Three-dimensional and One-dimensional Models for Cylinder

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Abstract: - In this paper, we develop mathematical models for 3-D, 2-D and one-dimensional hyperbolic heat equations (wave equation or telegraph equation) with inner source power, and construct their analytical solutions for the determination of the initial heat flux for cylindrical sample. In some cases, we give expression of wave energy. Some solutions of time inverse problems are obtained in the form of the first kind Fredholm integral equation, but others have been obtained in closed analytical form as series. We considered both direct and inverse problems at the time.

Key-Words: - Hyperbolic Equation, Ocean Energy, Steel Quenching, Green Function, Exact Solution, Inverse Problem, Fredholm integral equation, Series.

1 Introduction

Contrary to traditional method, the intensive quenching process uses environmentally friendly highly agitated water or low concentration of water/mineral salt solutions [1]-[3]. Traditionally, classical heat conduction equation is used for the mathematical description of the intensive quenching process. We have proposed to use hyperbolic heat equation [5]-[21] for more realistic description of the intensive quenching (IQ) process (especially for the initial stage of the process). Models of systematic hyperbolic heat equation, their mathematical research and solutions are discussed in monograph [22].

The idea of the usage of hyperbolic heat equation can be easily transferred to completely different sector of application - to the generation of electricity in sea or ocean by usage of wave energy [4] and [24]. It is important to note, that Ekergard and his co-authors [23] examine the development of the system in time, describing the equipment with ordinary differential equation. Here we describe the equipment in development of both - in time as well as in spatial arrangement of equipment using the multi-dimensional hyperbolic heat equation. Wave power plant has to work for long time period in moving environment – waves, see [24].

Therefore, it is important to examine not only the development of equipment in time, but also the movement of its different components [19]-[21]. Wave energy generator models can be viewed in both Cartesian coordinate and cylindrical co-ordinates. We investigate the rectangular models in papers [6]-[9], [14]-[21]. Generators of cylindrical form have been investigated in our papers [5], [12]. For three, two and one-dimensional cylinder we dedicate this paper.

We have constructed various one and two-dimensional analytical exact and approximate solutions for IQ processes in our previous papers [6]-[21], [25], [26].

We consider three-dimensional, two-dimensional and one-dimensional statements for non-homogeneous equation with non-homogeneous boundary conditions. Such statements allow constructing mathematical models for wave power plants in connection with other equipment, for example, with wind power. Boundary conditions could be of different types; thus, they allow us to use Green function method.

In recent years, we have been able to generalize the Green's function method to areas, which consist of several canonical connected sub-areas, and thus we have obtained the exact solutions for the L-, T- and Π-type areas [5]-[21], [25],[26]. We have investigated domains consisting of two cylinders [12] and two-layer sphere [13]. For the cylinder with fin, the solution was obtained for stationary case and hyperbolic heat transfer equation.

2 Mathematical Formulation of 3-D Problem for IQP or Wave Power

We noted already in the introduction that Professor M. Leijon, see [23], examined the development of system in time. Here, we offer to consider the description of system in time and space. For this purpose, instead of the ordinary differential equation, we consider the following partial differential equation:

$$\frac{\partial^2 U}{\partial t^2}=a_\tau^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2}+\frac{\partial^2 U}{\partial z^2}\right]-$$

$$-CU+F(r,\varphi,z,t),r\in[0,R],\varphi\in[0,2\pi], \qquad (1)$$

$$z\in[0,l],t\in[0,T],C\geq 0,a_\tau^2=\frac{a^2}{\tau_r},a^2=\frac{k}{c\rho}.$$

Here, c is specific heat capacity, k - heat conductivity coefficient, ρ - density, τ_r - relaxation time. The source term $F(r,\varphi,z,t)$ can be from different parts of the same device or outer source, for example, wind source.

In the case of wave energy, we can assume different non-homogeneous boundary conditions. Important is to formulate boundary conditions (3), (4) and (5) in the heat energy transfer form [12], [13]:

$$r\frac{\partial U}{\partial r}\bigg|_{r=0}=0, \qquad (2)$$

$$\left(R\frac{\partial U}{\partial r}+k_1U\right)\bigg|_{r=R}=Rg_1(\varphi,z,t),k_1=\frac{Rh_3}{k}. \qquad (3)$$

$$\left(\frac{\partial U}{\partial z}-k_2U\right)\bigg|_{z=0}=g_2(r,\varphi,t),k_i=\frac{h_i}{k}i=2,3, \qquad (4)$$

$$\left(\frac{\partial U}{\partial z}+k_3U\right)\bigg|_{z=l}=g_3(r,\varphi,t). \qquad (5)$$

Here, h_i is heat exchange coefficient. We have heat exchange with environment on all the other sides of device. In fact, it is possible to look at other types of boundary conditions: first (Dirichlet) and second (Neumann) type. The initial conditions for the function $U(r,\varphi,z,t)$ are assumed in the following form:

$$U|_{t=0}=U_0(r,\varphi,z), \qquad (6)$$

$$\frac{\partial U}{\partial t}\bigg|_{t=0}=U_1(r,\varphi,z). \qquad (7)$$

In the steel quenching, the condition (7) can be unrealistic from the practical point of view. The initial heat flux must be determined theoretically. As additional condition, we assume that either the temperature distribution or the heat fluxes distribution at the end of process is known:

$$U|_{t=T}=U_T(r,\varphi,z), \qquad (8)$$

$$\frac{\partial U}{\partial t}\bigg|_{t=T}=U_T^1(r,\varphi,z). \qquad (9)$$

The formulation of the three-dimensional mathematical model is important for wave energy generator [4]. It is good see from the above point on the fig. 1:

