assignment problem by constant output feedback or by dynamic output feedback.

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