Least squares spectral collocation method for solving identification problems in a Lake pollution model over a complex domain

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Abstract: In this paper, We modeled the behavior of pollution concentration in a lake by a parabolic equation. The domain of the lake is reduced to 2D-dimension in space and characterized by some obstacles inside and the circumference is a polygonal form. First, The mathematical model obtained contains unknown parameters which have to be determined and then the approximative solution of the mathematical model has to be estmimated. For this purpose, we approximate the system of equations by means of discrete differential operators adapted to the complexity of the domain based on Least Squares Spectral Collocation Method, LSSCM. To test our numerical scheme, we consider some experimental data. After computations, we obtain optimals values of unknown parameters and the approximative solution of the mathematical model. The 2D and 3D-dimension in space graphics of the solution are presented. The analysis of the graphs allows us to better identify the source of the pollution, the concentration of the pollution and the direction of the propagation in lake. We conclude that the use of Least squares spectral collocation method to solve pollution problem over a complex domain was successful.

Key–Words: Lake Pollution, Least-squares formulation, finite elements, Collocation point methods, differentiation matrices, complex domain

1 Introduction

In this paper, we describe the propagation of pollution concentration in a lake by a non-linear partial differentiation equation. This mathematical model is acheived by taking into account the physical, chemical and biological properties of water. The system of equations obtained contain some unknown parameters coming from the modelling process.

The mathematical problem we are facing is: can we recover both unknown parameters and the solution of the system equation (if there exists) knowing some measurements of pollution concentration in a subset of the lake? this is an identification problem. A such problem has been investigated by many autors both theoretical and numerical aspects and has been applyed in diverse fields of reseach; see [19] and[24] for more details. For an application in physic, see [7] and [22], the authors work on identification problem arising in single-photon emission computerized tomography. In identification problems, the measurements and where they have been collected play an important role so in [3], [1] and [2], the determination of some characteristic sources is based on boundary data. But there are restrictions on the number and type of sources that can be identified from boundary data. For more information see [2]. In some articles, [4], [5], [6] and [8], the authors present numerical methods to identify the source terms in a model of pollution propagation in surface water. This topic is close to our present research. But we do not limit the identification problem to the source term.

Two most popular methods are often used in identification problems: the least squares method (see [9], [16] and [30]) and the sentinel method (see [21], [23] and [27]). In many papers, both numerical and theoretical aspects for these methods are developed using smooth domains.

This paper deals with a computational method for determining unknown terms of a non linear partial differential equation including unknown parameters. Theses unknown terms may be in initial, boundary conditions or source terms. Furthermore, to be close to the reality, we assume that the computational domain has a complex geometry shape (see Figure 1). We intend to take into account the geometry complexity as well as the non linear term from which blow up property can occur. In this case, the computation of PDEs requires a special treatment both for meshing domain and the discretization of the equation system. To this end, we use LSSCM over a complex domain in the current work.The most important papers where LSSCM has been developed are given by [13], [15], [17] and [26]. This paper extends a previous work, [26], by solving both a parabolic equation and identification problem.

This paper is organized as follow: In section 2, we present a mathematical model of the pollution propagation and the description of the domain over which the problem is posed. Then the least squares spectral collocation formulation of parameters identification problem is posed. In section 3, we present the discretization process of both system equations and the domain. In the section 4, we present some numerical results and give some remarks in the last section.

2 Setting of the problem

The polluted lakes generally contain chemical wastes like nitrate and phosphate coming from industries, agricultural runoff and waste water from cities. These pollutants kill fishes and other aquatic animals by reducing the rate of dissolved oxygen in the water. So, there exists a correlation between the rate of dissolved oxygen and the rate of pollution in the water and that is why the quantity of pollution is valued from the quantity of dissolved oxygen that pollutants need for their chemical and biological reactions. This quantity is estimated in BOD (Biologic Oxygen Demand) and COD (Chemical Oxygen Demand) respectively (see [12] and [18]). We suppose in the remaining of this paper that $C(t, \mathbf{x})$ (Kg/m^3) denotes the concentration of the pollution in the Lake measured in COD at the time t and $\mathbf{x}$-position.

To take into account the fluid properties, we describe the pollutant propagation by considering its concentration $C(t, \mathbf{x})$. According to some assumptions the following terms will be considered to describe our model:

- A diffusion term, $k.div\,[a(\mathbf{x})\nabla C(t, \mathbf{x})]$, where k is a constant, $a(\mathbf{x})$ denotes the diffusion of chemical substances in the water. In the case of the Lake, $a(\mathbf{x})$ may be considered as a constant term so that the diffusion term becomes $K.\Delta C(t, \mathbf{x})$. K is the coefficient of diffusion (m^2/s).
- A transport term, $\overrightarrow{u}\nabla C(t, \mathbf{x}),$ where $\overrightarrow{u}$ denoted the velocity of the fluid. In absence of wind, considered as the only means of transportation here, we can assumed that $\overrightarrow{u} = \overrightarrow{0}$.
- A reaction term, $\lambda C(t, \mathbf{x}) - \mu \left|C\right|^p (t, \mathbf{x})$, where λ and μ are coefficients that describe the characteristics of the reaction process. It shows chemical and biochemical interactions in the fluid so two types of reactions may be considered: the first term increases the rate of pollution and the second term reduces the pollution rate.
- A source term $f(t, \mathbf{x})$. It brings polluted substances in the liquid. We have two cases of source term. The pollutants which come from land surface (agricultural runoff, waste water) denoted by $\xi(t, \mathbf{x})$ and those which are situated at the bottom of the Lake (sediment, Heavy metals). Let consider we have N_s sources terms at $\mathbf{x}_i-$ coordinate. So, the general formulation of the source can be taken like