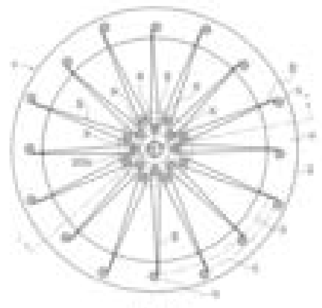

FIG. 2

Fig. 1. The look from the above point of cylindrical piezoelectric generator from patent [4].

For 3-D mathematical model is important that solution in $\varphi-$direction is continuous and smooth. These 2 conditions are important for the reduction of 3-D model to 2-D model by conservative averaging method [25], [26], [31] and [32] (see later in the section 5):

$$U|_{\varphi=0}=U|_{\varphi=2\pi}, \qquad (10)$$

$$\frac{\partial U}{\partial \varphi}\bigg|_{\varphi=0}=\frac{\partial U}{\partial \varphi}\bigg|_{\varphi=2\pi}. \qquad (11)$$

3 Solution of 3-D Problem

First, we assume that we have non-homogeneous Klein-Gordon equation-with source term: $C\geq 0$. The solution in three-dimensional problem is in the following form:

$$U(r,\varphi,z,t)=H(r,\varphi,z,t)+\int_0^R\xi d\xi\int_0^{2\pi}d\varsigma\times$$

$$\int_0^l U_1(\xi,\eta,\varsigma)G(r,\varphi,z,\xi,\eta,\varsigma,t)d\eta+\int_0^{2\pi}d\varsigma\times \qquad (12)$$

$$\int_0^R\xi d\xi\int_0^l U_0(\xi,\eta,\varsigma)\frac{\partial}{\partial t}G(r,\varphi,z,\xi,\eta,\varsigma,t)d\eta.$$

Here are source term and boundary conditions:

$$H(r,\varphi,z,t)=a_\tau^2R^2\int_0^t d\tau\int_0^{2\pi}d\varsigma\times$$

$$\int_0^l g_1(\eta,\varsigma,\tau)G(r,\varphi,z,R,\eta,\varsigma,t-\tau)d\eta - a_\tau^2\int_0^t d\tau \times$$
$$\int_0^{2\pi} d\varsigma \int_0^R \xi g_2(\xi,\varsigma,\tau)G(r,\varphi,z,\xi,0,\varsigma,t-\tau)d\xi + a_\tau^2$$
$$\times\int_0^t d\tau \int_0^{2\pi} d\varsigma \int_0^R \xi g_3(\xi,\varsigma,\tau)G(r,\varphi,z,\xi,l,\varsigma,t-\tau)d\xi$$
$$+\int_0^t d\tau \int_0^{2\pi} d\varsigma \times \tag{13}$$
$$\int_0^R \xi d\xi \int_0^l F(\xi,\eta,\varsigma,\tau)G(r,\varphi,z,\xi,\eta,\varsigma,t-\tau)d\eta.$$

The Green function [27] - [29] for initial-boundary problem for Klein-Gordon equation is known ([29]):

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \sum_{n=0}^\infty\sum_{m=1}^\infty\sum_{s=1}^\infty \frac{1}{\pi}\times$$
$$\frac{A_n\mu_{nm}^2 J_n(\mu_{nm}r)J_n(\mu_{nm}\xi)}{(\mu_{nm}^2R^2+k_1^2R^2-n^2)[J_n(\mu_{nm}R)]^2}\times \tag{14}$$
$$\frac{\cos[n(\varphi-\eta)]h_s(z)h_s(\zeta)\sin(\lambda_{nms}t)}{\|h_s\|^2\lambda_{nms}}.$$

Here, $J_n(\xi)-$ is Bessel's function and

$$\lambda_{nms} = \sqrt{a_\tau^2(\mu_{nm}^2+\beta_s^2)+C},$$
$$A_n = \begin{cases}1, if\ n=0,\\ 2, if\ n>0;\end{cases}$$
$$h_s(z) = \cos(\beta_s z) + \frac{k_2}{\beta_s}\sin(\beta_s z),$$
$$\|h_s\|^2 = \frac{k_3(\beta_s^2+k_2^2)}{2\beta_s^2(\beta_s^2+k_3^2)} + \frac{k_2}{2\beta_s^2} + \frac{l}{2}\left(1+\frac{k_2^2}{\beta_s^2}\right).$$

The eigenvalues μ_{nm},β_s are positive roots of the transcendental equations:

$$\mu J_n'(\mu R)+k_1 J_n(\mu R)=0, \frac{tg(\beta l)}{\beta} = \frac{k_2+k_3}{\beta^2-k_2k_3}.$$

