A concise guide to finitary and infinitary levels of expressive power of first-order logic

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Abstract: In this work, we give a short review of recent results concerning expressive power of first-order logic. We characterize the isomorphism type of the Tarski-Lindenbaum algebra of predicate calculus of a finite rich signature under finitary and infinitary semantic layers of model-theoretic properties. Results presented in this work characterize two levels of expressive power of first-order predicate logic.

Author's statement (*just for the reviewer*): Currently, a number of absolutely new results is obtained by the author. They characterize expressive power of first-order logic. A few preliminary works have been already published; a few must appear soon; a series of works is prepared for publication. Purpose of this work is to give a short review of used technical methods and basic results in this direction obtained recently by the author.

Key–Words: First-order logic, incomplete theory, Tarski-Lindenbaum algebra, model-theoretic property, semantic type of a theory.

Introduction

First-order predicate logic represents a formal basis of mathematics. Furthermore, this logic is used in investigations on artificial intelligence and some applied branches of formal logic. In this review, we give a definition to the concept of a model-theoretic property that is adequate to real practice of investigations in model theory. Based on this definition, we characterize two levels of expressive power of first-order predicate logic. These results can be regarded as an approach to the solution of a general question of expressive power of formulas of first-order logic.

0 Preliminaries

We consider theories in first-order predicate logic *with equality* and use general concepts of model theory, algorithm theory, constructive models, and Boolean algebras that can be found in [1], [2], and [3]. Special concepts used in this paper can be found in [4]. Generally, *incomplete theories* are considered. In the work, the signatures are considered only, which admit Gödel's numbering of the formulas. Such a signature is called *enumerable*. Normally, the notion of first-order definability is considered in model theory. Along with this, there is a thinner concept of first-order $\exists\cap\forall$-definability, that is, presentability via formulas that are equivalent to both an $\exists$-formula and a $\forall$-formula in the theory.

The following notations are used: $FL(\sigma)$ is the set of all formulas of signature σ, $FL_k(\sigma)$ is the set of all formulas of signature σ with free variables $x_0,\ldots,x_{k-1}$, $SL(\sigma)$ is the set of all sentences of signature σ. By *GR*, we denote *graph theory* of signature $\sigma_{GR}=\{\Gamma^2\}$ defined by axioms $(\forall x)\neg\Gamma(x,x)$, $(\forall x)(\forall y)[\Gamma(x,y)\leftrightarrow\Gamma(y,x)]$, while *GRE* is an extension of *GR* defined by the extra axioms $(\exists x,y)\Gamma(x,y)$ and $(\exists x,y)\big[(x\neq y)\,\&\,\neg\Gamma(x,y)\big]$.

A finite signature is called *rich*, if it contains at least one n-ary predicate or function symbol for $n\geqslant 2$, or two unary function symbols. For signatures σ_1 and σ_2, σ_1 is *covered by* σ_2, written $\sigma_1\leqslant\sigma_2$, if there is a mapping $\lambda:\sigma_1\to\sigma_2$ such that we have for all $\mathfrak{s}\in\sigma_1$: (a) $\mathfrak{s}$ and $\lambda(\mathfrak{s})$ are symbols of the same type (either predicates, or functions, or constants); (b) arity of $\mathfrak{s}\leqslant$ arity of $\lambda(\mathfrak{s})$ whenever $\mathfrak{s}$ is either a predicate or function symbol. The following statement takes place for an arbitrary finite signature σ:

$$\begin{aligned}&\sigma \textit{ is rich} \Leftrightarrow \\ &\{P^2\}\leqslant\sigma \textit{ or } \{f^1,h^1\}\leqslant\sigma \textit{ or } \{g^2\}\leqslant\sigma.\end{aligned} \qquad (0.1)$$

We denote by σ^∞ a fixed maximum large enumerable signature. Namely, σ^∞ contains countably many constant symbols, symbols of propositional variables, and predicate and function symbols of each arity $n\geqslant 1$. We use a fixed Gödel numbering Φ_k, $k\in\mathbb{N}$, for the set of sentences of a fixed signature σ, and Φ^{∞}_k, $k\in\mathbb{N}$, for the set of sentences of the infinite signature σ^{∞}. Let σ be a signature, and Σ be a subset of $SL(\sigma)$. Denote by $[\Sigma]^{\sigma}$ a theory of signature σ generated by Σ as a set of its axioms. By $[\Sigma]^{\star}$, we denote a theory of a signature $\sigma'\subseteq\sigma$ generated by the set Σ as a set of its axioms, where σ' contains only those symbols from σ that occur in formulas of the set Σ. Based on the Post numbering of the family of all computably enumerable sets W_n, $n\in\mathbb{N}$, [2], we construct an effective numbering for the classes of all theories. There are two versions of indices. If a theory T of signature σ is defined by the axioms $\{\Phi_i\,|\,i\in W_m\}$, the number m is called a *computably enumerable index* or simply *c.e. index* of T. Now, let $m\in\mathbb{N}$. Consider the set of axioms $\Sigma=\{\Phi^{\infty}_i\,|\,i\in W_m\}$ and construct a theory $T=[\Sigma]^{\star}$. The number m is called a *weak computably enumerable index* or simply *weak c.e. index* of T. As for a finitely axiomatizable theory F, it is defined by a single axiom Φ. A Gödel number n of this sentence Φ is said to be a *Gödel number* or *strong index* of F.

Theories T and S of signatures τ and σ such that $\tau\cap\sigma=\varnothing$ are called *first-order* $\exists\cap\forall$*-equivalent* or *algebraically isomorphic*, written as $T\approx_a S$, if there is a theory H of signature $\tau\cup\sigma$ such that $T=H\restriction\tau$, $S=H\restriction\sigma$; moreover, σ-symbols are $\exists\cap\forall$-definable in H relative to τ-symbols via an effective scheme of expressions, while τ-symbols are $\exists\cap\forall$-definable in H relative to σ-symbols via an effective scheme of expressions. The theories T and S are called *first-order equivalent* or *isomorphic*, written as $T\approx S$, if similar relations are satisfied with normal first-order definability instead of $\exists\cap\forall$-definability. It is obvious that $T\approx_a S\Rightarrow T\approx S$, for all theories T and S.

We introduce a *primitive* (technical) version of the concept of a model-theoretic property. Denote by $\mathbb{C}$ the class of all complete theories of arbitrary enumerable signatures. By a *property of model type*, or *model property* for short, we mean an arbitrary class $\mathfrak{p}$ of complete theories of enumerable signatures that is closed under isomorphisms of the form

$$T_0\approx T_1\;\Rightarrow\;(T_0\in\mathfrak{p}\Leftrightarrow T_1\in\mathfrak{p}),\ \textit{for all}\ T_0,T_1\in\mathbb{C}.$$

By a *property of algebraic type*, or *algebraic property* for short, we mean an arbitrary class $\mathfrak{p}$ of complete theories of enumerable signatures that is closed under isomorphisms of the form

$$T_0\approx_a T_1\;\Rightarrow\;(T_0\in\mathfrak{p}\Leftrightarrow T_1\in\mathfrak{p}),\ \textit{for all}\ T_0,T_1\in\mathbb{C}.$$

We denote by *ML* the set of all properties of model type and by *AL* the set of all properties of algebraic type. An inclusion $ML\subseteq AL$ is obvious. A subset $L\subseteq ML$ is called a *semantic layer of model properties*, while a subset $L\subseteq AL$ is called a *semantic layer of algebraic properties*. The inclusion $ML\subseteq AL$ ensures that any model semantic layer can be regarded as an algebraic semantic layer.

Let L be a semantic layer of model-theoretic properties. For theories T and S, an entry $T\equiv_L S$ will denote that there is a computable isomorphism $\mu:\mathcal{L}(T)\to\mathcal{L}(S)$ between the Tarski-Lindenbaum algebras of these theories such that, for any complete extension T' of T and corresponding complete extension S' of S, $S'=\mu(T')$, the theories T' and S' have identical model-theoretic properties within the layer L. In the case when $T\equiv_L S$ holds according to this definition, we say that T and S are *semantically similar* under the semantic layer L.

