Fault tolerance of hypercubes and folded hypercubes

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Abstract: Connectivity is a vital metric to explore fault tolerance and reliability of network structure based on a graph model. There are many kinds of connectivity to measure the fault tolerance and reliability of networks, such as classic connectivity, super connectivity, extraconnectivity. In this paper we focus on the number of components of graph which is called component connectivity. Let $G = (V, E)$ be a connected graph. A r-component cut of G is a set of vertices whose deletion results in a graph with at least r components. r-component connectivity $c\kappa_r(G)$ of G is the size of the smallest r-component cut. The r-component edge connectivity $c\lambda_r(G)$ can be defined similarly. In this paper, we determine the r-component edge connectivity of hypercubes and folded hypercubes:(1) $c\lambda_2(Q_n) = \lambda(Q_n) = n$ for $n \geq 2$. (2) $c\lambda_3(Q_n) = 2n - 1$ for $n \geq 2$. (3) $c\lambda_4(Q_n) = 3n - 2$ for $n \geq 2$. (4) $c\lambda_2(FQ_n) = n + 1$ for $n \geq 3$. (5) $c\lambda_3(FQ_n) = 2n + 1$ for $n \geq 3$. (6) $c\lambda_4(FQ_n) = 3n + 1$ for $n \geq 3$.

Key–Words: Interconnection networks; Fault tolerance; r-component edge connectivity

1 Introduction

A network is often modeled by a graph $G = (V, E)$ with the vertices representing nodes such as processors or stations, and the edges representing links between the nodes. One fundamental consideration in the design of networks is reliability [2, 9]. Let $G = (V, E)$ be a connected graph, $N_G(v)$ the neighbors of a vertex v in G (simply $N(v)$), $E(v)$ the edges incident to v. Moreover, for $S \subset V$, $G[S]$ is the subgraph induced by S, $N_G(S) = \cup_{v\in S}N(v) - S$, $N_G[S] = N_G(S) \cup S$, and $G - S$ denotes the subgraph of G induced by the vertex set of $V \setminus S$. If $u, v \in V$, $d(u, v)$ denotes the length of a shortest (u, v)-path. For $X, Y \subset V$, denote by $[X, Y]$ the set of edges of G with one end in X and the other in Y. For graph-theoretical terminology and notation not defined here we follow [1]. All graphs considered in this paper are simple, finite and undirected.

A r-component cut of G is a set of vertices whose deletion results in a graph with at least r components. r-component connectivity $c\kappa_r(G)$ of G is the size of the smallest r-component cut. The r-component edge connectivity $c\lambda_r(G)$ can be defined correspondingly. We can see that $c\kappa_{r+1}(G) \geq c\kappa_r(G)$ for each positive integer r. The connectivity $\kappa(G)$ is the 2-component connectivity $c\kappa_2(G)$. The r-component (edge) connectivity was introduced in [3] and [11] independently. Fábrega and Fiol introduced extraconnectivity in [5]. Let $F \subseteq V$ be a vertex set, F is called extra-cut, if $G - F$ is not connected and each component of $G - F$ has more than k vertices. The extraconnectivity $\kappa_k(G)$ is the cardinality of the minimum extra-cuts.