$$f(t, \mathbf{x}) = \xi(t, \mathbf{x}) + \sum_{i}^{N_s} \lambda_i \widehat{\xi_i}(t) \times \delta(\mathbf{x} - \mathbf{x}_i),$$

where $\delta(\mathbf{x} - \mathbf{x}_i)$ represents the Dirac function associated to $\mathbf{x}_i$.

The domain considered in this problem has two types of boundary: Γ_{int}, the circomference of obstacles inside the domain and Γ_{out}, the limit of Lake domain. Then the boundary of the domain, $\Gamma = \Gamma_{int} \cup \Gamma_{out}$ (see Figure 1). For the sake of simplicity, we suppose that there are no exchange of pollution concentration through the boundaries. So, we are concerned with a Dirichlet condition

$$C\,|_\Gamma = 0$$

and the initial condition is

$$C(0, \mathbf{x}) = g(\mathbf{x}; \tau).$$

We summary the behavior of pollution concentration C in a lake by the parabolic system below

$$\begin{aligned}
\frac{\partial C(t, \mathbf{x})}{\partial t} &= \xi \Delta C(t, \mathbf{x}) + \eta C(t, \mathbf{x})) - \mu \left|C\right|^p (t, \mathbf{x})) \\
&\quad + f(t, \mathbf{x}; \lambda), \qquad (t, \mathbf{x}) \in]0, T] \times \Omega, \\
C(t, \mathbf{x}) &= 0, \qquad (t, \mathbf{x}) \in]0, T] \times \Gamma, \\
C(0, \mathbf{x}) &= g(\mathbf{x}; \tau), \qquad \mathbf{x} \in \Omega
\end{aligned} \tag{1}$$

where

- $\Omega \subset \mathbb{R}^2$ and Γ is the boundary of Ω.
- $]0, T]$, $T > 0$, is the spending time for the experience.
- The reals $\xi, \eta, \mu, p, \lambda$ and τ are unknown parameters and strictly positives.

- The structure of $f(t, \mathbf{x}; \lambda)$, the source of pollution and $g(t, \mathbf{x}; \tau)$, the initial concentration are known.

We are concerned with the identification problem that consists of determining the unknown parameters ξ, η, μ, p, λ and τ in the system (1). For theoretical aspects, on can refer to [14], [20] and [21] for the existence and uniqueness conditions of global solution for the system (1).

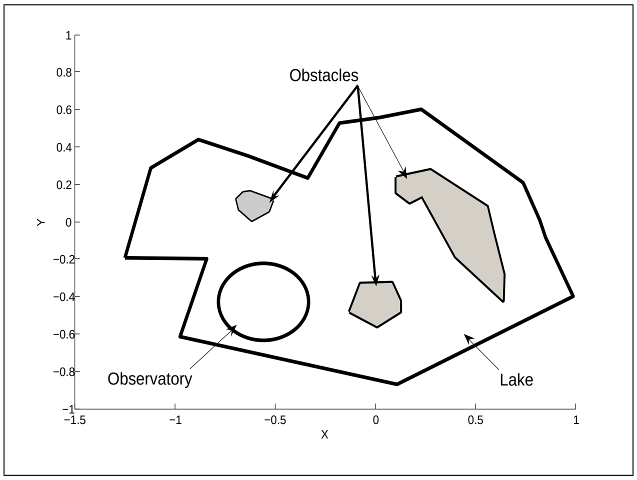

Figure 1: *The domain of the Lake.*

3 Discrete differential operators

The solving of (1) requires accurate approximation of derivatives and particularly on such domain. Least squares spectral collocation method combines both the standard least squares method and collocation method. For this purpose, a macro mesh is used to mesh the whole domain. LSSCM consists in solving (1) into each triangular finite elements and the global solution is obtain by applying the standard least squares method after the assembly process. The convergence of LSSCM has been hugely study by previous authors, we mentioned at the beginning, in smooth domain. The case of complex domains has been studied in [26] for elliptic equations. To better understand this method, we describe first the discrete differentiation matrix associated with Fekete([25]) or Gauss-Lobatto([29]) points in the triangles.

3.1 Discrete differential operators

Let us consider as standard triangle the following

$$\widehat{\Theta} = \{(r, s), -1 \leq r, s \leq 1;\ r + s \leq 0\}. \quad (2)$$

We denote by $p_j^{\alpha,\beta}(s)$ the Jacobi polynomials of (α, β)-order and degree j [?]. It is well known these polynomial have $j + 1$ zeros and are distributed arbitrary over $]-1, 1[$. In the case of standard quadrangle, the Gauss-Lobatto points are generated using tensor product (see [28]). In the case of a triangle, one can build a suitable transformation function to translate the Gauss-Lobatto points from a quadrangle to a standard triangle [29]. It is proved in [10] and [11] that these collocation points are close to the Fekete ones. Let consider $P_N(\widehat{\Theta})$ the space of polynomials of degree less than N over $\widehat{\Theta}$. In the remaining of this paper we shall by note $\{\phi_k\}_{1\leq k\leq N}$ a basis of $P_N(\widehat{\Theta})$.

