Structuring digital plane by the 8-adjacency graph with a set of walks

JOSEF ŠLAPAL
Brno University of Technology
Institute of Mathematics
Technicka 2, 616 69 Brno
CZECH REPUBLIC
slapal@fme.vutbr.cz

Abstract: In the digital plane $\mathbb{Z}^2$, we define connectedness induced by a set of walks of the same lengths in the 8-adjacency graph. The connectedness is shown to satisfy a digital analogue of the Jordan curve theorem. This proves that the 8-adjacency graph with a set of walks of the same lengths provides a convenient structure on the digital plane $\mathbb{Z}^2$ for the study of digital images.

Key Words: Digital plane, 8-adjacency graph, walk, connectedness, Jordan curve theorem

1 Introduction

Discrete mathematics has many applications not only in mathematics itself but also in numerous other disciplines. This is caused by the rapid computerization, hence discretization, of most modern technologies used in our everyday life. For example, graph theory provides powerful tools for solving various types of problems, particularly those that occur in image processing. Indeed, digital topology, a theory that was founded for studying the topological and geometric properties of digital images, is based on graph theory rather than topology (cf. [4-5, 7-8]). One of the basic tasks of digital topology is to find a convenient structure on the digital plane $\mathbb{Z}^2$ allowing us to study and process digital images. In the classical approach to digital topology, adjacency graphs with the vertex set $\mathbb{Z}^2$ are used to provide such structures, namely the well-known 4- and 8-adjacency graphs. A problem connected with adopting this approach is that neither 4-adjacency nor 8-adjacency graph allows for an analogue of the Jordan curve theorem (recall that the classical Jordan curve theorem states that a simple closed curve separates the real, i.e., Euclidean, plane into precisely two components). This problem is usually solved by using a combination of the two adjacencies and most of the graphical software is based on employing such a combination.

In 1990, E.D. Khalimsky, R. Kopperman and P.R. Meyer [2] proposed a new approach to digital topology based on using a single structure on $\mathbb{Z}^2$, the so-called Khalimsky topology. This approach, which has been developed by many authors (see, e.g., [6] and [9]), is equivalent to the one based on using a particular graph, the connectedness graph of the Khalimsky topology, for structuring $\mathbb{Z}^2$. It the present note, we build on the classical approach to digital topology. We show that, to obtain a convenient structure on the digital plane, we may employ the 8-adjacency graph together with a set of paths of the same lengths. Such a structure proved to have an advantage over the Khalimsky topology.

In [10], graphs with path partitions are studied where the path partitions considered are nothing but certain sets of walks. It was shown in [10] that path partitions provide graphs with a special connectedness that allows for using these graphs as convenient background structures on the digital spaces for the study of digital images. In the present paper, in difference to [10], we employ sets of walks, which are more general than path partitions, and we restrict our considerations to the 8-adjacency graph on $\mathbb{Z}^2$.

2 Preliminaries

For the graph-theoretic terminology, we refer to [1]. We will work with (simple) graphs, i.e., pairs $G = (V,E)$ where $V \neq \emptyset$ is a set, the so-called vertex set of $G$, and $E \subseteq \{(x,y) \in V \times V \mid x \neq y\}$ is the so-called set of edges. Two vertices $x, y \in V$ are said to be adjacent if $(x,y) \in E$. A walk in $G$ is a (finite) sequence $(x_n \mid i \leq n)$, i.e., $(x_0, x_1, \ldots, x_n)$, of vertices of $V$ such that $x_i$ is adjacent to $x_{i+1}$ whenever $i < n$; the non-negative integer $n$ is called the length of the walk $(x_n \mid i \leq n)$. A walk $(x_n \mid i \leq n)$ in $G$ is called a path if $x_i \neq x_j$ whenever $i, j \leq n$, $i \neq j$, and it is
called a circle if \( x_i \neq x_j \) whenever \( i, j < n, i \neq j \), and \( x_0 = x_n \).