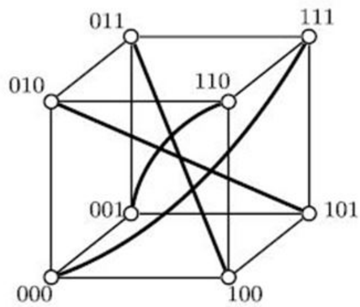

Fig.1 FQ_3

The hypercube $Q_n = (V, E)$ with $|V| = 2^n$ and $|E| = n2^{n-1}$. Every vertex can be represent by an n-bit binary string. Two vertices are adjacent if and only if their binary string representation differs in only one bit position. The n-dimensional folded hypercube FQ_n is proposed by El-Amawy and Latifi [4]. FQ_n is obtained from Q_n by adding 2^{n-1} edges, called complementary edges, each of them is between vertices $x = (x_1, \cdots, x_n)$ and $\overline{x} = (\overline{x}_1, \cdots, \overline{x}_n)$, where $\overline{x}_i = 1 - x_i$. Obviously, FQ_n is obtained from Q_n by adding a perfect matching M where $M = \{(x, \overline{x}) : x \in V(Q_n)\}$. Because Q_n can be expressed as $Q^0_{n-1} \odot Q^1_{n-1}$, where Q^0_{n-1} and Q^1_{n-1} are two $n-1$-dimensional hypercubes with the prefix 0 and 1 of each vertex, respectively. Furthermore, Q_n can be viewed as $G(Q^0_{n-1}, Q^1_{n-1}, M_0)$, where $M_0 = \{(0u, 1u) : 0u \in V(Q^0_{n-1}), 1u \in V(Q^1_{n-1})\}$. Similarly, FQ_n can be viewed as $G(Q^0_{n-1}, Q^1_{n-1}, M_0 + \overline{M})$, where $M_0 = \{(0u, 1u) : 0u \in V(Q^0_{n-1}), 1u \in V(Q^1_{n-1})\}$ and $\overline{M} = \{(0u, 1\overline{u}) : 0u \in V(Q^0_{n-1}), 1\overline{u} \in V(Q^1_{n-1})\}$.

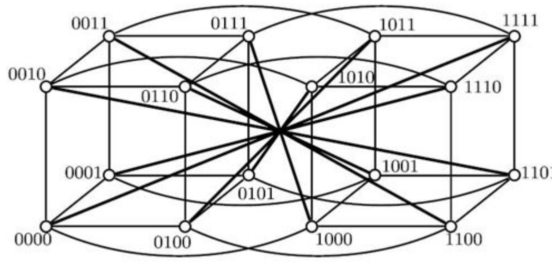

Fig.2 FQ_4

FQ_n is $(n+1)$-regular and $(n+1)$-connected. Moreover, FQ_n is a Cayley graph. It has diameter $\lceil n/2 \rceil$, about a half of the diameter of Q_n [4]. Thus, the folded hypercube FQ_n is an enhancement on the hypercube Q_n.

The fault tolerance analysis of hypercubes and folded hypercubes has recently attracted the attention of many researchers [6, 7, 10],[12]-[18],[20, 21]. In [8], Hsu et al. determined the r-component connectivity of the hypercube Q_n for $r = 2, 3, \cdots, n+1$. In [19], Zhao et al. determined the r-component connectivity of the hypercube Q_n for $r = n+2, n+3, \cdots, 2n-4$. In this paper, we obtain that:

(1) $c\lambda_2(Q_n) = \lambda(Q_n) = n$ for $n \geq 2$.
(2) $c\lambda_3(Q_n) = 2n-1$ for $n \geq 2$.
(3) $c\lambda_4(Q_n) = 3n-2$ for $n \geq 2$.
(4) $c\lambda_2(FQ_n) = n+1$ for $n \geq 3$.
(5) $c\lambda_3(FQ_n) = 2n+1$ for $n \geq 3$.
(6) $c\lambda_4(FQ_n) = 3n+1$ for $n \geq 3$.

2 Component connectivity of hypercubes and folded hypercubes

For the sake of convenience, we denote the vertex whose ith coordinate of the binary string representation different from v's by v_i. Similarly, v_{ij} is the vertex whose n-bit binary string which differs in the jth position with v_i. Clearly, $v_{ii} = v$.

Lemma 2.1. *[18] Any two vertices of Q_n have exactly two common neighbors for $n \geq 3$ if they have any.*

Lemma 2.2. *[17] Any two vertices of FQ_n have exactly two common neighbors for $n \geq 4$ if they have.*

Corollary 2.3. *For any two vertices $x, y \in V(Q_n)(n \geq 3)$ or $V(FQ_n)(n \geq 4)$,*

(1) if $d(x,y) = 2$, then they have exactly two common neighbors;

(2) if $d(x,y) \neq 2$, then they do not have common neighbors.

Lemma 2.4. *Let x and y be any two vertices of $V(Q_n)(n \geq 3)$ such that have two common neighbors.*

(1) If $x \in V(Q^0_{n-1}), y \in V(Q^1_{n-1})$, then the one common neighbor is in Q^0_{n-1}, and the other one is in Q^1_{n-1}.