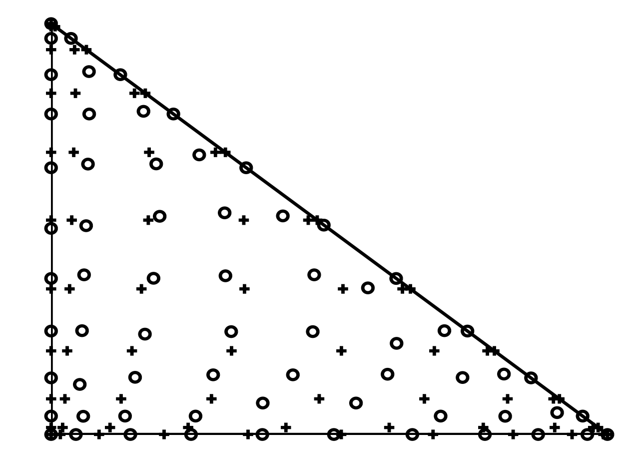

Figure 2: *The Lobatto triangle nodes (+) and associated Fekete nodes (o) over the reference triangle $\widehat{\Theta}$.*

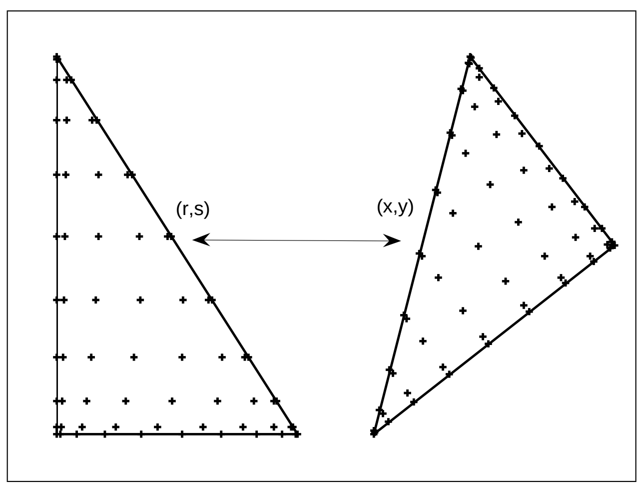

Figure 3: *Distribution of collocation points from reference triangle (left) to arbitrary triangle (right).*

3.2 Differentiation over the reference triangle

For any continuous function $u(r, s)$ on the reference triangle $\widehat{\Theta}$, we can write its spectral approximation in the space $P_N(\Theta)$ by

$$u^N(r,s)=\sum_{k=1}^{N} U_k\phi_k(r,s) \tag{3}$$

where the coefficients U_k are obtained using collocation equations at collocation points that we denote by $\widehat{z}_m(r,s)$:

$$u^N(\widehat{z}_m)=\sum_{k=1}^{N} U_k\phi_k(\widehat{z}_m), \quad m=1,2,...,N. \tag{4}$$

Setting $U(t)=\left(u^N(\widehat{z}_1),\ u^N(\widehat{z}_2),\ ...,\ u^N(\widehat{z}_N)\right)'$ and $C(t)=(U_1(t),\ U_2(t),\ ...,\ U_N(t))'$ the coefficient vectors, where $(..)'$ designs the transposed vector. The equation yields

$$C=V^{-1}\times U \tag{5}$$

where V is the Vandermonde matrix over collocation points. That is a matrix whose components are $\phi_k(\widehat{z}_m)$. According to (3), the derivatives in s and in r directions at collocation points $\widehat{z}_m$ are then given by

$$\partial_r u^N(\widehat{z}_m)=\sum_{k=1}^{N} U_k\times\partial_r\phi_k(\widehat{z}_m)$$

and

$$\partial_s u^N(\widehat{z}_m)=\sum_{k=1}^{N} U_k\times\partial_s\phi_k(\widehat{z}_m)$$

respectively. We introduce two differentiation matrices V^r and V^s, of the size $N\times N$ respectively in $r-$ and $s-$direction respectively, whose components are $V^r_{ij}=\partial_r\phi_j(\widehat{z}_i)$ and $V^s_{ij}=\partial_s\phi_j(\widehat{z}_i)$ respectively. Denoting U_r and U_s, the vector values of the differential approximations in $r-$ and $s-$ direction respectively at collocation points, we obtain

$$U_r=D^r\times U \quad \text{and} \quad U_s=D^s\times U \tag{6}$$

where we have set

$$D^r=V^r\times V^{-1} \text{ and } D^s=V^s\times V^{-1}. \tag{7}$$

3.3 Differentiation over an arbitrary triangle

Any derivative over an arbitrary triangle is derived from the reference triangle according to the bijective transformation such that:

$$u(r,s)=u(x(r,s),y(r,s)). \tag{8}$$

Applying the derivative rule in r (respectively s) direction, we have

$$\begin{cases} \partial_r u &= (\partial_r x)\,\partial_x u+(\partial_r y)\,\partial_y u,\\ \partial_s u &= (\partial_s x)\,\partial_x u+(\partial_s y)\,\partial_y u.\end{cases} \tag{9}$$