There is an interesting situation if both additional conditions (8), (9) are known. In this case, we introduce new time argument by formula:

$$\tilde{t} = T-t. \tag{15}$$

The formulation for new function $V(r,\varphi,z,\tilde{t})$ with time variable $\tilde{t}$ is the following:

$$\frac{\partial^2 V}{\partial \tilde{t}^2} = a_\tau^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial\varphi^2}+\frac{\partial^2 V}{\partial z^2}\right]-$$
$$-CV+F(r,\varphi,z,T-\tilde{t}),$$
$$V\big|_{\tilde{t}=0} = U_T(r,\varphi,z), \left.\frac{\partial V}{\partial\tilde{t}}\right|_{\tilde{t}=0} = -U_T^1(r,\varphi,z), \tag{16}$$
$$\left.\left(R\frac{\partial V}{\partial r}+k_1V\right)\right|_{r=R} = Rg_1(\varphi,z,T-\tilde{t}),$$
$$\left.\left(\frac{\partial V}{\partial z}-k_2V\right)\right|_{z=l} = g_2(r,\varphi,T-\tilde{t}),$$
$$\left.\left(\frac{\partial V}{\partial z}+k_3V\right)\right|_{z=0} = g_3(r,\varphi,T-\tilde{t}).$$

Solution of the inverse problem looks similar to (12):

$$V(x,y,z,\tilde{t}) = H(x,y,z,\tilde{t}) - \int_0^R \xi d\xi \int_0^{2\pi} d\varsigma \times$$
$$\int_0^l U_T^1(\xi,\eta,\varsigma)G(r,\varphi,z,\xi,\eta,\varsigma,\tilde{t})d\eta + \int_0^{2\pi} d\varsigma \times \tag{17}$$
$$\int_0^R \xi d\xi \int_0^l U_T(\xi,\eta,\varsigma)\frac{\partial}{\partial\tilde{t}}G(r,\varphi,z,\xi,\eta,\varsigma,\tilde{t})d\eta.$$

There is no problem to transform the expression for $H(x,y,z,\tilde{t})$ in the following form:

$$H(x,y,z,\tilde{t}) = a_\tau^2 R\int_{T-\tilde{t}}^T d\tau \int_0^{2\pi} d\varsigma \times$$
$$\int_0^l g_1(\eta,\varsigma,\tau)G(r,\varphi,z,R,\eta,\varsigma,T-\tau)d\eta - a_\tau^2\int_{T-\tilde{t}}^T d\tau$$
$$\int_0^{2\pi} d\varsigma \int_0^R g_2(\xi,\varsigma,\tau)G(x,y,z,\xi,0,\varsigma,T-\tau)d\xi + a_\tau^2\times$$
$$\int_{T-\tilde{t}}^T d\tau \int_0^{2\pi} d\varsigma \int_0^R g_3(\xi,\varsigma,\tau)G(x,y,z,\xi,b,\varsigma,T-\tau)d\xi +$$
$$\int_{T-\tilde{t}}^T d\tau \int_0^{2\pi} d\varsigma \times$$
$$\int_0^R d\xi \int_0^l F(\xi,\eta,\varsigma,\tau)G(r,\varphi,z,\xi,\eta,\varsigma,T-\tau)d\eta.$$

We have the expression for the heat flux in time from (17):

$$\frac{\partial}{\partial\tilde{t}}V(r,\varphi,z,\tilde{t}) = \frac{\partial}{\partial\tilde{t}}H(r,\varphi,z,\tilde{t})+$$

$$\int_0^R \xi d\xi \int_0^{2\pi} d\varsigma \int_0^l V_1(\xi,\eta,\varsigma)\frac{\partial}{\partial \tilde t} G(r,\varphi,z,\xi,\eta,\varsigma,\tilde t)d\eta +$$
$$\int_0^{2\pi} d\varsigma \int_0^R \xi d\xi \int_0^l V_0(\xi,\eta,\varsigma)\frac{\partial^2}{\partial \tilde t^2} G(r,\varphi,z,\xi,\eta,\varsigma,\tilde t)d\eta.$$

From last expression at $\tilde t = T$ and equality (17), we have solution for the time inverse problem:

$$U_T^1(r,\varphi,z) = -\frac{\partial}{\partial \tilde t} H\left(r,\varphi,z,\tilde t\right)\Big|_{\tilde t = T} - \tag{18}$$
$$\int_0^R \xi d\xi \int_0^{2\pi} d\varsigma \int_0^l V_1(\xi,\eta,\varsigma)\frac{\partial}{\partial \tilde t} G(r,\varphi,z,\xi,\eta,\varsigma,\tilde t)\Big|_{\tilde t=T} d\eta -$$
$$\int_0^{2\pi} d\varsigma \int_0^R \xi d\xi \int_0^l V_0(\xi,\eta,\varsigma)\frac{\partial^2}{\partial \tilde t^2} G(r,\varphi,z,\xi,\eta,\varsigma,\tilde t)\Big|_{\tilde t=T} d\eta.$$

Very interesting is wave energy [24] as you can see in [30]:

$$I_0\left(t\right) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{\sin^2\left(\lambda_{nms}t\right)}{\lambda_{nms}^2}.$$