(2) If $x, y \in V(Q^0_{n-1})$ or $V(Q^1_{n-1})$, then the two common neighbors are in Q^0_{n-1} or Q^1_{n-1}.

Proof. (1) Let $x = 0u$ and $y = 1u_i$. Then x, y have two common neighbors $1u, 0u_i$. According to Lemma 2.1, the result holds.

(2) Let $x = 0u$ and $y = 0v$. Since they have two common neighbors, we assume that they are $0u_i, 0u_j$. And $0u_{ij}$ has two neighbors $0u_i, 0u_j$. According to Lemma 2.1, $y = 0v = 0u_{ij}$. □

Analogue to Lemma 2.4, we have

Lemma 2.5. *For any two vertices $x, y \in V(FQ_n)(n \geq 4)$, $FQ_n = G(Q^0_{n-1}, Q^1_{n-1}, M_0 + \overline{M})$, and x and y have two common neighbors.*

(1) If $x \in V(Q^0_{n-1}), y \in V(Q^1_{n-1})$, then one of the common neighbors is in Q^0_{n-1}, and the other one is in Q^1_{n-1}.

(2) If $x, y \in V(Q^0_{n-1})$ or $V(Q^1_{n-1})$, then both of the common neighbors are in Q^0_{n-1} or Q^1_{n-1}.

Theorem 2.6. $c\lambda_2(Q_n) = \lambda(Q_n) = n$ *for* $n \geq 2$.

Theorem 2.7. $c\lambda_3(Q_n) = 2n-1$ *for* $n \geq 2$.

Proof. Take an edge $e = uv$, then $|E(u) \cup E(v)| = 2n-1$. And $Q_n - E(u) - E(v)$ has at least 3 connected components. That is $c\lambda_3(Q_n) \leq 2n-1$.

Next we will show that $c\lambda_3(Q_n) \geq 2n-1$ by induction. It is easy to check it is true for $n = 2, 3, 4$. So we suppose $n \geq 5$ and assume it is true for all $k < n$. We will prove that is true for $k = n$.

Let $F \subseteq E(Q_n)$ with $|F| \leq 2n-2$, and $Q_n - F$ has at least 3 components. Now since $Q_n = Q^0_{n-1} \odot Q^1_{n-1}$, we have $|E(Q^0_{n-1}) \cap F| \leq n-1$ or $|E(Q^1_{n-1}) \cap F| \leq n-1$, say, $|E(Q^0_{n-1}) \cap F| \leq n-1$. Since $\lambda(Q_{n-1}) = n-1$, we have two cases.

Case 1. $Q^0_{n-1} - F$ is not connected.

Then $|E(Q^0_{n-1}) \cap F| = n-1$ and $Q^0_{n-1} - F$ has only two components.

If $Q^1_{n-1} - F$ is not connected, then $|E(Q^1_{n-1}) \cap F| = n-1$. That is $[Q^0_{n-1}, Q^1_{n-1}] \cap F = \varnothing$. But each vertex of $Q^1_{n-1} - F$ is connected to one component of $Q^0_{n-1} - F$. Hence $Q_n - F$ has only two components, a contradiction.

Note that $|[Q^0_{n-1}, Q^1_{n-1}]| = 2^{n-1} > n-1 (n \geq 5)$. If $Q^1_{n-1} - F$ is connected, then $Q^1_{n-1} - F$ is connected to one component of $Q^0_{n-1} - F$. Hence $Q_n - F$ has only two components, a contradiction.

Case 2. $Q^0_{n-1} - F$ is connected.

If $Q^1_{n-1} - F$ is connected, then we are done. We assume that $Q^1_{n-1} - F$ is not connected. And $Q^1_{n-1} - F$ has at most one isolated vertex since $|F| \leq 2n-2$.

If $Q^1_{n-1} - F$ has at least 3 components, from the inductive hypothesis, then $|E(Q^1_{n-1}) \cap F| \geq 2n-3$. Hence at most one of components of $Q^1_{n-1} - F$ is not connected to $Q^0_{n-1} - F$, $Q_n - F$ has at most two components, a contradiction.