Let us denote by $U_x(t)$ and $U_y(t)$ the vector values of the differential approximation in x and y directions respectively at collocation points. Then, from (7) and (9) we deduce

$$\begin{pmatrix}U_x\\U_y\end{pmatrix}=G^{-1}\times\begin{pmatrix}D^r\\D^s\end{pmatrix}\times U \tag{10}$$

where the matrix G is associated to the system (9) over collocation points. Setting $\mathbb{D}=G^{-1}\times\binom{D^r}{D^s}$ then the differentiation matrices over an arbitrary triangle in $x-$direction, and $y-$direction are obtained by extracting the matrix $\mathbb{D}$ from the first to the N^{th} column and from the $(N+1)^{th}$ to $2N^{th}$column respectively :

$$D^x=\mathbb{D}(1:N,\ :\ ) \quad \text{and} \quad D^y=\mathbb{D}(N{+}1:2N,\ :\ ). \tag{11}$$

The second order differentiation matrices on an arbitrary triangle are obtained in the similar way. Next we shall denote by D^{xx} and D^{yy} the second differentiation matrix in $x-$direction and $y-$direction respectively and by $\mathbb{L}$ the discrete Laplacian, where

$$\mathbb{L}=D^{xx}+D^{yy}. \tag{12}$$

3.4 Differentiation over the entire domain

We are concerned in this paper to $2D-$ space and the triangular finite elements mesh. To define the differentiation matrix over the whole domain like the previous sections, we explain briefly the collocation points assembly process. Let denote by $\{\Theta_k\}_{k=1,...,N_\Theta}$ the set of elementary triangle in the domain of length N_Θ and $\mathbf{X}_{loc}$ the set of collocation points coming from each Θ_k of length N_{loc}. It's clear that $\mathbf{X}_{loc}$ contains some collocation points repeated more than ones, like certain points on edges and nodes (see Figure 4). To avoid the repeated points, we achieve a second set, $\mathbf{X}_{glob}$, of length N_{glob} ($N_{loc}>N_{glob}$). We have the following relation

$$\mathbf{X}_{glob}=\mathbb{Z}\times\mathbf{X}_{loc}$$

where $\mathbb{Z}$ is the assembly matrix of size $N_{glob}\times N_{loc}$. A such matrix were defined in [29] for hybrid elements(triangular and quadrilateral elements). The components of $\mathbb{Z}$ are 0 or 1.

Let U_{glob} the vector solution at $\mathbf{X}_{glob}$ collocation points.

$$U_{glob}=\left(u_1,u_2,...,u_{N_{glob}}\right)'.$$

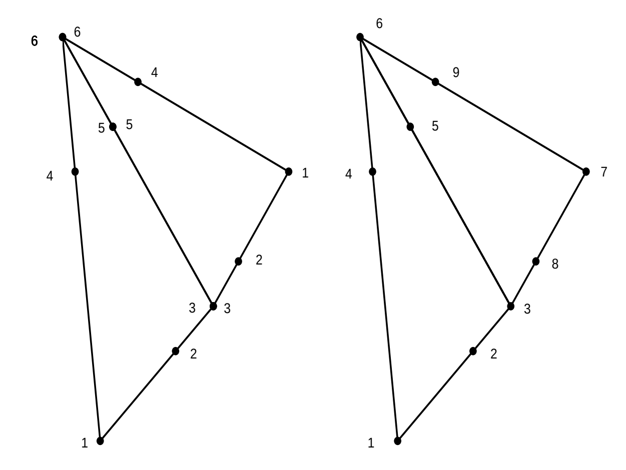

Figure 4: *(left) Local numbering of two elementary triangles, the size of $\mathbf{X}_{loc}$ is twelve(12) collocation points per triangle. (right) Global numbering of two elementary triangles, the size of $\mathbf{X}_{glob}$ is nine(9) collocation points.*

We define the first derivative matrix in x-direction of U_{glob} over the whole domain by the following relation:

$$\partial_x U = \mathbb{H}_x \times \mathbb{Z} \times U_{glob}$$

where $\mathbb{H}_x$ is a diagonal matrix of size $d(N)^2 = (N \times N_\Theta)^2$

$$\mathbb{H}_x = \begin{pmatrix} D_x^1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_x^{N_\Theta} \end{pmatrix} \quad (13)$$

where D_x^k represent the local first derivative matrix in x-direction over an elementary triangle Θ_k. Then we deduce the global differentiation matrix in x-direction, given by:

$$\mathbb{D}_x = \mathbb{H}_x \times \mathbb{Z}. \quad (14)$$

In the same ways, we define below the first derivative matrix in y-direction and the second derivative matrix in x- and y-direction respectively over the whole domain by:

$$\mathbb{D}_y = \mathbb{H}_y \times \mathbb{Z}; \quad (15)$$

$$\mathbb{D}_{xx} = \mathbb{H}_{xx} \times \mathbb{Z}; \quad (16)$$

and

$$\mathbb{D}_{yy} = \mathbb{H}_{yy} \times \mathbb{Z} \quad (17)$$

where $\mathbb{H}_y$, $\mathbb{H}_{xx}$ and $\mathbb{H}_{yy}$ are like (13) where D_x^i are replaced by D_y^i, D_{xx}^i and D_{yy}^i respectively , $i = 1, ..., N_\Theta$.