4 Solution of 3-D problem with constant initial conditions

In the previous section, we have constructed some three-dimensional solutions for direct and time inverse problems for hyperbolic heat equation. Initial conditions enough often are constant functions [17], [20]. In this case, it is possible to solve the solutions in the form of series. For simplicity, we look the homogeneous boundary conditions:

$$U(r,\varphi,z,t) = U_1 \int_0^R \xi d\xi \int_0^{2\pi} d\varsigma \times$$
$$\int_0^l G(r,\varphi,z,\xi,\eta,\varsigma,t)d\eta + U_0 \int_0^{2\pi} d\varsigma \times \tag{19}$$
$$\int_0^R \xi d\xi \int_0^l \frac{\partial}{\partial t} G(r,\varphi,z,\xi,\eta,\varsigma,t)d\eta =$$
$$= U_0 G_0 + U_1 G_1.$$

We use the Green function (14) in a bit different form:

$$G(r,\varphi,z,\xi,\eta,\zeta,t) = \frac{1}{\pi}\times$$
$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n\left(\mu_{nm}r\right)J_n\left(\mu_{nm}\xi\right)}{\left(\mu_{nm}^2R^2 + k_1^2R^2 - n^2\right)\left[J_n\left(\mu_{nm}R\right)\right]^2}\times$$
$$\frac{\left[\cos(n\varphi)\cos(n\eta)+\sin(n\varphi)\sin(n\eta)\right]}{\left\|h_s\right\|^2}$$
$$\frac{h_s\left(z\right)h_s\left(\zeta\right)\sin\left(\lambda_{nms}t\right)}{\lambda_{nms}}. \tag{20}$$

After integration, the function G_0 can be obtained in the following form:

$$G_0 = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n\left(\mu_{nm}r\right)\cos(\lambda_{nms}t)}{\left(\mu_{nm}^2R^2 + k_1^2R^2 - n^2\right)\left[J_n\left(\mu_{nm}R\right)\right]^2}\times$$
$$\frac{\left[\cos(n\varphi)\sin(nl)+\sin(n\varphi)(1-\cos(n\eta))\right]}{n\left\|h_s\right\|^2}h_s\left(z\right)\times$$
$$\frac{\sin(2\pi\beta_s)+\dfrac{k_2}{\beta_s}\left(1-\cos(2\pi\beta_s)\right)}{\beta_s}\int_0^R \xi J_n\left(\mu_{nm}\xi\right)d\xi.$$

Similarly, we can transform the function G_1:

$$G_1 = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{A_n\mu_{nm}^2 J_n\left(\mu_{nm}r\right)\sin(\lambda_{nms}t)}{\left(\mu_{nm}^2R^2 + k_1^2R^2 - n^2\right)\left[J_n\left(\mu_{nm}R\right)\right]^2}\times$$
$$\frac{\left[\cos(n\varphi)\sin(nl)+\sin(n\varphi)(1-\cos(n\eta))\right]}{n\left\|h_s\right\|^2}h_s\left(z\right)\times$$
$$\frac{\sin(2\pi\beta_s)+\dfrac{k_2}{\beta_s}\left(1-\cos(2\pi\beta_s)\right)}{\beta_s}\int_0^R \xi J_n\left(\mu_{nm}\xi\right)d\xi.$$

In this paper, we can show that time reverse problem with two final time conditions is not ill-posed problem and can be solved similarly as time direct problem. It was shown in our paper [15] that time reverse problem for rectangular sample can be solved without some numerical problem. It is well known that for inverse problem it is not easy to calculate the solution [25].

5 Solution of Two-dimensional Problem

Two-dimensional problem can be obtained in two ways. First way is standard: we use monograph [29] for the two-dimensional solution and Green function. The mathematical model is in the form:

$$\frac{\partial^2 U}{\partial t^2} = a_\tau^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{\partial^2 U}{\partial z^2}\right] - CU + F(r,z,t),$$
$$r\in\left[0,R\right], z\in\left[0,l\right], t\in\left[0,T\right], C\geq 0,$$
$$r\frac{\partial U}{\partial r}\Big|_{r=0} = 0, \left(\frac{\partial U}{\partial r}+k_1U\right)\Big|_{r=R} = g_1\left(z,t\right),$$

$$\left(\frac{\partial U}{\partial z}-k_2U\right)\bigg|_{z=0}=g_2\left(r,t\right), \qquad (21)$$

$$\left(\frac{\partial U}{\partial z}+k_3U\right)\bigg|_{z=l}=g_3\left(r,t\right),$$

$$U\big|_{t=0}=U_0(r,z),\frac{\partial U}{\partial t}\bigg|_{t=0}=U_1(r,z).$$

Of course, the temperature distribution and the heat fluxes distribution at the end of process is given:

$$U\big|_{t=T}=U_T(r,z),\frac{\partial U}{\partial t}\bigg|_{t=T}=U_T^1(r,z).$$

The solution of the two-dimensional problem is in following form:

$$U(r,z,t)=H\left(r,z,t\right)+\int_0^R\xi d\xi\times$$
$$\int_0^l U_1(\xi,\eta)G(r,z,\xi,\eta,t)d\eta+ \qquad (22)$$
$$\int_0^R\xi d\xi\int_0^l U_0(\xi,\eta)\frac{\partial}{\partial t}G(r,z,\xi,\eta,t)d\eta.$$