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be graphs. We say that \( G_1 \) is a subgraph of \( G_2 \) if \( V_1 \subseteq V_2 \) and \( E_1 \subseteq E_2 \). If, moreover, \( V_1 = V_2 \), then \( G_1 \) is called a factor of \( G_2 \). A subgraph \( G_1 = (V_1, E_1) \) of a graph \( G_2 = (V_2, E_2) \) is said to be its induced subgraph if \( E_1 = E_2 \cap \{ \{ x, y \} : x, y \in V_1 \} \). The cartesian product of \( G_1 \) and \( G_2 \) is the graph \( G_1 \times G_2 = (V_1 \times V_2, E) \) where \( E = \{ \{ (x_1, x_2), (y_1, y_2) \} : (x_1, x_2), (y_1, y_2) \in V_1 \times V_2, \{ x_1, y_1 \} \in E_1, \{ x_2, y_2 \} \in E_2 \} \) and the strong product of \( G_1 \) and \( G_2 \) is the graph \( G_1 \odot G_2 = (V_1 \times V_2, E') \) where, for any \( \{ (x_1, x_2), (y_1, y_2) \} \) with \( (x_1, x_2), (y_1, y_2) \in V_1 \times V_2, \{ x_1, y_1 \} \in E_1, \{ x_2, y_2 \} \in E_2 \) and if and only if one of the following three conditions is fulfilled:

1. \( \{ x_1, y_1 \} \in E_1 \) and \( \{ x_2, y_2 \} \in E_2 \).
2. \( x_1 = y_1 \) and \( \{ x_2, y_2 \} \in E_2 \).
3. \( \{ x_1, y_1 \} \in E_1 \) and \( x_2 = y_2 \).

Thus, \( G_1 \times G_2 \) is always a factor of \( G_1 \odot G_2 \).

3 Graphs with walk sets

Given a graph \( G \) and a positive integer \( n \), we denote by \( P_n(G) \) the set of all walks of length \( n \) in \( G \). For every set of walks (briefly, walk set) \( B \subseteq P_n(G) \), we put

\[
B^{-1} = \{ (x_1, i) \leq n \in P_n(G); \ (x_{n-i} = i \leq n) \in B \},
\]

\[
B = \{ (x_1, i) \leq m \in P_m(G); \ 0 < m \leq n \text{ and there exists } (y_1, i \leq n) \in B \text{ such that } x_i = y_i \text{ for every } i \leq m \} \text{ (so that } B \subseteq B^* \text{), and}
\]

\[
B^* = B \cup B^{-1}.
\]

Let \( G_1 \) be a graph and \( B_1 \subseteq P_{n_1}(G_1) \) for every \( j \in \{1, 2\} \). We put \( B_1 \otimes B_2 = B \) where \( B \subseteq P_{n_1}(G_1 \otimes G_2) \) is the subset such that, for any \( ((x_0, y_0), ..., (x_n, y_n)) \) \( \in P_{n_1}(G_1 \otimes G_2), \ (x_0, y_0), ..., (x_n, y_n) \) \( \in B \) if and only if one of the following three conditions is satisfied:

1. \( \{ x_0, ..., x_n \} \in B_1 \) and \( \{ y_0, ..., y_n \} \in B_2 \).
2. \( x_0 = ... = x_n \) and \( \{ y_0, ..., y_n \} \in B_2 \).
3. \( \{ x_0, ..., x_n \} \in B_1 \) and \( y_0 = ... = y_n \).

Definition 1 Let \( G = (V, E) \) be a graph and \( B \subseteq P_n(G) \). A sequence \( C = (x_i) \leq m \) \( m > 0 \), of vertices of \( V \) is called a \( B \)-walk in \( G \) if there is an increasing sequence \( (i_k, k \leq p) \) of non-negative integers with \( i_0 = 0 \) and \( i_p = m \) such that \( i_k - i_{k-1} \leq n \) and \( (x_i, i_k - i_{k-1} \leq i \leq i_k) \in B^* \) for every \( k \) with \( 0 < k \leq p \). The sequence \( (i_k, k \leq p) \) is said to be a binding sequence of \( C \).