Let $\mu:\mathcal{L}(T)\to\mathcal{L}(S)$ be an isomorphism of the Tarski-Lindenbaum algebra of a theory T of signature τ in the Tarski-Lindenbaum algebra of a theory S of signature σ. We can naturally define a correspondence between the extensions of these theories (including both complete and incomplete theories) by the following rules:

$$\begin{array}{ll}\texttt{(a)} & T'\supseteq T\mapsto S'\supseteq S,\ \textit{by rule}\ S'=\mu(T'),\\ \texttt{(b)} & S'\supseteq S\mapsto T'\supseteq T,\ \textit{by rule}\ T'=\mu^{-1}(S').\end{array}\qquad(0.2)$$

Lemma 0.1. *Let μ be an isomorphism between $\mathcal{L}(T)$ and $\mathcal{L}(S)$. The following statements take place*:

(a) *the mappings $T'\mapsto\mu(T')$ and $S'\mapsto\mu^{-1}(S')$ defined by the rules* (0.2) *are mutually inverse to each other*,
(b) *for any theory $T'\supseteq T$, T' is a complete extension of T $\Leftrightarrow$ $\mu(T')$ is a complete extension of S*,
(c) *theory T' is an f.a. over T $\Leftrightarrow$ theory $\mu(T')$ is an f.a. over S*,
(d) *theory T' is a c.a. over T $\Leftrightarrow$ theory $\mu(T')$ is a c.a. over S*,

PROOF. Use methods of Boolean algebras. □

1 Cartesian-type interpretations

We use a simplest concept of an *interpretation* of a theory T_0 in the region $U(x)$ of a theory T_1, [5]. We use some special classes of interpretations, such as effective, faithful, auto-free, model-bijective, and isostone interpretations, [6]. In this section, we introduce a technical class of interpretations presenting finitary methods in first-order logic.

Given a signature σ and a finite sequence of formulas of this signature of either of the following forms:

$$\begin{array}{ll}\texttt{(a)} & \varkappa=\langle\varphi_1^{m_1}/\varepsilon_1,\varphi_2^{m_2}/\varepsilon_2,\ldots,\varphi_s^{m_s}/\varepsilon_s\rangle,\\ \texttt{(b)} & \varkappa=\langle\varphi_1^{m_1},\varphi_2^{m_2},\ldots,\varphi_s^{m_s}\rangle\end{array}\qquad(1.1)$$

where φ_k is a formula with m_k free variables, $\varepsilon_k(\bar{y}_k,\bar{z}_k)$ is a formula with $2m_k$ free variables such that $\mathrm{Len}\,\bar{y}_k = \mathrm{Len}\,\bar{z}_k = m_k$; moreover, (1.1)(b) is just a simpler notation instead of the common entry (1.1)(a) in the case when $\varepsilon_k(\bar{y}_k,\bar{z}_k)$ coincides with $\bar{y}_k = \bar{z}_k$ for all $k \leqslant s$.

Starting from a model $\mathfrak{M}$ of signature σ together with a tuple $\varkappa$ of any of the forms (1.1)(a,b), we are going to construct a new model $\mathfrak{M}_1 = \mathfrak{M}\langle\varkappa\rangle$ of signature

$$\sigma_1 = \sigma \cup \{U^1, U^1_1, U^1_2, \ldots, U^1_s\} \cup \{K_1^{m_1+1}, \ldots, K_s^{m_s+1}\} \qquad (1.2)$$

as follows. As the universe, we take

$$|\mathfrak{M}_1| = |\mathfrak{M}| \cup A_1 \cup A_2 \cup \ldots \cup A_s,$$

where all specified parts are pairwise disjoint sets. On the set $|\mathfrak{M}|$, all symbols of signature σ are defined exactly as they were defined in $\mathfrak{M}$; in the remainder, they are defined trivially; predicate $U(x)$ distinguishes $|\mathfrak{M}|$; predicate $U_k(x)$ distinguishes A_k; the other predicates are defined by specific rules depending on the case. In the case (1.1)(b), each predicate K_k in (1.2) should be defined so that it would represent a one-to-one correspondence between the set of tuples $\{\bar{a} \mid \mathfrak{M} \models \varphi_k(\bar{a})\}$ and the set $A_k = U_k(\mathfrak{M}_1)$. Turn to the most common case (1.1)(a). Denote by $\mathrm{Equiv}(\varepsilon_k,\varphi_k)$ a sentence stating that ε_k is an equivalence relation on the set of tuples distinguished by the formula $\varphi_k(\bar{x})$ in $\mathfrak{M}$. In this case, (m_k+1)-ary predicate K_k should be defined so that it would represent a one-to-one correspondence between the quotient set $\{\bar{a} \mid \mathfrak{M} \models \varphi_k(\bar{a})\}/\varepsilon'_k$ and the set $U_k(\mathfrak{M}_1)$, where

$$\varepsilon'_k(\bar{y},\bar{z}) = \varepsilon_k(\bar{y},\bar{z}) \vee \neg\,\mathrm{Equiv}(\varepsilon_k,\varphi_k). \qquad (1.3)$$

The aim of replacement of ε_k by ε'_k using $\mathrm{Equiv}(\varepsilon_k,\varphi_k)$ is to provide total definiteness of the operation of an extension $\mathfrak{M}\langle\varkappa\rangle$ independently of whether the formulas ε_k represent equivalence relations in corresponding domains or not. In the case (1.1)(a), $\mathfrak{M}\langle\varkappa\rangle$ is said to be a *Cartesian-quotient extension of* $\mathfrak{M}$, while in the case (1.1)(b), the model $\mathfrak{M}\langle\varkappa\rangle$ is said to be a *Cartesian extension of* $\mathfrak{M}$ *by a sequence of formulas* $\varkappa$.

Expand the operation of an extension (initially defined for models) on theories. Given a theory T and a tuple $\varkappa$ of the form (1.1). Using a fixed signature (1.2) for extensions of models, we define a new theory $T' = T\langle\varkappa\rangle$ as follows

$$T' = \mathrm{Th}(K),\ K = \{\mathfrak{M}\langle\varkappa\rangle \mid \mathfrak{M} \in \mathrm{Mod}(T)\}.$$

In the case (1.1)(a) it is called a *Cartesian-quotient extension*, while in the case (1.1)(b) it is called a *Cartesian extension of T by a sequence $\varkappa$*.

Normally, we consider passages $T \mapsto T\langle\varkappa\rangle$ for which the sequence (1.1) satisfies the following technical condition:

$$\varphi_k(\bar{x}_k) \textit{ and } \varepsilon_k(\bar{y}_k,\bar{z}_k) \textit{ are } \exists\cap\forall\textit{-presentable}, \qquad (1.4)$$
$$\textit{for all } k \leqslant s.$$

Denote by $\mathcal{KD}(\sigma)$ and $\mathcal{KC}(\sigma)$ the sets of tuples of formulas of signature σ of the forms, respectively, (1.1)(a) and (1.1)(b), while $\mathcal{KD}$ and $\mathcal{KC}$ are unions of these sets for all possible (enumerable) signatures σ. We denote by $\mathcal{KC}_{\exists\cap\forall}$ the set of all tuples (1.1)(b) satisfying (1.4). Furthermore, $\mathcal{KD}^{\varepsilon}_{\exists\cap\forall}$ is the set of all tuples (1.1)(a) satisfying (1.4).

In theory $T\langle\varkappa\rangle$, the region $U(x)$ represents a model of theory T. Particularly, the transformation $T \mapsto T\langle\varkappa\rangle$ defines a natural interpretation $I_{T,\varkappa}$ of T in $T\langle\varkappa\rangle$. It is called a *special Cartesian-quotient* interpretation. Similar definition applies to the other case of the tuple $\varkappa$; thereby, the concepts of a *special Cartesian* interpretation is also defined. Considering theories up to an algebraic isomorphism, we may use simpler term *Cartesian-quotient* or, respectively, *Cartesian* interpretation.

Lemma 1.1. *Given a theory T of an enumerable signature σ and a sequence of formulas $\varkappa \in \mathcal{KD}(\sigma)$. Special Cartesian-quotient interpretation $I_{T,\varkappa} : T \rightarrowtail T\langle\varkappa\rangle$ is effective, faithful, auto-free, model-bijective, and isostone. In particular, interpretation $I_{T,\varkappa}$ determines a computable isomorphism $\mu_{T,\varkappa} : \mathcal{L}(T) \to \mathcal{L}(T\langle\varkappa\rangle)$ between the Tarski-Lindenbaum algebras.*

The following statement is established based on first-order combinatorial properties of Cartesian extensions of theories:

Lemma 1.2. *The following relation defined on the class of all theories*

$$T \cong_a S \Leftrightarrow_{dfn} (\exists\varkappa'\varkappa'' \in \mathcal{KC}_{\exists\cap\forall})\left[\,T\langle\varkappa'\rangle \approx_a S\langle\varkappa''\rangle\,\right] \qquad (1.5)$$

is reflexive, symmetric, and transitive (*i.e., it is an equivalence relation*).