The known boundary conditions and source term are in the function $H\left(r,z,t\right)$:

$$H\left(r,z,t\right)=-a_\tau^2\int_0^t d\tau\int_0^R g_2\left(\xi,\tau\right)G(r,z,\xi,0,t-\tau)d\xi$$
$$a_\tau^2\int_0^t d\tau\int_0^l g_1\left(\eta,\tau\right)G(r,z,R,\eta,t-\tau)d\eta+ \qquad (23)$$
$$+a_\tau^2\int_0^t d\tau\int_0^R g_3\left(\xi,\tau\right)G(r,z,\xi,l,t-\tau)d\xi+$$
$$\int_0^t d\tau\int_0^R d\xi\int_0^l F\left(\xi,\eta,\tau\right)G(r,z,\xi,\eta,t-\tau)d\eta.$$

The Green function for two-dimensional problem is in the form [29]:

$$G(r,z,\xi,\eta,t)=\frac{1}{\pi R^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu_n^2J_0\left(\frac{\mu_n r}{R}\right)J_0\left(\frac{\mu_n\xi}{R}\right)}{\left(k_1^2R^2+\mu_n^2\right)J_0^2\left(\mu_n\right)}\times$$
$$\frac{\varphi_m\left(z\right)\varphi_m\left(\eta\right)}{\left\|\varphi_m\right\|^2\sqrt{\lambda_m}}\exp\left[-\left(a_\tau^2\lambda_m^2+\frac{a_\tau^2\mu_n^2}{R^2}\right)t\right],$$

$$\varphi_m\left(z\right)=\cos\left(\lambda_m z\right)+\frac{k_2}{\lambda_m}\sin\left(\lambda_m z\right), \qquad (24)$$

$$\left\|\varphi_m\right\|^2=\frac{k_3\left(\lambda_m^2+k_2^2\right)}{2\lambda_m^2\left(\lambda_m^2+k_3^2\right)}+\frac{k_2}{2\lambda_m^2}+\frac{l}{2}\left(1+\frac{k_2^2}{\lambda_m^2}\right).$$

The eigenvalues μ_n,λ_m are positive roots of the transcendental equations:

$$\mu J_1\left(\mu\right)+k_1RJ_0\left(\mu\right)=0,\frac{tg\left(\lambda l\right)}{\lambda}=\frac{k_2+k_3}{\lambda^2-k_2k_3}.$$

We will obtain the solution for two-dimensional problem as it was done in our papers [25], [26] and [31] by method of conservative averaging:

$$V(r,z,t)=\frac{1}{2\pi}\int_0^{2\pi}U(r,\varphi,z,t)d\varphi.$$

We integrate the main differential equation (1) in the direction $\varphi\in\left[0,2\pi\right]$:

$$\frac{\partial^2V}{\partial t^2}=a_\tau^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{\partial^2V}{\partial z^2}\right]+$$
$$\frac{1}{2\pi r^2}\frac{\partial U}{\partial\varphi}\bigg|_{\varphi=0}^{\varphi=2\pi}-CV+f(r,z,t),$$

$$f(r,z,t)=\frac{1}{2\pi}\int_0^{2\pi}F(r,\varphi,z,t)d\varphi.$$

The equality (11) gives two-dimensional equation:

$$\frac{\partial^2V}{\partial t^2}=a_\tau^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{\partial^2V}{\partial z^2}\right]- \qquad (25)$$
$$CV+f(r,z,t).$$

Formulas (21) are initial and boundary conditions for this equation.

6 Solution of One Dimensional Problem

We will start with a formulation of the mathematical model of the steel cylinder which is relatively thin in z directions: $l<<R.$ In accordance with the conservative averaging method [25], [26], we introduce the following integral averaged value (one space-dimensional function) for the two-dimensional formulation:

$$u(r,t)=\left(l\right)^{-1}\int_0^l U(r,z,t)dz,$$
$$\overline{f}(r,t)=\left(l\right)^{-1}\int_0^l f(r,z,t)dz. \qquad (26)$$

We integrate equation (27) in the direction z:

$$\frac{\partial^2 u}{\partial t^2} = a_\tau^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \right] + \\ + \frac{1}{l} \frac{\partial U}{\partial z} \bigg|_{z=0}^{z=l} - Cu + \overline{f}(r,t). \tag{27}$$

The boundary conditions (23) for new function $u(r,t)$ are as follows:

$$\frac{1}{l} \frac{\partial U}{\partial z} \bigg|_{z=0} = k_2 U(r,0,t) + g_2(r,t), \frac{1}{l} \frac{\partial U}{\partial z} \bigg|_{z=l} = \\ -k_3 U(r,l,t) + g_3(r,t).$$

We look for thin cylinder, it means that we have:

$$U(r,0,t) = U(r,l,t) \cong u(r,t).$$

Finally, we transform the equation (29) to Klein-Gordon equation form:

$$\frac{\partial^2 u}{\partial t^2} = a_\tau^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \right] - cu + \tilde{f}(r,t), \\ c = C + k_2 + k_3, \\ \tilde{f}(r,t) = \overline{f}(r,t) - g_2(r,t) + g_3(r,t). \tag{28}$$