Definition 2 Let \( G = (V, E) \) be a graph and \( B \subseteq P_n(G) \). A set \( A \subseteq V \) is said to be \( B \)-connected in \( G \) if any two different vertices of \( G \) belonging to \( A \) can be joined by a \( B \)-walk in \( G \) contained in \( A \). A maximal \( B \)-connected set in \( G \) is called a \( B \)-component of \( G \). A \( B \)-walk in \( G \) is called a \( B \)-circle if \( x_i \neq x_j \) for all \( i, j < m \) with \( i \neq j \) and \( x_0 = x_m \).

Given a graph \( G = (V, E) \), a walk set \( B \subseteq \mathcal{P}_n(G) \), and a subset \( X \subseteq V \), we say that \( X \) separates \( G \) into precisely two \( B \)-components if the induced subgraph \( H \) of \( G \) with the vertex set \( V \setminus X \) consists of precisely two \( B \)-components, where \( B' \subseteq \mathcal{P}_n(H) \) is the walk set given by \( B' = \mathcal{P}_n(G) \cap \{ \{ x_i \} \leq n \}; \ x_i \in V \setminus X \) for all \( i < n \).

Proposition 3 Let \( G = (V_1, E_1) \) be a graph, \( B_1 \subseteq \mathcal{P}_n(G_1) \), and \( Y_j \subseteq V_j \) be a subset for every \( j \in \{1, 2\} \). If \( Y_j \) is \( B_j \)-connected in \( G_j \) for every \( i \in \{1, 2\} \), then \( Y_1 \times Y_2 \) is \( B_1 \otimes B_2 \)-connected in \( G_1 \otimes G_2 \).

Proof: Let \( Y_j = \{ y_i \} \leq p_j \) \( B_j^* \) for every \( j \in \{1, 2\} \). For each \( j \in \{1, 2\} \), there is a walk \( (x_1^j, i) \leq n_j \in B \) such that \( y_i = x_i^j \) for all \( i \leq p_j \) or \( y_i = x_i^{p_j} \) for all \( i \leq p_j \). Let \( y \in \{ y_i^1 \} \leq p_1 \times \{ y_i^2 \} \leq p_2 \) be an arbitrary element. Then, for each \( j \in \{1, 2\} \), there is a non-negative integer \( q_j, q_j < p_j \) such that \( y = (y_{q_j}, y_{q_j}^2) \). Then, clearly, either \( (y_{q_j}^1, i) \leq q_1 \) or \( (y_{q_j}^2, q_1) \leq q_1 \) is an element of \( B_j^* \) with the first member \( y_{q_j}^1 \), and the last one \( y_{q_j}^2 \). Denote this element of \( B_j^* \) by \( (x_i^j, i) \leq r_j \) and put \( C_j = ((z_i^j, i) \leq p_j \times \{ y_i^j \} \leq p_j \), with the first member \( y_j \), and with \( z_{r_j} = x_{r_j} \). Clearly, either \( (y_{q_j}^2, i) \leq q_2 \) or \( (y_{q_j}^2, i) \leq p_2 \) is an element of \( B_j^* \) with the first member \( y_{q_j}^2 \) and the last one \( x_{r_j} \). Denote this element of \( B_j^* \) by \( (x_i^j, i) \leq r_j \) and put \( C_j = ((z_i^j, i) \leq p_j \times \{ y_i^j \} \leq p_j \), with the first member \( y_j \), and with the last one \( x_{r_j} \). We have shown that any point of \( \{ y_i^1 \} \leq p_1 \times \{ y_i^2 \} \leq p_2 \) can be connected with the point \( (x_{r_j} \times x_{r_j}^2) \) by a \( B_1 \otimes B_2 \)-walk in \( G_1 \otimes G_2 \) contained in \( \{ y_i^1 \} \leq p_1 \times \{ y_i^2 \} \leq p_2 \) and \( x_{r_j} \times x_{r_j}^2 \). Thus, \( Y_1 \times Y_2 \) is \( B_1 \otimes B_2 \)-connected in \( G_1 \otimes G_2 \) whenever \( Y_1 \in B_1^* \) and \( \forall y_i \in B_2^* \).
element of $B^1_j$ whenever $k \leq q_j$. For every $j \in \{1, 2\}$, putting $C^1_j = \{x^1_k; 1 \leq i \leq i_{k+1}\}$, we get $\{x^1_i; i \leq p_j\} = \cup_{k < q_j} C^1_k$. Therefore, $\{x^1_i; i \leq p_1\} \times \{x^1_i; i \leq p_2\} = \cup_{k < q_1} \cup_{k < q_2} (C^1_k \times C^2_k)$, where $C^1_k \times C^2_k$ is $B_1 \otimes B_2$-connected in $G_1 \otimes G_2$ whenever $k < q_j$, $j = 1, 2$, by the previous part of the proof. Thus, for any $k_1 < q_1$, $(C^1_{k_1} \times C^2_{k_2})$ is a finite sequence of $B_1 \otimes B_2$-connected sets in $G_1 \otimes G_2$ with nonempty intersection of every consecutive pair of them. Hence, the set $\cup_{k_1 < q_1} \cup_{k_2 < q_2} (C^1_{k_1} \times C^2_{k_2})$ is $B_1 \otimes B_2$-connected in $G_1 \otimes G_2$. Consequently, the set $\cup_{k_1 < q_1} \cup_{k_2 < q_2} (C^1_{k_1} \times C^2_{k_2})$ is $B_1 \otimes B_2$-connected in $G_1 \otimes G_2$. Therefore, $Y_1 \times Y_2$ is $B_1 \otimes B_2$-connected in $G_1 \otimes G_2$ whenever $Y_1$ is a $B_1$-walk in $G_1$ and $Y_2$ is a $B_2$-walk in $G_2$.