Further properties of Cartesian-type extensions of theories and Cartesian-type interpretations can be found in [6] and [4].

2 Scheme of finitary and infinitary semantic layers

DEFINITION 2.A. We introduce the following notations for particular semantic layers that are relevant in this direction:

(A) *ASL* = the set of model-theoretic properties $\mathfrak{p} \in AL$ preserved by any special Cartesian interpretation $I_{T,\xi} : T \rightarrowtail T\langle\xi\rangle$ for an *arbitrary computably axiomatizable theory T of an enumerable signature σ*

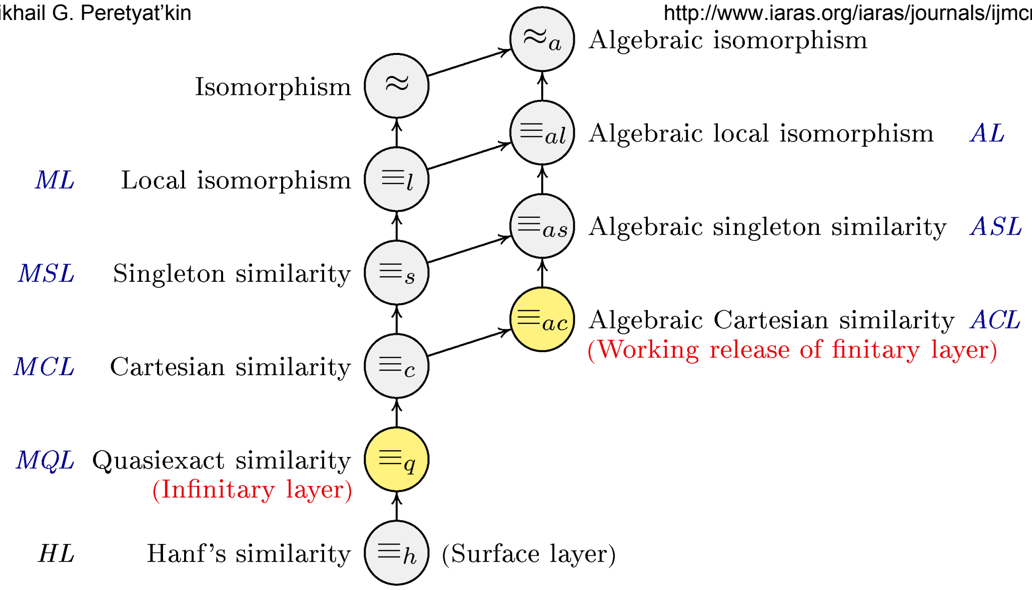

Fig. 1. A scheme of semantic layers of model-theoretic properties

and an arbitrary finite tuple $\xi=\langle\varphi_1,...,\varphi_s\rangle$ of sentences of signature σ satisfying (1.4).

(B) $MSL = ASL \cap ML$.

(C) $ACL =$ the set of model-theoretic properties $\mathfrak{p} \in AL$ preserved by any special Cartesian interpretation $I_{T,\xi} : T \rightarrowtail T\langle\xi\rangle$ for an *arbitrary computably axiomatizable theory T of an enumerable signature σ* and an arbitrary tuple $\xi = \langle\varphi_1^{m_1},\dots,\varphi_s^{m_s}\rangle$ of formulas of signature σ satisfying (1.4).

(D) $MCL = ACL \cap ML$.

(E) $ADL =$ the set of model-theoretic properties $\mathfrak{p} \in AL$ preserved by any special Cartesian-quotient interpretation $I_{T,\xi} : T \rightarrowtail T\langle\xi\rangle$ for an *arbitrary computably axiomatizable theory T of an enumerable signature σ* and an arbitrary tuple $\xi = \langle\varphi_1^{m_1}/\varepsilon_1,\dots,\varphi_s^{m_s}/\varepsilon_s\rangle$ of formulas of signature σ satisfying (1.4).

(F) $MDL = ADL \cap ML$.

Layer *ACL* is said to be the (*algebraic*) *Cartesian* semantic layer; it plays the role of a *working release* of the *finitary semantic layer*. By *MCL* we denote its model version called the *model Cartesian* layer. Layer *ADL* is said to be the (*algebraic*) *Cartesian-quotient* semantic layer; it plays the role of a *maximalistic release* of the *finitary semantic layer*. By *MDL*, we denote its model version called the *model-type Cartesian-quotient* layer.

Fig. 1 presents a scheme of inclusions between the semantic layers and corresponding similarity relations relevant for first-order combinatorics. Arrows point out relatively stronger similarity relations and relatively wider semantic layers of model-theoretical properties. Two relations $\approx$ and $\approx_a$ in the top are relations of isomorphism of theories, where $\approx$ means a *model isomorphism* or simply *isomorphism*, while $\approx_a$ means an *algebraic isomorphism* or $\exists\cap\forall$-*presentable equivalence* between theories, cf. Preliminaries. Although $\approx$ and $\approx_a$ are not similarity relations, they are included in the scheme for the sake of completeness. The entries $\equiv_c$, $\equiv_{ac}$, etc., are short forms for semantic similarity relations $\equiv_{MCL}$, $\equiv_{ACL}$ with semantic layers *MCL*, *ACL*, etc., that were defined above. The inclusions $MDL \subseteq MCL$ and $ADL \subseteq ACL$ are also valid although they are not presented in the scheme in Fig. 1.

The layer *MQL* consists of the model-theoretic properties preserved by all interpretations in the class $IQuasi \cup ICartes$ between computably axiomatizable theories, where *IQuasi* is the set of all quasiexact interpretations, while *ICartes* is the set of all Cartesian interpretations. The layer *MQL* is supported by a regular version of the universal construction of finitely axiomatizable theories, [7]. The Hanf layer *HL* is an empty set $\varnothing$. Corresponding semantic similarity relation $\equiv_\varnothing$, alternatively $\equiv_h$, is called *Hanf's isomorphism* because William Hanf was the first investigator who studied such relations between theories just in relation to the problem of expressive possibilities of first-order logic.

3 A definition to the concept of a model-theoretic property

We are going to discuss approaches to the problem of classification of complete theories modulo coincidence of their model-theoretic properties. Two complete theories are said to be equivalent if their real model-theoretic properties are identical:

$$T_1 \overset{\text{MT}}{\simeq} T_2 \Leftrightarrow_{dfn} \tag{3.1}$$
$$(\forall \textit{ real model-theoretic property } \mathfrak{p})$$
$$\big[T_1 \in \mathfrak{p} \Leftrightarrow T_2 \in \mathfrak{p}\big].$$

Accordingly, any classes of complete theories closed under $\overset{\text{MT}}{\simeq}$ are said to be *real* model-theoretic properties. Thus, to define the concept of a real model-theoretic property it is necessary to find available dependencies (called reasoning) between complete theories of the following form

$$T_1 \simeq_{\text{x}} T_2 \Rightarrow T_1 \overset{\text{MT}}{\simeq} T_2, \tag{3.2}$$

that have significance in the practice of working in model theory.

Two most important reasoning (for complete theories) are:

$$\text{(a) } T \approx_a S \Rightarrow T \overset{\text{MT}}{\simeq} S, \tag{3.3}$$
$$\text{(b) } T\langle\varkappa\rangle = S \Rightarrow T \overset{\text{MT}}{\simeq} S, \textit{ for any } \varkappa \in \mathcal{KC}_{\exists\cap\forall}.$$

General significance of the reasoning (3.3)(a) is obvious. Nevertheless, lots of researchers follow the naive approach considering any classes of complete theories, even if they are not closed under isomorphisms of theories. To avoid this common irregular situation, we will assume (by default) that any considered class of complete theories first should be closed under algebraic isomorphisms of theories by the rule

$$\mathfrak{p} \mapsto \mathfrak{p}^* = [\mathfrak{p}]_{\approx_a} = \tag{3.4}$$
$$\{T \in \mathbb{C} \mid (\exists T' \in \mathfrak{p})[T \approx_a T']\}.$$

This correction rule is said to be a *normalization prestage* of the definition we are going to introduce.

Now, we give a *generic definition* to the concept of a model-theoretic property.

DEFINITION 3.A. [GENERIC DEFINITION OF A MODEL-THEORETIC PROPERTY].