The main differential equation together with boundary conditions and initial conditions from (25) are in following form:

$$\left(\frac{\partial u}{\partial r} + k_1 u \right) \bigg|_{r=R} = \overline{g}_1(t), \\ \overline{g}_1(t) = (l)^{-1} \int_0^l g_1(z,t) dz, \\ u|_{t=0} = u_0(r), \frac{\partial u}{\partial t} \bigg|_{t=0} = u_1(r), \\ u_0(t) = (l)^{-1} \int_0^l U_0(z,t) dz, \\ u_1(t) = (l)^{-1} \int_0^l U_1(z,t) dz. \tag{29}$$

Solution of this problem is expressed by Green function, see [35]:

$$u(r,t) = \int_0^R u_0(\xi) \frac{\partial}{\partial t} G(r,\xi,t) d\xi + \\ \int_0^R u_1(\xi) G(r,\xi,t) d\xi + \\ a_\tau^2 \int_0^t \overline{g}_1(\tau) G(r,R,t-\tau) d\tau + \\ \int_0^t d\tau \int_0^R \tilde{f}(\xi,\tau) G(r,\xi,t-\tau) d\xi. \tag{30}$$

Green function from [29] is in the form:

$$G(r,\xi,t) = \frac{2\xi}{R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2 J_0\left(\frac{\mu_n r}{R}\right)}{\left(k_1^2 R^2 + \mu_n^2\right) J_0^2(\mu_n)} \times \\ J_0\left(\frac{\mu_n \xi}{R}\right) \frac{\sin\left(t\sqrt{\lambda_n}\right)}{\sqrt{\lambda_m}}, \lambda_n = \frac{a_\tau^2 \mu_n^2}{R^2} + c. \tag{31}$$

The eigenvalues μ_n are positive roots of the transcendental equation:

$$\mu J_1(\mu) - k_1 R J_0(\mu) = 0.$$

Another situation is for cylinder with small diameter: $R << l.$ From (25), we define a new function $v(z,t)$:

$$v(z,t) = \frac{1}{R^2} \int_0^R r V(r,z,t) dr, \\ \hat{f}(z,t) = \frac{1}{R^2} \int_0^R r f(r,z,t) dr.$$

We integrate the modified differential equation (25) in r direction:

$$\frac{\partial^2 rV}{\partial t^2} = a_\tau^2 \left[\frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 rV}{\partial z^2} \right] - CrV + rf(r,z,t).$$

This gives:

$$\frac{\partial^2 v}{\partial t^2} = a_\tau^2 \frac{\partial^2 v}{\partial z^2} + r \frac{\partial V}{\partial r} \bigg|_{r=0}^{r=R} - Cv + \hat{f}(z,t).$$

The boundary condition in the r direction gives:

$$r \frac{\partial U}{\partial r} \bigg|_{r=R} = -k_1 R U(R,t) + R g_1(z,t), \\ r \frac{\partial U}{\partial r} \bigg|_{r=0} = 0.$$

Finally, we have a one-dimensional Klein-Gordon partial differential equation:

$$\frac{\partial^2 v}{\partial t^2} = a_\tau^2 \frac{\partial^2 v}{\partial z^2} - dv + g(z,t), \\ d = C + k_1 R, g(z,t) = \hat{f}(z,t) + R g_1(z,t). \tag{32}$$

The boundary conditions and initial conditions from (23) can be rewritten in the following form:

$$\left(\frac{\partial v}{\partial z} - k_2 v \right) \bigg|_{z=0} = g_2(t), \\ \left(\frac{\partial v}{\partial z} + k_3 v \right) \bigg|_{z=l} = g_3(t), \\ v|_{t=0} = v_0(z), \frac{\partial v}{\partial t} \bigg|_{t=0} = v_1(z). \tag{33}$$

Here, the new averaged functions are:

$$v_0(z)=\frac{1}{R^2}\int_0^R rU_0(r,z)dr, v_1(z)=\frac{1}{R^2}\int_0^R rU_1(r,z)dr,$$

$$g_2(t)=\frac{1}{R^2}\int_0^R rg_2(r,t)dr, g_3(t)=\frac{1}{R^2}\int_0^R rg_3(r,t)dr.$$

We have solution in the following form:

$$u(z,t)=\int_0^l v_0(\eta)\frac{\partial}{\partial t}G(z,\eta,t)d\eta+$$
$$\int_0^l v_1(\eta)G(z,\eta,t)d\eta-$$
$$a_\tau^2\int_0^t g_2(\tau)G(z,0,t-\tau)d\tau+ \tag{34}$$
$$a_\tau^2\int_0^t g_3(\tau)G(z,l,t-\tau)d\tau+$$
$$\int_0^t d\tau\int_0^l g(\eta,\tau)G(z,\eta,t-\tau)d\eta.$$