4 Least squares spectral collocation formulation

Using the discretization method explain above, the discrete system of (1) can be written as follow

$$\boxed{\begin{array}{rcl} \frac{d\underline{C}(t)}{dt} - \xi \mathbb{L} \times \underline{C}(t) + \eta \underline{C}(t) - \mu \left|\underline{C}\right|^p (t) & = & \underline{f}(t;\lambda) \\ D_b \times \underline{C}(t) & = & 0 \\ \underline{C}(0) & = & \underline{g}(t;\tau) \end{array}} \quad (18)$$

where $t \in]0, T]$,

$$\underline{C}(t) = \left(C(t, \mathbf{x}_1), ..., C(t, \mathbf{x}_{d(N)})\right)', \quad (19)$$

$$\underline{f}(t;\lambda) = \left(f(t, \mathbf{x}_1; \lambda), ..., f(t, \mathbf{x}_{d(N)}; \lambda)\right)', \quad (20)$$

$$\underline{g}(t;\tau) = \left(g(t, \mathbf{x}_1; \tau), ..., g(t, \mathbf{x}_{d(N)}; \tau)\right)', \quad (21)$$

the $\mathbf{x}_k$, $k = 1, ..., d(N)$ are the collocation points. D_b designs the discrete operator which selects the boundary points. The residual of the discrete system (18) is valued by the following operator

$$\begin{array}{rcl} \mathscr{L}(\underline{v}, \underline{C}) & = & \left\| \frac{d\underline{C}(t)}{dt} - \xi \mathbb{L} \times \underline{C}(t) + \eta \underline{C}(t) \right. \\ & & \left. -\mu \left|\underline{C}\right|^p (t) - \underline{f}(t;\lambda) \right\|^2 . \end{array} \quad (22)$$

Let denote the vector of unknown parameters by

$$\underline{v} = (\xi, \eta, \mu, p, \lambda, \tau). \quad (23)$$

To estimate $\underline{v}$ we need some data measurements of C which govern the system (1) over the whole domain Ω. It seems to be difficult in reality to get such data. In this paper, we are concerned with the following type of inverse problem: Having some data about the solution over a sudomain of Ω, we want to estimate $\underline{v}$ and the numerical solution $\underline{C}(t)$ in the entire domain. To solve this problem we have been inspired by the sentinel method developped by J.L. Lions [21]. We denote by C_{obs} the so called data and $\Omega_{obs} \subset \Omega$ the observatory, a subdomain of Ω (Figure 6) where the measurement has been done. We denote by D_{obs} and D_b the matrix which select the observatory respectively the boundary collocation points from whole collocation points of Ω such that

$$\begin{array}{rcl} D_{obs} \times \underline{C} & = & C|_{x \in \Omega_{obs}} \\ D_b \times \underline{C} & = & C|_{x \in \Gamma}. \end{array} \quad (24)$$

We are concerned here with the determination of $\underline{C}(t, \underline{v})$ solution of (18) and the unknown vector (23).

We summary the least squares spectral collocation method of (18) as

$$\boxed{\begin{array}{c}\text{find } \underline{v}^*, \underline{C}^* \text{ solutions of} \\ J_{\widehat{\beta}}(\underline{v}^*, \underline{C}^*) = \min_{(\underline{v}, \underline{C}) \in \mathbb{R}^6 \times \mathbb{R}^{d(N)}} J_{\widehat{\beta}}(\underline{v}, \underline{C})\end{array}} \qquad (25)$$

where

$$\begin{aligned} J_{\widehat{\beta}}(\underline{v}, \underline{C}) &= \sum_{k=1}^{[T/\Delta t]} \|D_{obs} \times \underline{C}(k\Delta t) - \underline{C}_{obs}(k\Delta t)\|^2 \\ &+ \sum_{k=1}^{[T/\Delta t]} \|D_b \times \underline{C}(k\Delta t)\|^2 + \widehat{\beta} \times \|\underline{v} - \widehat{\underline{v}}\|^2 \\ &+ \mathscr{L}(\underline{v}, \underline{C}) \end{aligned} \qquad (26)$$

and $\underline{C}_{obs}(t)$ is a measurement vector obtain at $\mathbf{x} \in \Omega_{obs}$ at time t. A positive constraint is established on $\underline{C}$. $\widehat{\beta}$ stand for Tikhonov regularization parameter and $\widehat{\underline{v}}$ is an a priori information, (27). We look for values of $\underline{v} \in]0, 2]$ according to this information

$$\boxed{\begin{array}{ccccc} 0 & < & \xi & < & 10^{-3} \\ 10^{-6} & < & \eta & < & 0,4 \\ 0,1 & < & \mu & < & 6 \\ 0,034 & < & p & < & 2 \\ 10^{-6} & < & \lambda & < & 0,5 \\ 10^{-2} & < & \tau & < & 1,6 \end{array}} \qquad (27)$$

We summary the computational process through an algorithmic scheme (Figure 5).

5 Numerical experimentation

Collecting Data: For experimental purpose to test our numerical scheme, we built data, $\underline{C}_{obs}(t)$ as:

Figure 8: firstly, We consider graphs of pollution at five arbitrary points, O_i, $i = 1, ..., 5$ (Observation points, Figure 6) situated in Ω_{obs},

Figure 7: secondly, we collect the value of each curve at 24 axis points representing the time intervals (per days) and

finally we fit the data over the whole collocation points include to Ω_{Obs}.