Let $Y_j$ be a $B_j$-connected set in $G_j$ for every $j \in \{1, 2\}$ and let $(x_1, x_2), (y_1, y_2) \in G_1 \otimes G_2$ be arbitrary points. Then, for each $j \in \{1, 2\}$, there is a $B_j$-walk $(z^1_j; i \leq p_j)$ in $G_j$ joining the points $x_j$ and $y_j$ which is contained in $Y_j$. The set $\{z^1_j; i \leq p_1\} \times \{y^2_j; i \leq p_2\}$ contains the points $(x_1, x_2)$ and $(y_1, y_2)$ and is a $B_1 \otimes B_2$-connected set in $G_1 \otimes G_2$ by the previous part of the proof. Thus, there is a $B_1 \otimes B_2$-walk $C$ in $G_1 \otimes G_2$ joining the points $(x_1, x_2)$ and $(y_1, y_2)$ which is contained in $\{z^1_j; i \leq p_1\} \times \{y^2_j; i \leq p_2\} \subset Y_1 \times Y_2$. $C$ is contained in $Y_1 \times Y_2$, too, and so $Y_1 \times Y_2$ is $B_1 \otimes B_2$-connected in $G_1 \otimes G_2$. The proof is complete. 

4 8-adjacency graph with a set of walks

Recall that the 8-adjacency graph on $\mathbb{Z}^2$ is the graph $(\mathbb{Z}^2, A_8)$ where $A_8 = \{(x_1, y_1), (x_2, y_2); (x_1, y_1), (x_2, y_2) \in \mathbb{Z}^2, \max\{|x_1 - x_2|, |y_1 - y_2|\} = 1\}$. In the sequel, $G$ will denote the 8-adjacency graph on $\mathbb{Z}^2$. It is evident that $G = Z_2 \otimes Z_2$ where $Z_2$ is the 2-adjacency graph on $\mathbb{Z}$, i.e., the graph $(\mathbb{Z}, A_2)$ where $A_2 = \{(p, q); p, q \in \mathbb{Z}, |p - q| = 1\}$.