Therefore we assume that $Q^1_{n-1} - F$ has only two components. But $2^{n-1} - (2n-2) > 0 (n \geq 5)$, $Q_n - F$ has at most two components, a contradiction. □

Theorem 2.8. *$c\lambda_4(Q_n) = 3n - 2$ for $n \geq 2$.*

Proof. Take a path $P_3 = uvw$. Then $|E(u) \cup E(v) \cup E(w)| = 3n-2$. And $Q_n - E(u) - E(v) - E(w)$ has at least 4 connected components. That is $c\lambda_4(Q_n) \leq 3n-2$.

Next we will show that $c\lambda_4(Q_n) \geq 3n-2$ by induction. It is easy to check it is true for $n = 2, 3, 4$. So we suppose $n \geq 5$ and assume this is true for all $k < n$. We will prove that is true for $k = n$.

Let $F \subseteq E(Q_n)$ with $|F| \leq 3n-3$, and $Q_n - F$ has at least 4 components. Now since $Q_n = Q^0_{n-1} \odot Q^1_{n-1}$, we have $|E(Q^0_{n-1}) \cap F| \leq [3n/2] - 2$ or $|E(Q^1_{n-1}) \cap F| \leq [3n/2] - 2$, say, $|E(Q^0_{n-1}) \cap F| \leq [3n/2] - 2$. Since $c\lambda_3(Q_{n-1}) = 2n-3 > [3n/2] - 2 (n \geq 5)$, $Q^0_{n-1} - F$ has at most two components.

Case 1. $Q^0_{n-1} - F$ is connected.

If $Q^1_{n-1} - F$ has at least 4 components, then $c\lambda_4(Q_{n-1}) \geq 3n-5$ by the inductive hypothesis. We need delete at most two edges again. Since each vertex of Q^1_{n-1} has a neighbor in Q^0_{n-1} and $|[Q^0_{n-1}, Q^1_{n-1}]| = 2^{n-1} > 2 (n \geq 5)$, $Q_n - F$ has at most 3 components, a contradiction.

Suppose $Q^1_{n-1} - F$ has at most 3 components. Because of $|[Q^0_{n-1}, Q^1_{n-1}]| = 2^{n-1} - (3n-3) > 0 (n \geq 5)$, $Q_n - F$ has at most 3 components, a contradiction.

Case 2. $Q^0_{n-1} - F$ has only two connected components.

Then $|E(Q^0_{n-1}) \cap F| \geq \lambda(Q_{n-1}) = n-1$ and $|E(Q^1_{n-1}) \cap F| \leq 2n-2$. Note that $c\lambda_3(Q_{n-1}) = 2n-3$.

If $Q^1_{n-1} - F$ has at least 3 components, then $|E(Q^1_{n-1}) \cap F| \geq 2n-3$ and $|E(Q^0_{n-1}) \cap F| \leq n$. But $|[Q^0_{n-1}, Q^1_{n-1}] \cap F| \leq 1$ and $2^{n-1} > 1 (n \geq 5)$, $Q_n - F$ has at most two components, a contradiction.

Hence $Q^1_{n-1} - F$ has at most two components. We have $|[Q^0_{n-1}, Q^1_{n-1}]| > 3n-3 (n \geq 5)$, and $Q_n - F$ has at most 3 components, a contradiction. □

And because the hypercube Q_n is the subgraph of the folded hypercube FQ_n, we can apply the similar method to FQ_n. Hence we have the following theorem.

Theorem 2.9. *(1) $c\lambda_2(FQ_n) = \lambda(FQ_n) = n + 1$ for $n \geq 3$.*

(2) $c\lambda_3(FQ_n) = 2n + 1$ for $n \geq 3$.

(3) $c\lambda_4(FQ_n) = 3n + 1$ for $n \geq 3$.

3 Conclusions

The component connectivity is a generalization of classical connectivity of graphs. The hypercube network Q_n has proved to be one of the most popular interconnection networks since it has a simple structure and has many good properties. The folded cube FQ_n is a variant of Q_n. We determined the r-component (edge) connectivity of Q_n and FQ_n for $r = 2, 3, 4$. Future research on this topic would compute the component connectivity of different topologies.

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