The expression of source function f is given by

$$f(t, \mathbf{x}) = 10\lambda sin(\pi t + 1)\delta_{\mathbf{x}_c}(\mathbf{x})$$

where $\delta_{\mathbf{x}_c}(\mathbf{x})$ is a Dirac function define in $\mathbf{x}_c$, here $\mathbf{x}_c = (0, 0)$, such that

$$\delta_{\mathbf{x}_c}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{x}_c \\ 0 & \text{otherwise.} \end{cases}$$

The initial function is given by

$$g(x) = \tau. \left(x^2 + y^2 - .06\right) \delta_{\mathbf{x}_c}(\mathbf{x}).$$

We use MATLAB 7.01 graphic interface to compute the algorithm which has some solvers associated to our numerical scheme. So, for the achievement of (25) we used " *fmincon*" solver and for (22) we used the "*ode45*" solver.

Results: By using the data built in Figure 7, for $0 \leq t \leq 24$ and after the computation of (25), the optimal values of $\underline{v}$ obtained are:

$$\boxed{\begin{array}{rc} \xi = & 58.10^{-4} \\ \eta = & 10^{-6} \\ \mu = & 6 \\ p = & 0.3373 \\ \lambda = & 0.4999 \\ \tau = & 1.4680 \end{array}}$$

The approximative values of the parameters are the main results of the numerical process. Its identify clearly the parabolic equation (1):

The expression of source function f is determined

$$f(t, \mathbf{x}) = 10\lambda sin(\pi t + 1)\delta_{\mathbf{x}_c}(\mathbf{x}) \quad \textit{with} \quad \lambda = 0.4999 \qquad (28)$$

the initial function is identified

$$g(x) = \tau. \left(x^2 + y^2 - .06\right) \delta_{\mathbf{x}_c}(\mathbf{x}) \quad \textit{with} \quad \tau = 1.4680$$

and the system of equation which governed our pollution model is dertermined:

$$\begin{aligned} \frac{\partial C(t, \mathbf{x})}{\partial t} &= \xi \Delta C(t, \mathbf{x}) + \eta C(t, \mathbf{x})) - \mu \left|C\right|^p (t, \mathbf{x})) \\ &\quad + f(t, \mathbf{x}; \lambda), \qquad (t, \mathbf{x}) \in]0, T] \times \Omega, \\ C(t, \mathbf{x}) &= 0, \qquad (t, \mathbf{x}) \in]0, T] \times \Gamma, \end{aligned} \qquad (29)$$

with

$$\xi = 58.10^{-4}; \quad \eta = 10^{-6}; \quad \mu = 6 \quad \textit{and} \quad p = 0.3373. \qquad (30)$$

Then one can approximate the solution of (29) by any suitable partial differential equation solver. The approximative solution is given by $\underline{C}^*$.

For time goes from 0 to 20, we plot in two dimensions (Figures 10 and 11) and in three dimensions (Figure 9) the graphs of the approximative solution $\underline{C}^*$ so that we can appreciate the evolution of the experimental solution. We have selected four pictures at arbitrary time ($t = 2$, $t = 4$, $t = 6$ and $t = 20$). So in:

Figure 9: Firsly, We observe the growing of the volume of the concentration when the time of the experience is increasing. Secondly, the shape of the solution contains some holes located at the same positions of our obstacles introduced in the domain. That means, there are no pollutions in these places and it confirms our hypotheses. The groth of $\underline{C}^*$ is consistent the graph of data (Figure 8). Finally, at each time, we remark somewhere in the graph a highest value of the pollution concentration, it shows the position of the source term as predicted in 28.

Figure 10 and 11: we decide to plot in $2D-$ dimension to point out the value of $\underline{C}^*$ around the boundaries. When the time increases, We remark that the value of pollution concentration around the boundaries(circumference, obstacles) increase. In particular, in Figure 10 there is an accumulation of streamline around the boundaries. In Figure 11, the color of the domain moves from blue ($t = 2$) to green ($t = 20$). the source term position is identified by the heighest value, the red color and The surfaces of obstacles are maintained in blue color, the lowest value. Here again, the source term is clearly identify and the growth of pollution is in phase with the previous graphs.

Figure 12: we are interested by the direction of the propagation of the pollution, so we have presented the gradient of $\underline{C}^*$ in $2D-$ dimension. The arrows show the senses of the propagation in the domain. When the time increases the arrows move from the source position throughout the entire domain excepted in the obstacles. Also, most of the arrows are directed to the boundaries. That confirms the accumulation of pollution around the boundaries in Figure 10.

6 Concluding remarks

In this paper, we acheive a convenient algorithm to study the propagation of pollutants in a lake over a complex domain. The succes of the numerical scheme is done by coupling the technique of differentiation operators over finite elements and the least squares spectral collocation method. One particularity of this algoritm, it does not need the data over the entire domain. It works with few data collected in a subdomain of the lake like in sentinel method [21]. The numerical test with experimental data gives an excellent results that identify clearly the mathematical model, localyze the source terme and the direction of pollutants. the data are consistence with our results obtained.