Initially, we have to point out a set of relations of reasoning of the form (for complete theories)

$$\simeq_{\text{x}}^{(i)},\ i \in I \tag{3.5}$$

that we intend to accept as a basis of the definition. The relation $\overset{\text{MT}}{\simeq}$, cf. (3.1), will be presented by the relation $\simeq_{\text{x}}^*$ obtained by the operation of closure of the system of relations (3.5) up to an equivalence relation. Accordingly, the class of all real model-theoretic properties will be presented by the following expression:

$$AreaL = \{\mathfrak{p} \subseteq \mathbb{C} \mid \mathfrak{p} \textit{ is closed under } \simeq_{\text{x}}^*\}. \tag{3.6}$$

To check up, whether a set $\mathfrak{p} \subseteq \mathbb{C}$ is a model-theoretic property, first, a normalization pre-stage $\mathfrak{p} \mapsto \mathfrak{p}^*$ should be performed; then, the condition $\mathfrak{p}^* \in AreaL$ is to be checked. If the result is positive, we qualify $\mathfrak{p}$ as a *real model-theoretic property*; moreover, a specifying term "$\mathfrak{p}$ is a model-theoretic property up to closure under isomorphisms" may be used. Otherwise, if the test $\mathfrak{p}^* \in AreaL$ fails, $\mathfrak{p}$ is qualified as a class that is not a real model-theoretic property.

End of the definition.

Notice that, an inverse dependence of the set of real model-theoretic properties on the accepted set of reasoning $\simeq_{\text{x}}^{(i)}$, $i \in I$ takes place. Indeed, let the pointed out set defines an equivalence relation $\simeq_{\text{x}}^*$ playing the role of the relation $\overset{\text{MT}}{\simeq}$, thus, defining the layer *AreaL*. Assume that, as the base for a new definition, some larger set of reasoning $\simeq_{\text{x}}^{(i)}$, $i \in I^+$, $I^+ \supseteq I$, is taken. It is obvious that the inclusion $\simeq_{\text{x}}^* \subseteq \simeq_{\text{x}}^+$ must take place; i.e., each class of the new equivalence $\simeq_{\text{x}}^+$ consists of a number of classes of the initial equivalence $\simeq_{\text{x}}^*$. Thereby, we have $AreaL^+ \subseteq AreaL$ because $AreaL^+$ consists of the sets of complete theories closed under equivalence $\simeq_{\text{x}}^+$ having larger classes in comparison with those of the initial equivalence $\simeq_{\text{x}}^*$.

The following (pragmatic) variant of the definition is accepted as preferable:

`Reference_Block` P-version of Def. 3.A (3.7)

As a set of reasoning, we accept the relation (3.3)(a) together with a series of relations (3.3)(b) for all $\varkappa \in \mathcal{KC}_{\exists\cap\forall}$. The relation $\approx_a$ on the class of all complete theories defined by expression (1.5) in Lemma 1.2 is the closure of this system of relations. Thus, within this approach, relation $\overset{\text{MT}}{\simeq}$ coincides with $\approx_a$. Accordingly, in view of the scheme of semantic layers in Fig. 1, we obtain the following chain of inclusions:

$$AreaL = ACL \subseteq ASL \subseteq AL. \tag{3.8}$$

By default, we also suppose that, to check Definition 3.A for a set $\mathfrak{p} \subseteq \mathbb{C}$, a normalization transformation (3.4) should be performed initially.

`End_Ref`

An important statement concerning different versions of Definition 3.A.

Lemma 3.1. *Suppose that a variant α of definition of a real model-theoretic property is given with reasoning consisting of the relation $(3.3)(a)$ and a series of relations $(3.3)(b)$ for all $\varkappa \in \mathcal{KC}_{\exists\cap\forall}$ together with a definite set of additional relations of the form (3.2). Then, the following chain of inclusions takes place*:

$$AidealL^\alpha \subseteq AreaL^\alpha \subseteq ACL \subseteq ASL \subseteq AL, \tag{3.9}$$

where the ideal semantic layer $AideaL^{\alpha}$ corresponds to the potential possibility of an extension of the accepted system of reasoning α with some new rules of the form (3.2) *that can appear and could be accepted in the future within the system α.*

PROOF. From the principle of inverse dependence we mentioned earlier. □

The following systems of reasoning to the definition of the concept of a real model-theoretic property are possible. Let an arbitrary set $\mathfrak{p} \subseteq \mathbb{C}$ be given. At the *naive approach*, any set of complete theories is considered as a model-theoretic property; the *primitive approach*, cf. Preliminaries, requires that $\mathfrak{p}$ should be closed under isomorphisms of theories; the *pragmatic approach*, cf. (3.7), requires that $\mathfrak{p}$ is closed under isomorphisms, Cartesian extensions, and back transitions in the operation of Cartesian extensions of theories; at last, the *maximalistic approach* requires that $\mathfrak{p}$ is closed under isomorphisms, Cartesian-quotient extensions, and back transitions in the operation of Cartesian-quotient extensions of theories, i.e., the reasoning $T\langle\varkappa\rangle = S \Rightarrow T \overset{\mathrm{MT}}{\simeq} S$, for all $\varkappa \in \mathit{KD}^{\varepsilon}_{\exists\cap\forall}$, is accepted that is wider in comparisons with (3.3)(b).

We list a few comparison statements characterizing different approaches. Main results concerning expressive possibilities of first-order logic are obtained based on the operation of Cartesian extensions of theories generating the reasoning (3.3)(b); thus, addition of reasoning with Cartesian-quotient extensions of theories would be superfluous since it leads to relatively smaller semantic layer of model-theoretic properties in view of the principle of inverse dependence. It is also of importance that the operation of Cartesian extensions of theories is more adequate to general model theory whereas Cartesian-quotient extensions have a more algebraic accent. As for the maximalistic approach, its motivation is an idea to reach a maximum fundamental significance based on the Cartesian-quotient extensions of theories representing the class of all finitary first-order methods even despite definite decreasing of the semantic layer of controlled model-theoretic properties.

Notice that, some other approaches to the definition of the concept of a real model-theoretic property are possible which can be based on other principles different from those accepted within the scheme we have described. For comparison of our approach with the other potentially possible ones, some discussion is required concerning advantages and lacks of each of the alternative approaches.

4 Signature reduction procedures jointly with the universal construction

The works of L. Kalmar [8], R.L Vaught [9, Sec.4], and W.Hanf [10] represent earlier signature reduction methods between first-order theories. In this section, we describe three possible type of signature reduction procedures based on a collection of special transformations between theories. We call them *elementary transformations* or *stages*. Actually, these transformations represent known in the common practice signature reduction methods. Full scheme of interaction between the elementary stages is shown in Fig. 2. There are three entries and a single exit in the scheme.

Now, we turn to the further details.

4.1 *Finite-to-finite signature reduction procedure*. A theory of an arbitrary finite signature is transformed into a theory of any pre-specified finite rich signature. This type of transformation is realized via an `Entry1` in the scheme in Fig. 2.

First, we formulate the main statement in a compact form:

Theorem 4.1. [Finite-to-finite signature reduction statement: a compact form] *Given two finite rich signatures σ_1 and σ_2. Effectively in their Godel numbers, it is possible to construct a sequence of formulas $\varkappa = \langle\varphi_1^{m_1}, \ldots, \varphi_s^{m_s}\rangle$ of signature σ_1 satisfying* (1.4) *and a sentence ψ of signature σ_2 together with an algebraic isomorphism $PC(\sigma_1)\langle\varkappa\rangle \approx_a PC(\sigma_2)[\psi]$.*

Now, we give an extended form of the same statement. By *σFinRich*, we denote the set of all finite rich signatures, $T\mathfrak{S}^{\phi}$ denotes the set of all theories of arbitrary finite signatures, while *ICartes*$^{\phi}$ denotes the set of all Cartesian interpretations between the theories of finite signatures.