Let $B \subset P_2(Z_2)$ be the set given as follows:

$B = \{x_i; i \leq 2\} \subset P_2(Z_2)$; there exists an odd number $i \in \mathbb{Z}$ such that $x_i = 2l + i$ for all $i \leq 2$ or $x_i = 2l - i$ for all $i \leq 2$.

Using results of the previous section, we may propose a new structure on the digital plane convenient for the study of digital images. Such a structure is obtained as the 8-adjacency graph on $\mathbb{Z}^2$ (i.e., the strong product of two copies of the 2-adjacency graph on $\mathbb{Z}$) with the walk set given by the strong product of two copies of the walk set $B$.

Since the digital line $\mathbb{Z}$ is evidently $B$-connected in the graph $Z_2$, the digital plane $\mathbb{Z}^2$ is $B$-connected in the 8-adjacency graph $G$ by Proposition 3.

We denote by $H$ the factor of the 8-adjacency graph $G$ whose edges are those $\{(x_1, y_1), (x_2, y_2)\} \in A_8$ that satisfy one of the following four conditions for some $k \in \mathbb{Z}$:

- $x_1 - y_1 = x_2 - y_2 = 4k$,
- $x_1 + y_1 = x_2 + y_2 = 4k$,
- $x_1 = x_2 = 4k$,
- $y_1 = y_2 = 4k$.

A section of the graph $H$ is demonstrated in Figure 1 where only the vertices $(4k, 4l)$, $k, l \in \mathbb{Z}$, are marked out (by bold dots) and thus, on every edge drawn between two such vertices, there are 3 more (non-displayed) vertices, so that such an edge represents 4 edges in the graph $H$.

**Definition 4** A $B \otimes B$-circle $J$ in the graph $G$ is said to be *fundamental* if it is a circle in $H$ and, whenever $(4k + 2, 4l + 2) \in J$ for some $k, l \in \mathbb{Z}$, one of the following two conditions is true:

\[
((4k + 2) - 1, (4l + 2) + 1, (4k + 2) - 1, (4l + 2) + 1) \subseteq J,
\]

\[
((4k + 2) + 1, (4l + 2) + 1, (4k + 2) + 1, (4l + 2) + 1) \not\subseteq J.
\]

The fundamental circles are just the circles in $H$ that turn at the bold vertices only as demonstrated in Figure 1.

**Theorem 5** If $J$ is a fundamental circle in the graph $G$, then $J$ separates $G$ into precisely two $B \otimes B$-components, one finite and the other infinite, such that the union of any of them with $J$ is a $B \otimes B$-connected set in $G$.

**Sketch of proof:** For every point $z = (4k + 2, 4l + 2)$, $k, l \in \mathbb{Z}$, each of the following four subsets of $\mathbb{Z}^2$ is called a fundamental triangle (given by $z$):
\{(r, s) \in \mathbb{Z}^2; 4k \leq r \leq 4k + 4, 4l \leq s \leq 4l + 4, s \leq r + 4l - 4k\},
\{(r, s) \in \mathbb{Z}^2; 4k \leq r \leq 4k + 4, 4l \leq s \leq 4l + 4, s \geq 4k + 4l + 4 - r\},
\{(r, s) \in \mathbb{Z}^2; 4k \leq r \leq 4k + 4, 4l \leq s \leq 4l + 4, s \geq r + 4l - 4k\},
\{(r, s) \in \mathbb{Z}^2; 4k \leq r \leq 4k + 4, 4l \leq s \leq 4l + 4, s \leq 4k + 4l + 4 - r\}.

Clearly, the edges of any fundamental triangle form a \(B \otimes B\)-circle in \(G\). It may be shown that every fundamental triangle is \(B \otimes B\)-connected in \(G\) and so is also every set obtained from a fundamental triangle by subtracting some of its edges.

We will say that a (finite or infinite) sequence \(S\) of fundamental triangles is a \emph{tiling sequence} if the members of \(S\) are pairwise different and every member of \(S\), excluding the first one, has an edge in common with at least one of its predecessors. Given a tiling sequence \(S\) of fundamental triangles, we denote by \(S'\) the sequence obtained from \(S\) by subtracting, from every member of the sequence, all its edges that are not shared with any other member of the sequence. By the first part of the proof, for every tiling sequence \(S\) of fundamental triangles, the set \(\bigcup\{T; T \in S\}\) is \(B \otimes B\)-connected in \(G\) and the same is true for the set \(\bigcup\{T; T \in S'\}\).