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Step 1 Read the following parameters:

- n : Mesh parameter,
- T : Experience time,
- $\triangle t$: A unit of time,
- N : Degree of interpolation polynomials,
- ε : Precision parameter.

Step 2 Mesh the domains

- Ω : Whole domain,
- Ω_{obs} : Observatory.

Step 3 Compute the vectors and matrices associated to the mesh.

- Z : Assembly matrix,
- D_{bord} : Matrix which select the boundary points, computed according to (24),
- D_{obs} : Matrix which select the observatory points, computed according to (24),
- $\mathbb{L}$: Discrete differential operator approaching the Laplacian such as (12)
- x_{loc} : Vector of local collocation points, obtained according to (2),
- $x_{glob} = Z \times x_{loc}$: Vector of global collocation points,
- $x_{bord} = D_{bord} \times x_{glob}$: Vector of boundary points,
- $x_{obs} = D_{obs} \times x_{glob}$: Vector of observatory points.

Step 4 Define the structure of vectorial functions:

- $\underline{f}(t, \lambda)$ according to (20),
- $\underline{g}(t, \tau)$ according to (21),

Step 5 Define the residual operator

$$\mathscr{L}(\underline{v}, \underline{C}) = \left\| \frac{d\underline{C}(t)}{dt} - \xi \mathbb{L} \times \underline{C}(t) + \eta \underline{C}(t) - \mu \left| \underline{C} \right|^p (t) - \underline{f}(t; \lambda) \right\|^2 .$$

Step 6 Compute the regularizing parameter $\widehat{\beta}$.

Step 7 Define the least squares spectral collocation operator

$$\begin{aligned} J_{\widehat{\beta}}(\underline{v}, \underline{C}) \quad = \quad & \sum_{k=1}^{[T/\Delta t]} \| D_{obs} \times \underline{C}(k\Delta t) - \underline{C}_{obs}(k\Delta t) \|^2 + \sum_{k=1}^{[T/\Delta t]} \| D_b \times \underline{C}(k\Delta t) \|^2 \\ & + \widehat{\beta} \times \| \underline{v} - \widehat{\underline{v}} \|^2 + \mathscr{L}(\underline{v}, \underline{C}). \end{aligned}$$

Step 8 Choose an arbitrary value $\underline{v}_0$.

Step 9 Achieve the following problem

$$J_{\widehat{\beta}}(\widehat{\underline{v}}, \widehat{\underline{C}}) = \min_{\underline{v}, \underline{C}} \; J_{\widehat{\beta}}(\underline{v}, \underline{C}).$$

Figure 5: *Algorithm of least squares spectral collocation scheme.*

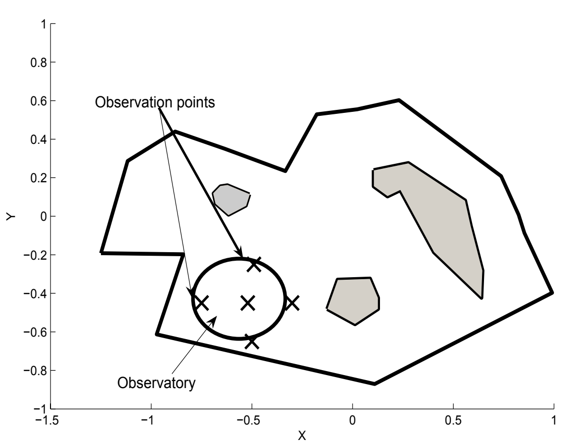

Figure 6: *Presentation of observation domain* Ω_{obs} *and the points,* O_i *used for the measurements* .

	Measures			
	O1	O2	O3	O4
t_1	0	0	0	0
t_2	0.0341	1.6315	2.4895	2.4303
t_3	0.1227	0.2481	1.5021	0.4914
t_4	0.2471	0.5347	1.0590	1.6734
t_5	0.4863	0.6602	0.7808	1.2885
t_6	0.8996	2.0906	2.6617	4.3541
t_7	1.9562	3.1453	2.7454	3.3367
t_8	4.7598	4.7221	5.4121	4.7685
t_9	8.0019	8.3292	9.1448	9.5251
t_{10}	10.1644	10.3390	11.0510	10.0418
t_{11}	10.2478	10.0611	11.3513	11.3318
t_{12}	9.9489	10.6747	11.3433	11.2951
t_{13}	11.0906	10.5023	11.6931	11.6932
t_{14}	11.6184	13.8016	12.5992	12.2813
t_{15}	11.0771	10.9407	10.9209	12.0159
t_{16}	12.0918	12.2057	12.0490	10.1750
t_{17}	14.0955	15.1623	13.5582	13.9864
t_{18}	14.1742	14.2334	14.4907	15.3864
t_{19}	15.1680	15.0724	14.0159	14.7469
t_{20}	15.3715	14.5391	15.9543	16.5321
t_{21}	16.5278	16.8222	16.0171	16.0574
t_{22}	16.0663	14.7302	15.2589	15.9360
t_{23}	16.2276	16.9419	17.1612	17.7301
t_{24}	14.0069	15.6304	14.7085	14.4529

Figure 7: *The data obtained at four (4) observation points (see also figure 8).*