Theorem 4.2. [Finite-to-finite signature reduction procedure: a common form] *It is possible to determine a finite-to-finite signature reduction procedure, also called a finite signature transformation. It is presented by a mapping of the form*

$$\mathrm{Redu}: T\mathfrak{S}^{\phi} \times \sigma\mathit{FinRich} \rightarrow T\mathfrak{S}^{\phi} \times \mathit{ICartes}^{\phi}$$

satisfying all demands specified in the following statement:

Let T be a theory of a finite signature τ and σ be an arbitrary finite rich signature. Applying the mapping Redu *we obtain*

$$\mathrm{Redu}(T, \sigma) = (S, I),$$

where S is a theory of signature σ, while I is an interpretation of T in S, such that the following assertions are satisfied:

`Reference_Block` (4.1)

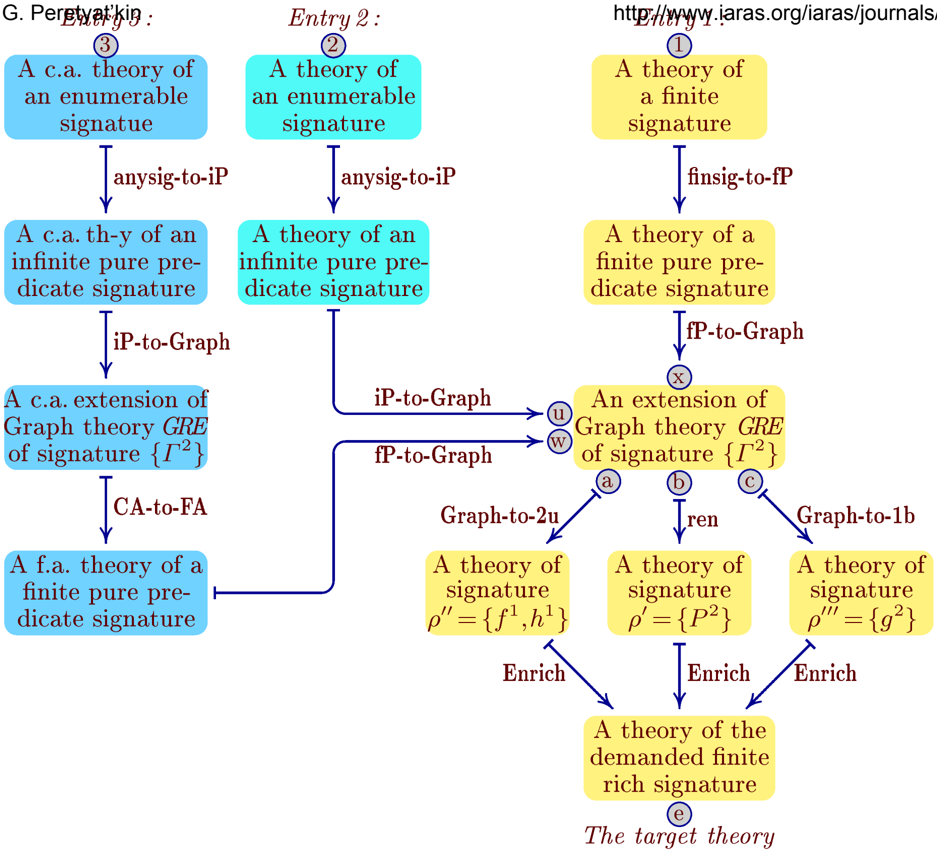

Fig. 2. Signature reduction procedures and the universal construction

(a) *I is an $\exists\cap\forall$-presentable Cartesian interpretation of theories (thereby, the interpretation I defines a computable isomorphism $\mu:\mathcal{L}(T)\to\mathcal{L}(S)$ preserving model-theoretic properties of the semantic layer ACL),*

(b) *T is c.a. $\Leftrightarrow$ S is c.a.; in the case when T is a c.a. theory, c.e. indices of both S and I are found effectively in a pair of parameters consisting of a c.e. index of the input theory T and a Gödel number of the target finite rich signature σ,*

(c) *T is f.a. $\Leftrightarrow$ S is f.a.; in the case when T is a f.a. theory, both a Gödel number of S and a c.e. index of I are found effectively in a pair of parameters consisting of Gödel numbers of the input theory T and the target finite rich signature σ.*

End_Ref

A SKETCH OF PROOF to Theorem 4.1. For the sake of simplicity, we prove the following more common statement:

$$(\exists\varkappa\in\mathcal{KC}_{\forall\cap\exists}\ \textit{effectively in } \sigma_1 \textit{ and } \sigma_2)\quad (4.2)$$
$$(\forall \textit{f.a. theory } T\supseteq PC(\sigma_1))$$
$$(\exists \textit{f.a. theory } S\supseteq PC(\sigma_2))\,\big[T\langle\varkappa\rangle\approx_a S\big].$$

Then, Theorem 4.1 is a particular case of the statement (4.2) with $T = PC(\sigma_1)$.

We start to prove (4.2). Given two finite rich signatures σ_1 and σ_2 together with a finitely axiomatizable theory T of signature σ_1. Our purpose is to describe a procedure of reduction of the theory T to a theory of the pre-specified finite rich signature σ_2.

Based on the property (0.1), we organize a signature reduction procedure consisting of two parts. In the first part, a reduction to any of three following "minimal" finite rich signatures

$$\rho'=\{P^2\},\ \rho''=\{f^1,h^1\},\ \rho'''=\{g^2\}\qquad (4.3)$$

is performed, while in the second part, a routine passage from either ρ' or ρ'' or ρ''' to the demanded finite rich signature σ_2 is performed depending on which of the cases $\rho'\leqslant\sigma_2$ or $\rho''\leqslant\sigma_2$ or $\rho'''\leqslant\sigma_2$ takes place.

For the finite-to-finite case of reduction, we use a natural set of transformations of theories consisting of five *elementary* transformations acting along the passage 1-x-e in Fig. 2. Their short specifications are described below:

finsig-to-fP — a transformation from a theory of a finite signature to a theory of a finite pure predicate signature with predicates of arity $\geqslant 1$. An n-ary function $f(x_1,\ldots,x_n)$ is replaced by a $(n+1)$-ary predicate presenting graphic of the function; a constant c

is replaced by a unary predicate distinguishing an element presenting the constant. Additionally, we should replace each nulary predicate by a unary one. This transformation defines an algebraic isomorphism of theories.

fP-to-Graph — a transformation from a theory of a finite pure predicate signature with predicates of arity $\geqslant 1$ to an extension of special graph theory *GRE* of signature $\{\Gamma^2\}$, cf. Stage *FG* in Theorem 5.10.1 in [7]. This transformation is a Cartesian extension of a theory, thus, it defines a Cartesian interpretation.

Graph-to-2u — a transformation from a theory of signature $\{\Gamma^2\}$ which is an extension of special graph theory *GRE* to a theory of signature with two unary functions; cf. Case 2 in Lemma 5.10.1 in [7]. This transformation is a Cartesian extension of a theory, thus, it defines a Cartesian interpretation.

Graph-to-1b — a transformation from a theory of signature $\{\Gamma^2\}$ which is an extension of a special graph theory *GRE* to a theory of signature with one binary function; cf. Case 1 in Lemma 5.10.1 in [7]. This transformation is a Cartesian extension of a theory, thus, it defines a Cartesian interpretation.

Enrich — a transformation from a theory of a signature matching one of the three cases (4.3) of a minimal finite rich signature to a theory of a given finite rich signature; cf. Stage *GL* in Table 5.8.1 in [7]. This transformation is an isomorphism of theories (notice that, a problem to assign values to constants possible in the target signature σ_2 is regularly solvable because the extension of Graph theory *GRE* preceding the stage Enrich has at least one distinguished element).

Fig. 2 represents a scheme of successive actions of the elementary transformations. We use circled digits and letters to point out some intermediate points in order to explain different variants of traversal through the scheme. `Entry1` of the scheme requires, as an input, a theory of a finite signature and yields an output theory of the demanded finite rich signature σ_2. We define Redu as a composition of transformations of theories along the passage 1-x-e in the scheme shown in Fig. 2. Each elementary stage is a Cartesian $\exists\cap\forall$-presentable interpretation. Thereby, the full passage that is a composition of these separate stages is also an a Cartesian $\exists\cap\forall$-presentable interpretation.

Theorem 4.1 is proved. □

Theorem 4.2 is easily deduced from Theorem 4.1 by applying elementary methods of c.e. Boolean algebras together with the properties of correspondence (0.2) considered in Lemma 0.1. □

Give a complementary statement.

Lemma 4.3. *Finite-to-finite signature reduction procedure we have described in Theorem* 4.1 *and Theorem* 4.2 *represents a particular case of the operation of a Cartesian extension of a theory.*

Moreover, interpretation I involved in the extension have the following properties:

(a) *I preserves all model-theoretic properties within the layer ACL*,

(b) *I preserves all model-theoretic properties within the real layer AreaL*,

(c) *in general case, the finite-to-finite signature reduction procedure does not preserve, both locally and globally, model-theoretic properties of $\exists\forall$-axiomatizability, $\forall$-axiomatizability, and $\exists$-axiomatizability; i.e., in some cases, these properties are not preserved by the procedure* Redu.