Green function in this case is ([29], [30]):

$$G(z,\eta,t)=\sum_{n=1}^{\infty}\frac{y_n(z)y_n(\varsigma)\sin\left(t\sqrt{a_\tau^2\lambda_n^2+d}\right)}{\|y_n\|^2\sqrt{a_\tau^2\lambda_n^2+d}},$$

$$y_n(z)=\cos(\lambda_n z)+\frac{k_2}{\lambda_n}\sin(\lambda_n z), \tag{35}$$

$$\|y_n\|=\frac{k_2}{2\lambda_n^2}\frac{\lambda_n^2+k_2^2}{\lambda_n^2+k_3^2}+\frac{k_2}{2\lambda_n^2}+\frac{l}{2}\left(1+\frac{k_2^2}{\lambda_n^2}\right).$$

The eigenvalues λ_n are positive roots of the transcendental equations:

$$\frac{tg(\lambda l)}{\lambda}=\frac{k_2+k_3}{\lambda^2-k_2k_3}.$$

We have two final conditions for both one-dimensional problems. Additional conditions for the problem (30), (31) are:

$$u\big|_{t=T}=u_T(r), \left.\frac{\partial u}{\partial t}\right|_{t=T}=u_T^1(r). \tag{36}$$

In its turn, additional conditions for the problem (32), (33) are:

$$v\big|_{t=T}=v_T(z), \left.\frac{\partial v}{\partial t}\right|_{t=T}=v_T^1(z). \tag{37}$$

7 Time Inverse One Dimensional Problem

We would like to continue the one-dimensional problem with time inverse formulation (17) for $\bar{u}(r,\tilde{t})$:

$$\frac{\partial^2\bar{u}}{\partial\tilde{t}^2}=a_\tau^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{u}}{\partial r}\right)\right]-c\bar{u}+\tilde{f}(r,T-\tilde{t}), \tag{38}$$
$$r\in(0,R), \tilde{t}\in(0,T],$$
$$\bar{u}\big|_{\tilde{t}=0}=u_T(r), \left.\frac{\partial\bar{u}}{\partial\tilde{t}}\right|_{\tilde{t}=0}=-u_T^1(r),$$
$$\left.\left(\frac{\partial\bar{u}}{\partial r}+k_1\bar{u}\right)\right|_{r=R}=\bar{g}_1(T-\tilde{t}),$$
$$\bar{u}\big|_{\tilde{t}=T}=u_0(r), \left.\frac{\partial\bar{u}}{\partial\tilde{t}}\right|_{\tilde{t}=T}=-u_1(r).$$

Solution is similar with the formula (30):

$$\bar{u}(r,\tilde{t})=\int_0^R u_T(\xi)\frac{\partial}{\partial\tilde{t}}G(r,\xi,\tilde{t})d\xi-$$
$$\int_0^R u_T^1(\xi)G(r,\xi,\tilde{t})d\xi+$$
$$a_\tau^2\int_0^{\tilde{t}}\bar{g}_1(T-\tilde{t}+\tau)G(r,R,\tilde{t}-\tau)d\tau+$$
$$\int_0^{\tilde{t}}d\tau\int_0^l\tilde{f}(\xi,T-\tilde{t}+\tau)G(r,\xi,\tilde{t}-\tau)d\xi.$$

The solution can be rewritten in the following form:

$$\bar{u}(r,\tilde{t})=\int_0^R u_T(\xi)\frac{\partial}{\partial\tilde{t}}G(r,\xi,\tilde{t})d\xi-$$
$$\int_0^R u_T^1(\xi)G(r,\xi,\tilde{t})d\xi+$$
$$a_\tau^2\int_0^{\tilde{t}}\bar{g}_1(T-\tau_1)G(r,R,\tau_1)d\tau_1+$$
$$\int_0^{\tilde{t}}d\tau_1\int_0^l\tilde{f}(\xi,T-\tau_1)G(r,\xi,\tau_1)d\xi. \tag{39}$$

For the heat flux, we have an expression:

$$\frac{\partial}{\partial\tilde{t}}\bar{u}(x,\tilde{t})=\int_0^l u_T(\xi)\frac{\partial^2}{\partial\tilde{t}^2}G(x,\xi,\tilde{t})d\xi$$
$$-\int_0^l u_T^1(\xi)\frac{\partial}{\partial\tilde{t}}G(x,\xi,\tilde{t})d\xi+$$
$$a_\tau^2\frac{\partial}{\partial\tilde{t}}\int_0^{\tilde{t}}\bar{g}_1(T-\tau_1)G(r,R,\tau_1)d\tau_1+$$
$$\frac{\partial}{\partial\tilde{t}}\int_0^{\tilde{t}}d\tau_1\int_0^l\tilde{f}(\xi,T-\tau_1)G(r,\xi,\tau_1)d\xi.$$

From here, a nice explicit representation of the necessary initial heat flux immediately follows:

$$v_0(x) = -\int_0^l v_T(\xi)\frac{\partial}{\partial \tilde{t}}G(x,\xi,\tilde{t})\Big|_{\tilde{t}=T}\,d\xi +$$
$$\int_0^l u_T(\xi)\frac{\partial^2}{\partial \tilde{t}^2}G(x,\xi,\tilde{t})\Big|_{\tilde{t}=T}\,d\xi + \bar{g}_1(0)G(r,R,T) +$$
$$+\int_0^l \tilde{f}(\xi,0)G(r,\xi,T)d\xi. \quad (40)$$