Let \(J\) be a fundamental circle in the graph \(G\). Then \(J\) constitutes the border of a polygon \(S_F \subseteq \mathbb{Z}^2\) consisting of fundamental triangles. More precisely, \(S_F\) is the union of some fundamental triangles such that any pair of them is disjoint or meets in just one edge in common. Let \(U\) be a tiling sequence of the fundamental triangles contained in \(S_F\). Since \(S_F\) is finite, \(U\) is finite, too, and we have \(S_F = \bigcup\{T; T \in U\}\). It may be shown that every fundamental triangle \(T \in U\) is \(B \otimes B\)-connected in \(G\). Thus, \(S_F\) is \(B \otimes B\)-connected in \(G\), too. Similarly, \(U'\) is a finite sequence with \(S_F - J = \bigcup\{T; T \in U'\}\) and we may show that every member of \(U'\) is \(B \otimes B\)-connected in \(G\). It follows that \(S_F - J\) is \(B \otimes B\)-connected in \(G\), too.

Further, let \(V\) be a tiling sequence of fundamental triangles which are not contained in \(S_F\). Since the complement of \(S_F\) in \(\mathbb{Z}^2\) is infinite, \(V\) is infinite, too. Put \(S_I = \bigcup\{T; T \in V\}\). It may be shown that every fundamental triangle \(T \in V\) is \(B \otimes B\)-connected in \(G\), so that \(S_I\) is \(B \otimes B\)-connected in \(G\), too. Similarly, \(V'\) is a finite sequence with \(S_I - J = \bigcup\{T; T \in V'\}\) and we may show that every member of \(V'\) is \(B \otimes B\)-connected in \(G\). Therefore, \(S_I - J\) is \(B \otimes B\)-connected in \(G\), too.

It may easily be seen that every \(B \otimes B\)-walk \(C = (z_i; i \leq k), k\) a positive integer, in \(G\) connecting a point of \(S_F - J\) with a point of \(S_I - J\) meets \(J\) (i.e., meets an edge of a fundamental triangle which is contained in \(J\)). Therefore, the set \(\mathbb{Z}^2 - J = (S_F - J) \cup (S_I - J)\) is not \(B \otimes B\)-connected in \(G\).

We have shown that \(J\) separates \(G\) into precisely two components \(S_F - J\) and \(S_I - J\). \(S_F - J\) finite and \(S_I - J\) infinite, with \(S_F\) and \(S_I\) \(B \otimes B\)-connected in \(G\).

\section{Conclusion}

We proposed a new structure of connectedness in the digital plane given by a sets of paths in the 8-adjacency graph. In Theorem 5, we showed that fundamental circles in the graph \(H\) (i.e., circles in the graph demonstrated in Figure 1) separate the digital plane \(\mathbb{Z}^2\) into precisely two components (with respect to the connectedness given by the set of paths in the 8-adjacency graph) so that they may be considered to be digital analogues of the Jordan curves in the Euclidean plane. The fundamental circles may consist of horizontal, vertical and diagonal parts and their advantage over the digital Jordan curves in the Khalimsky topology determined in [2] is that they may turn at the acute angle \(\frac{\pi}{4}\) - see Figure 1 (in the Khalimsky topology, the Jordan curves may never turn at the acute angle \(\frac{\pi}{4}\)). Since digital Jordan curves represent borders of objects in digital images (cf. [3]), the proposed connectedness structure in the digital plane given by the set \(B\) of walks in the 8-adjacency graph provides a richer variety of applications than the one provided by the Khalimsky topology.

\textbf{Acknowledgements:} This work was supported by Brno University of Technology from the Specific Research Programme, project no. FSI-S-17-4464.

\textbf{References:}