PROOF. Immediately, from proofs of Theorem 4.1 and Theorem 4.2. □

4.2 *Infinite-to-finite signature reduction procedure*. A theory of an arbitrary enumerable signature is transformed into a theory of any pre-specified finite rich signature. This type of signature reduction procedure is defined via an `Entry2` in Fig. 2. It is realized by the stages acting along the passage 2-u-e including those listed in Subsection 4.1 together with two additional transformations whose short specifications are presented below:

anysig-to-iP — a transformation from a theory of an arbitrary enumerable signature (either finite or infinite) into a theory of an infinite pure predicate signature with predicates of arity $\geqslant 1$; it is analogous to stage finsig-to-fP, but with addition a countable set of new trivially defined (dummy) predicates. A thin point is that, if an input theory is c.a. and is given by its weak c.e. index, the output theory is presented by a normal c.e. index. This transformation defines an algebraic isomorphism of theories.

iP-to-Graph — a transformation from a theory of an infinite pure predicate signature with predicates of arity $\geqslant 1$ to an extension of graph theory *GRE* of signature $\{\Gamma^2\}$ (main stage of the infinite-to-finite signature reduction procedure); cf. Stage *IG* in Theorem 5.9.1 in [7]. This transformation defines a quasiexact interpretation of theories.

We note an effective version of the infinite-to-finite signature reduction procedure:

Theorem 4.3. *Given a c.a. theory T and a finite rich signature σ_2. Effectively in a weak c.e. index of T, one can construct a c.a. theory S of signature σ_2 together with a quasiexact interpretation $I:T\rightarrowtail S$; in particular, the interpretation I defines a computable isomorphism $\mu:\mathcal{L}(T)\to\mathcal{L}(S)$ preserving model-theoretic properties of the infinitary semantic layer MQL.*

PROOF. Stage iP-to-Graph preserves layer *MQL* included in the layer *ACL*, cf. Fig. 1, that is preserved by the other stages in the passage 2-u-e in Fig. 2. □

4.3 *Transformation of theories for the universal* *construction*. A c.a. theory of an arbitrary enumerable signature given by its weak c.e. index is transformed into a f.a. theory of any pre-specified finite rich signature yielding its Godel number. This type of transformation is defined via an `Entry3` in Fig. 2. It is realized by the stages acting along the passage 3-w-e including those listed in Subsection 4.1 and Subsection 4.2 together with the following additional transformation:

CA-to-FA — a transformation from a computably axiomatizable theory of signature $\{\Gamma^2\}$ extending graph theory *GRE* into a finitely axiomatizable theory of a finite pure predicate signature (main stage of the universal construction); cf. Stage *GF* in Table 5.8.1 and Theorem 6.1.1 in [7]. An input theory is given by its c.e. index, while the output theory is presented by its Godel number. A standard release of this transformation defines a quasiexact interpretation of theories, thus, preserving the layer *MQL*. Notice that, there are simplified versions of the stage CA-to-FA preserving some proper sublayers of *MQL*.

We formulate the universal construction designed from the stage CA-to-FA:

Theorem 4.4. *Given a c.a. theory T and a finite rich signature σ_2. Effectively in a weak c.e. index of T, one can construct a f.a. theory F of signature σ_2 together with a quasiexact interpretation $I:T\rightarrowtail S$; in particular, the interpretation I defines a computable isomorphism $\mu:T\to F$ preserving the infinitary semantic layer MQL of model-theoretic properties (having a simplified version of the stage* CA-to-FA*, the layer of controlled properties will be smaller).*

PROOF. Stages iP-to-Graph and CA-to-FA preserve layer *MQL* included in layer *ACL*, cf. Fig. 1, that is preserved by the other stages in the passage 1-w-e in Fig. 2. □

REMARK 4.5. The works [11] and [4] describe a comparison method for semantic layer that is based on a representative list of model-theoretic properties. Applying this method, we can check that the power of the infinite-to-finite signature reduction procedure coincides with that of the universal construction, cf. Theorem 4.3 vs. Theorem 4.4. Any version of the universal construction can be considered as an advanced release of the infinite-to-finite signature reduction procedure. Thus, it would be unreal to expect that the former one can be more powerful in comparison with the latter one. This observation gives an informal substantiation to the fact that the power of an available standard version of the universal construction, [7], is actually maximum possible.

We should note that simple versions of the universal construction could also be useful. Indeed, high complexity of the universal construction represents a certain psychological barrier while studying results obtained on the base of this construction. The fact of availability of a mini-version of the universal construction that is more accessible for studying could reduce this barrier. Hereafter, we suppose that a fixed release of the universal construction is accepted, denoted by $\mathbb{U}$, that can control the following sublayer

$$MQL \subseteq MQL \qquad (4.4)$$

of the infinitary layer *MQL*. An example of a mini-version of the universal construction that is more simple in studying can be found in [12].

REMARK 4.6. Statement of Remark 4.5 establishes a close connection between the main stage of the infinite-to-finite signature reduction procedure and that of a standard release of the universal construction. This gives a good possibility to introduce a clear (understandable) definition to the concept of a quasiexact interpretation. First, we have to design a proof to the stage iP-to-Graph describing in detail properties of the involved interpretation I ensuring preservation of model-theoretic properties in infinitary semantic layer. After that, we have to extract a common description of the interpretation I such that it would be appropriate to both stages iP-to-Graph and CA-to-FA.

5 Virtual isomorphisms between the undecidable predicate calculi

The concept of a virtual extension of a theory can be found in [4] and [13]. We present the following statement:

Theorem 5.1. *Let σ_1 and σ_2 be arbitrary finite rich signatures. There are sequences $\varkappa_1$ and $\varkappa_2$ of formulas of the form (1.1)(b) in appropriate signatures satisfying (1.4) such that $PC(\sigma_1)\langle\varkappa_1\rangle \approx_a PC(\sigma_2)\langle\varkappa_2\rangle$.*

By using presentation of invertible multi-dimensional quotient interpretations via Cartesian-quotient extensions of theories, Lemma 6.8(a) in [6], we obtain that the main result of the work [14] represents a weak version of Theorem 5.1 with $\varkappa_1,\varkappa_2\in KD$ and $\approx$ instead of $\approx_a$.

By applying Theorem 5.1, we can establish the following fact:

Corollary 5.2. *Let σ_1 and σ_2 be finite rich signatures. There exists a computable isomorphism $\mu:\mathcal{L}(PC(\sigma_1))\to\mathcal{L}(PC(\sigma_2))$ that preserves model-theoretic properties from the semantic layer ACL (thereby, μ preserves all available model-theoretic properties). Moreover, μ preserves globally the following general-model properties: decidability, computable axiomatizability, to be a theory of* *a given degree of unsolvability, finite axiomatizability, and Π_n-axiomatizability, for any fixed $n \geqslant 2$.*

The part stating preservation globally of Π_n-axiomatizability, $n \geqslant 2$, gives the positive answer to an open question asked by V. L. Selivanov in 2007.

6 Semantic types of theories and operations on them

From the point of view of a semantic layer L, any computably axiomatizable theory T can be characterized by a 3-tuple $\big(\mathcal{L}(T),\gamma,\xi\big)$, where $\big(\mathcal{L}(T),\gamma\big)$ is its Tarski-Lindenbaum algebra with a Gödel numbering, while ξ is a mapping from the Stone space $St(\mathcal{L}(T))$ into the power-set $\mathcal{P}(L)=\{K|K\subseteq L\}$ which is defined as follows: for any T' in $St(\mathcal{L}(T))$ that is a complete extension of T, we put $\xi(T')=\{p\in L \mid T'$ *has the property* $p\}$. As a matter of fact, so defined 3-tuple $\pounds(T)=(\mathcal{L}(T),\gamma,\xi)$ represents a full abstract exposition of T in terms of the semantic layer L. We call this tuple *generalized Tarski-Lindenbaum algebra of the theory T under the semantic layer L*.

Generalizing the situation, we introduce a special class of objects to present isomorphism types of the generalized Tarski-Lindenbaum algebras under a semantic layer L of model-theoretic properties. Namely, consider an arbitrary 3-tuple of the form $\mathfrak{B}=(\mathcal{B},\nu,\xi)$, where $(\mathcal{B},\nu)$ is a c.e. Boolean algebra, while ξ is a mapping from Stone space $St(\mathcal{B})$ into the power-set $\mathcal{P}(L)$. So defined 3-tuple $(\mathcal{B},\nu,\xi)$ is called an *abstract semantic L-type*, or simply a *semantic type*.