8 Solution of 1-D problem with constant initial conditions

We would like to finish the one-dimensional solution with a simplification for constant initial conditions in the formulation (34)-(35):

$$v\big|_{t=0} = v_0(z) = v_0 = const,$$
$$\frac{\partial v}{\partial t}\Big|_{t=0} = v_1(z) = v_1 = const. \quad (41)$$

The solution of the time direct problem is the following if we assume $g(z,t) = g_2(t) = g_3(t) = 0$:

$$u(z,t) = v_0\int_0^l \frac{\partial}{\partial t}G(z,\xi,t)d\xi +$$
$$v_1\int_0^l G(z,\xi,t)d\xi = v_0 I_0 + v_1 I_1. \quad (42)$$

Intensive steel quenching process with initial conditions (43) is very natural [10]-[14]. We have the homogeneous equation (34) and the homogeneous boundary conditions. Next, we integrate Green functions in the formula (44):

$$I_0 = \int_0^l \frac{\partial}{\partial t}G(z,\xi,t)d\xi =$$
$$\sum_{n=1}^{\infty}\frac{y_n(z)y_n(l)\cos\left(t\sqrt{a_\tau^2\lambda_n^2+d}\right)}{\lambda_n\left\|y_n^2\right\|},$$
$$y_n(l) = \cos(\lambda_n l) + \frac{k_2}{\lambda_n}\sin(\lambda_n l);$$
$$I_1 = \int_0^l G(x,\xi,t)d\xi =$$
$$\sum_{n=1}^{\infty}\frac{y_n(z)y_n(l)\sin\left(t\sqrt{a_\tau^2\lambda_n^2+d}\right)}{\lambda_n\left\|y_n^2\right\|}. \quad (43)$$

It means finally that we have expression for temperature in the form of series:

$$u(x,t) = v_0 I_0(x,t) + v_1 I_1(x,t) = \sum_{n=1}^{\infty}\frac{y_n(z)y_n(l)}{\lambda_n\left\|y_n^2\right\|}\times$$
$$\left[v_0\cos\left(t\sqrt{a_\tau^2\lambda_n^2+d}\right) + v_1\sin\left(t\sqrt{a_\tau^2\lambda_n^2+d}\right)\right]. \quad (44)$$

Similarly, we can transform the derivative for heat flux in the form of series:

$$\frac{\partial u}{\partial t} = v_0\int_0^l \frac{\partial^2}{\partial t^2}G(z,\eta,t)d\eta +$$
$$v_1\int_0^l \frac{\partial}{\partial t}G(z,\eta,t)d\eta = v_0 J_0 + v_1 J_1.$$
$$J_0(x,t) = -\sum_{n=1}^{\infty}\frac{y_n(z)y_n(l)\sin\left(t\sqrt{a_\tau^2\lambda_n^2+d}\right)}{\left(\sqrt{a_\tau^2\lambda_n^2+d}\right)^{-1}\lambda_n\left\|y_n^2\right\|},$$
$$J_1(x,t) = \sum_{n=1}^{\infty}\frac{y_n(z)y_n(l)\cos\left(t\sqrt{a_\tau^2\lambda_n^2+d}\right)}{\lambda_n\left\|y_n^2\right\|}. \quad (45)$$

The solution with homogeneous boundary condition and without source term is in the following form:

$$u(r,t) = u_0\int_0^R \frac{\partial}{\partial t}G(r,\xi,t)d\xi + u_1\int_0^R G(r,\xi,t)d\xi =$$
$$u_0 K_0(r,t) + u_1 K_1(r,t).$$

Here:

$$K_0(r,t) = \sum_{n=1}^{\infty}\frac{2\mu_n^2 J_0\left(\frac{\mu_n r}{R}\right)}{R^2\left(k_1^2R^2+\mu_n^2\right)}\cos\left(\sqrt{\lambda_n}t\right)\times$$
$$\int_0^R J_0\left(\frac{\mu_n\xi}{R}\right)\xi d\xi = \sum_{n=1}^{\infty}\frac{2RJ_0\left(\frac{\mu_n r}{R}\right)}{\mu_n\left(k_1^2R^2+\mu_n^2\right)}\times$$
$$\cos\left(\sqrt{\lambda_n}t\right)\int_0^{\mu_1}\xi_1 J_0(\xi_1)d\xi_1, K_1(r,t) =$$
$$\sum_{n=1}^{\infty}\frac{2RJ_0\left(\frac{\mu_n r}{R}\right)}{\lambda_n\mu_n\left(k_1^2R^2+\mu_n^2\right)}\sin\left(\sqrt{\lambda_n}t\right)\int_0^{\mu_n}\xi_1 J_0(\xi_1)d\xi_1.$$

9 Conclusion

We have constructed some solutions for direct and time inverse problems for hyperbolic heat equation. The solutions for determination of initial heat flux are obtained either in the form of Fredholm integral equation of 1st kind with continuous kernel or in the closed analytical form – in the form of series or ordinary integrals.

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