Let $\mathfrak{B}_1=(\mathcal{B}_1,\nu_1,\xi_1)$ and $\mathfrak{B}_2=(\mathcal{B}_2,\nu_2,\xi_2)$ be two abstract semantic types under a semantic layer L. The types $\mathfrak{B}_1$ and $\mathfrak{B}_1$ are called *computably isomorphic* or *equivalent*, written $\mathfrak{B}_1\equiv_L\mathfrak{B}_2$, if there is a computable isomorphism $\mu:(\mathcal{B}_1,\nu_1)\to(\mathcal{B}_2,\nu_2)$ such that for any ultrafilter $\mathcal{F}_1\in\mathrm{St}(\mathcal{B}_1)$ and corresponding ultrafilter $\mathcal{F}_2\in\mathrm{St}(\mathcal{B}_2)$, $\mathcal{F}_2=\mu(\mathcal{F}_1)$, the equality $\xi_1(\mathcal{F}_1)=\xi_2(\mathcal{F}_2)$ takes place.

Let T be a theory and L be a semantic layer. Consider generalized Tarski-Lindenbaum algebra $(\mathcal{L}(T),\gamma,\xi)$ of T under the semantic layer L. If $\mathfrak{B}$ is a semantic type satisfying $(\mathcal{L}(T),\gamma,\xi)\equiv_L\mathfrak{B}$, we say that T *has the semantic type* $\mathfrak{B}$ *under* L, or that the *semantic type* $\mathfrak{B}$ *is presented* (*realized*) *in* T *under* L. By $\pounds(T)$, we denote the semantic type of a theory T under the full semantic layer *AL*, while $\pounds_L(T)$ stands for the semantic type of T under a semantic layer $L\subseteq AL$.

One can see that the concept of a semantic type together with the equivalence relation for such objects are in exact correspondence with the relation of semantic similarity of theories under a semantic layer. Namely, the following statement takes place:

Lemma 6.1. *Let T and S be theories of enumerable signatures and L be a semantic layer. Then, the following assertions are equivalent*:

(a) *T and S are semantically similar under L*,

(b) $\pounds(T)\equiv_L\pounds(S)$.

PROOF. Immediately, from definitions. □

Let $\mathfrak{B}$ be an abstract semantic type. The type $\mathcal{B}$ is said to be *finitely axiomatizable* or *$\mathcal{F}$-type*, if $\mathfrak{B}$ is realized in a finitely axiomatizable theory; $\mathcal{B}$ is *computably axiomatizable* or *$\mathcal{E}$-type*, if $\mathfrak{B}$ is realized in a computably axiomatizable theory.

Lemma 6.2. *Any $\mathcal{E}$-type under the layer $M\mathbb{Q}L\subseteq$ MQL, cf.* (4.4)*, is an $\mathcal{F}$-type under $M\mathbb{Q}L$.*

Given a semantic layer $L\subseteq AL$. Let a semantic type $\mathfrak{B}$ be presented in computably axiomatizable theory with a c.e. index n. In such case, the number n is called an *$\mathcal{E}$-index* of this type $\mathfrak{B}$, symbolically $\mathfrak{B}=\mathscr{E}_{\{n\}}$. Similarly, if a type $\mathfrak{B}$ is presented in finitely axiomatizable theory defined by a Gödel number n, the number n is called an *$\mathcal{F}$-index* of this type $\mathfrak{B}$, symbolically $\mathfrak{B}=\mathscr{F}_{\{n\}}$.

Define the operation of a direct product of two semantic types.

Let two semantic types $\mathfrak{T}_1=(\mathcal{B}_1,\nu_1,h_1)$ and $\mathfrak{T}_2=(\mathcal{B}_2,\nu_2,h_2)$ be given under a layer L. Define some new semantic type

$$\mathfrak{T}=(\mathcal{B},\nu,h)=(\mathcal{B}_1,\nu_1,h_1)\otimes(\mathcal{B}_2,\nu_2,h_2)=\mathfrak{T}_1\otimes\mathfrak{T}_2$$

under the layer L as follows. We put $(\mathcal{B},\nu)=(\mathcal{B}_1,\nu_1)\otimes(\mathcal{B}_2,\nu_2)$, while the assignment function h is determined by the rule

$$h(\mathcal{F})=\begin{cases} h_1(\mathcal{F}), \text{ if } \mathcal{F}\in\mathrm{St}(\mathcal{B}_1),\\ h_2(\mathcal{F}), \text{ if } \mathcal{F}\in\mathrm{St}(\mathcal{B}_2).\end{cases}$$

So defined operation $\mathfrak{T}_1\otimes\mathfrak{T}_2$ is called *direct product* of semantic types $\mathfrak{T}_1$ and $\mathfrak{T}_2$, while the function h is called *direct product* of functions h_1 and h_2, using for this entry $h=h_1\otimes h_2$.

Now, we introduce a natural operation of the *direct product* of a *sequence of semantic types*. Let $\mathfrak{B}_n=(\mathcal{B}_n,\nu_n,\xi_n)$, $n\in\mathbb{N}$, be a sequence of semantic types under a semantic layer L, and P be a complete theory of an enumerable signature that is used as an additional parameter in the operation.

Define a new semantic type

$$(\mathcal{B},\nu,\xi)=\bigotimes\nolimits_{n\in\mathbb{N}}^{[P]}\mathfrak{B}_n=\bigotimes\nolimits_{n\in\mathbb{N}}^{[P]}\big(\mathcal{B}_n,\nu_n,\xi_n\big)$$

as follows. We put $(\mathcal{B},\nu)=\bigotimes_{n\in\mathbb{N}}(\mathcal{B}_n,\nu_n)$, while the assignment operation ξ is determined by the following rule

$$\xi(\mathfrak{F})=\begin{cases} \xi_n(\mathfrak{F}), \textit{ if } \mathfrak{F}\in\mathrm{St}(\mathcal{B}_n),\ n\in\mathbb{N},\\ \mathrm{prop}(P), \ \textit{if}\ \mathfrak{F}=\hat{\mathfrak{F}}=\textit{Filter}\{\neg 1_i \mid i\in\mathbb{N}\},\end{cases}$$

where $\mathrm{prop}(P)$ is the set of model-theoretic properties associated with the complete theory P. The idea of the assignment function in the operation $\otimes$ for a sequence of semantic types is based on the following algebraic relation for the Boolean algebras

$$\mathrm{St}(\bigotimes_{i\in\mathbb{N}}\mathcal{B}_i) = \bigcup_{i\in\mathbb{N}}\mathrm{St}(\mathcal{B}_i)\cup\{\hat{\mathfrak{F}}\}.$$

7 Generalized Tarski-Lindenbaum algebra of undecidable predicate calculi

Now, we turn to principal statements characterizing the globalization structure of first-order predicate calculus of a finite rich signature under finitary and infinitary semantic layers in the form of some explicit formulas.

Remember notations introduced in this article:

- *AL* is the layer consisting of all model-theoretic properties of both model and algebraic types, cf. Preliminaries,
- *ACL* is the Cartesian semantic layer playing the role of a *working release* of the *finitary semantic layer*, cf. Section 2,
- *MQL* is the model quasiexact layer alternatively called the *infinitary semantic layer* cf. Section 2,
- $\mathcal{MQL}$ is a fixed sublayer of *MQL* supported by an accepted release of the universal construction as it was established in (4.4),
- $PC(\sigma)$ is the predicate calculus of signature σ considered as a first-order theory (defined by an empty set of axioms),
- $(\mathcal{L}(PC(\sigma)),\gamma,\xi)$ is the generalized Tarski-Lindenbaum algebra of predicate calculus $PC(\sigma)$; where γ is a fixed Gödel numbering of the set of sentences of signature σ, while $\xi:\mathrm{St}(PC(\sigma))\to\mathcal{P}(AL)$ is the mapping assigning model-theoretic properties to complete extensions of the theory $PC(\sigma)$,
- $\mathscr{F}_{\{n\}}$ is the *finitely axiomatizable* semantic type with an index n,
- $\mathscr{E}_{\{n\}}$ is the *computably axiomatizable* semantic type with an index n,
- The concept of an f-dense theory: a theory P of a finite signature σ is said to be f*-dense* under a semantic layer D if the following properties are satisfied: (a) theory P is complete and decidable, (b) for any $\Phi\in SL(\sigma)$ satisfying $P\vdash\Phi$, a sentence $\Psi\in SL(\sigma)$ and a computable isomorphism μ can be found, satisfying the following properties: $P\vdash\neg\Psi$, $\vdash\Psi\to\Phi$, and $\text{₤}([\Psi]^\sigma)\equiv_D\text{₤}(GRE)$ by means of μ; moreover, both a Gödel number of Ψ and an index of the isomorphism μ are found effectively from a Gödel number of the sentence Φ,
- The concept of an *inf*-dense theory is a generalization of the concept of an f-dense theory with using computably axiomatizable theories instead of finitely axiomatizable ones (details do not matter in this work).

We formulate the principal statement of the paper.

Theorem 7.1. [Globalization Theorem for First-Order Logic] *Let σ be a finite rich signature, and*

$$\text{₤}\big(PC(\sigma)\big) = \big(\mathcal{L}(PC(\sigma)),\gamma,\xi\big)$$

be the semantic type of the predicate calculus of signature σ. Let L and K be semantic layers s.t. $L\subseteq$ ACL and $K\subseteq\mathcal{MQL}$, P be an f-dense theory under the layer L, and R be an inf-dense theory under the layer K. An extra demand $K\subseteq L$ is also accepted in Part (C) involving both layers L and K.

The following assertions take place:

(A) [Finitary Globalization] *The following presentation takes place*:

$$\text{₤}(PC(\sigma))\equiv_L\mathfrak{B}^{ACL}_{fin} =_{dfn} \bigotimes^{[P]}_{n\in\mathbb{N}}\mathscr{F}_{\{n\}}, \tag{7.1}$$

(B) [Infinitary Globalization] *The following presentation takes place*:

$$\text{₤}(PC(\sigma))\equiv_K\mathfrak{B}^{\mathcal{MQL}}_{inf} =_{dfn} \bigotimes^{[R]}_{n\in\mathbb{N}}\mathscr{E}_{\{n\}}, \tag{7.2}$$

(C) [Interference] *Any computably axiomatizable semantic type under L is finitely axiomatizable under K. Moreover, there are total computable functions $q(n)$ and $v(n,t)$, such that q is a permutation of the set $\mathbb{N}$, and the following similarity relations are held for all $n\in\mathbb{N}$*:

$$\mathscr{E}_{\{n\}}\equiv_K\mathscr{F}_{\{q(n)\}};\ \textit{moreover, the function } (\lambda t)v(n,t) \textit{ presents an isomorphism corresponding to this similarity relation.} \tag{7.3}$$

Thereby, for an arbitrary f-dense under K theory S (that must automatically be inf-dense under K), the following similarity relation is satisfied

$$\bigotimes^{[S]}_{n\in\mathbb{N}}\mathscr{E}_{\{n\}}\equiv_K\bigotimes^{[S]}_{n\in\mathbb{N}}\mathscr{F}_{\{q(n)\}}, \tag{7.4}$$

such that corresponding Hanf's isomorphism μ maps member $\mathscr{E}_{\{n\}}$ onto member $\mathscr{F}_{\{q(n)\}}$ for all $n\in\mathbb{N}$, while a particular ultrafilter in the left-hand side of (7.4) is mapped onto a particular ultrafilter in the right-hand side,

(D) [FINITARY ADD/OMIT MEMBERS] *Given an arbitrary $\mathcal{F}$-type $\mathfrak{B}'$ under the layer L and an integer $k_0 \geqslant 0$. We have*

$$\mathfrak{B}^{ACL}_{fin} = \bigotimes^{[P]}_{n<\omega} \mathscr{F}\{n\} \equiv_L \mathfrak{B}' \otimes \bigotimes^{[P]}_{k_0 \leqslant m<\omega} \mathscr{F}\{m\}; \tag{7.5}$$

more precisely: having omitted a few product members and attached an extra member in the sequence involved in the operation (7.1)*, it is possible to define a computable isomorphism* μ *between the latter semantic type and the changed one, such that, a particular ultrafilter from the left-hand side of* (7.5) *is linked by* μ *with that available in the right-hand side of* (7.5),

(E) [INFINITARY ADD/OMIT MEMBERS] *Given an arbitrary $\mathcal{E}$-type $\mathfrak{B}''$ under the layer K and an integer $k_0 \geqslant 0$. We have*

$$\mathfrak{B}^{MQL}_{inf} = \bigotimes^{[R]}_{n<\omega} \mathscr{E}\{n\} \equiv_K \mathfrak{B}'' \otimes \bigotimes^{[R]}_{k_0 \leqslant m<\omega} \mathscr{E}\{m\}; \tag{7.6}$$

more precisely: having omitted a few product members and attached an extra member in the sequence involved in the operation (7.2)*, it is possible to define a computable isomorphism* μ *between the latter semantic type and the changed one, such that a particular ultrafilter from the left-hand side of* (7.6) *is linked by* μ *with that available in the right-hand side of* (7.6),

(F) [EFFECTIVENESS] *Transformations presented in the parts of this theorem are realized effectively in Gödel's numbers and/or c.e. indices of the objects involved in the construction. We can effectively find Gödel numbers and/or c.e. indices of all further objects appeared in the construction, such as c.e. index of a function, Gödel number or c.e. index of a semantic type, c.e. index of a computable sequence of semantic types, etc.*

8 Main applications of the Globalization Theorem

We show that the localized statements are immediate consequences of the globalization formulas.

Theorem 8.1. *Statement presenting the finite-to-finite signature reduction procedure, cf. Theorem* 4.1 *and Theorem* 4.2*, is an immediate consequence of Globalization Theorem* 7.1 (A).

Theorem 8.2. *An accepted version of the universal construction controlling the layer* (4.4)*, cf. Theorem* 4.4*, is an immediate consequence of Globalization Theorem* 7.1(C).

One more important application of Globalization Theorem.

The *pseudo-indecomposability* property was introduced due to Hanf [15], who considered this concept for the class of pure Boolean algebras. As an important application of the globalization formulas, we are going to study a version of this property expanded on the class of semantic types.

A semantic type $\mathfrak{B}$ under a semantic layer L is said to be *pseudo-indecomposable*, if for any $b \in \mathfrak{B}$, either $\mathfrak{B}[b]$ or $\mathfrak{B}[-b]$ is equivalent to $\mathfrak{B}$. A type $\mathfrak{B}$ is said to be *effectively pseudo-indecomposable* if it is pseudo-indecomposable and, effectively in the Godel number of an element $a \in |\mathfrak{B}|$, one can solve which of the two types $\mathfrak{B}[b]$ or $\mathfrak{B}[-b]$ is isomorphic to $\mathfrak{B}$; moreover, it is possible to get an index of corresponding isomorphism.

Theorem 8.3. *Let σ be a finite rich signature and*

$$\mathfrak{B}^* = \text{\textsterling}\big(PC(\sigma)\big) = \big(\mathcal{L}(PC(\sigma)), \gamma, \xi\big)$$

be the generalized Tarski-Lindenbaum algebra of the predicate calculus of signature σ under the layer AL. Then, $\mathfrak{B}^*$ is effectively pseudo-indecomposable under any semantic layer $L \subseteq ACL$.

PROOF. Immediately from the decomposition (7.1) in Theorem 7.1. □

Conclusion

William Hanf in [16] has solved the known problem of Alfred Tarski about the isomorphism type of the Tarski-Lindenbaum algebra of predicate calculus of a finite rich signature. Historical background of the Tarski problem is discussed in the works [17], [16], [18], [19], [14], [20], and [7]. Results of this review solve a generalized Tarski problem characterizing the Tarski-Lindenbaum algebra of any predicate calculus of a finite rich signature with the description of model-theoretic properties of complete extensions of the predicate calculus. As an immediate consequence, we can obtain most of the currently available results on expressive power of first-order logic. A new extended and advanced edition of the book [7] is prepared to publication that includes both the new results based on the two levels of expressiveness of first-order logic and a new clearer definition to the concept of a quasiexact interpretation (by the scheme presented in Remark 4.6) together with an advanced exposition of the universal construction

Cartesian extensions and finite-to-finite signature reduction procedures are examples of methods of *finitary first-order combinatorics*, while infinite-to-finite signature reduction procedures and an available version of the universal construction of finitely axiomatizable theories are examples of methods of *infinitary first-order combinatorics*. From this point of view, methods and results of [16] and [21] correspond to the infinitary level of expressiveness of first-order logic. On the other hand, methods and results of [18], [19], [14], and [22] correspond to the finitary level of expressiveness of this logic. These two groups of works are based on different approaches, and both deserve to be studied independently, possibly, supplemented by their comparison and benchmarking.

Summing up, it is possible to say that the definition of the concept of a model-theoretic property together with its application to the globalization formulas can be regarded as an attempt to solve the general question of expressive power of formulas of first-order logic. The results can be of interest in pure logic and model theory as well as in applied logic and some branches of computer science.

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