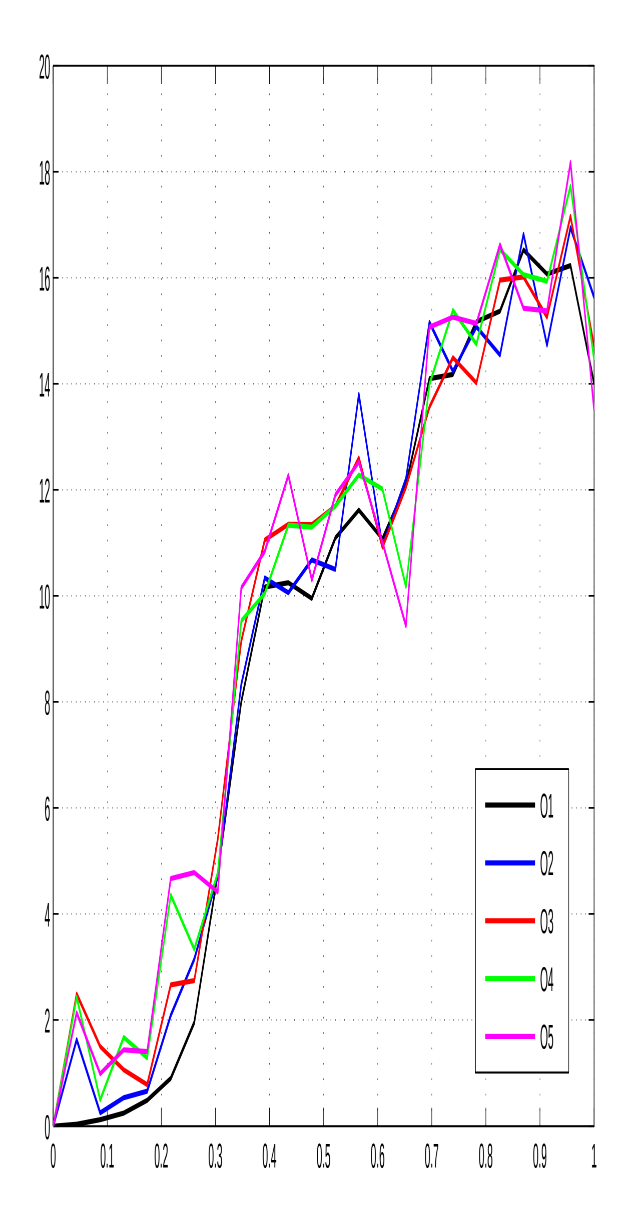

Figure 8: *The measurement curves at 5 observation points* , O_1, O_2, O_3, O_4 and O_5

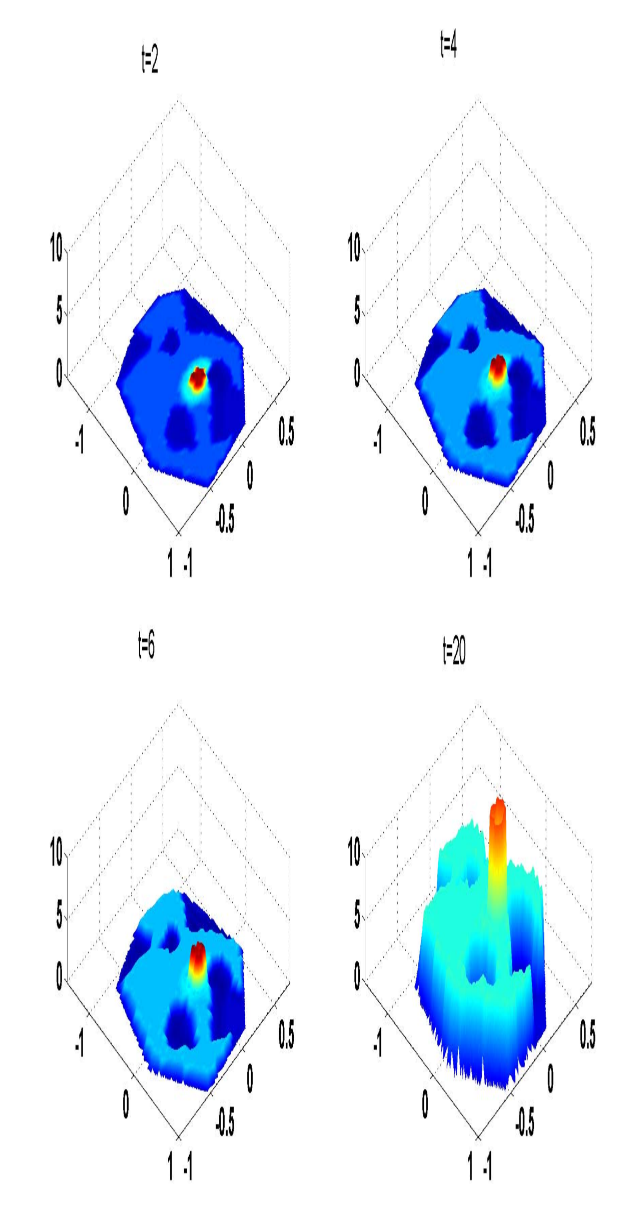

Figure 9: *Concentration curve in* $3D$ *dimension at times* $t = 2$ *respectively* $t = 4$, $t = 6$ *and* $t = 20$.

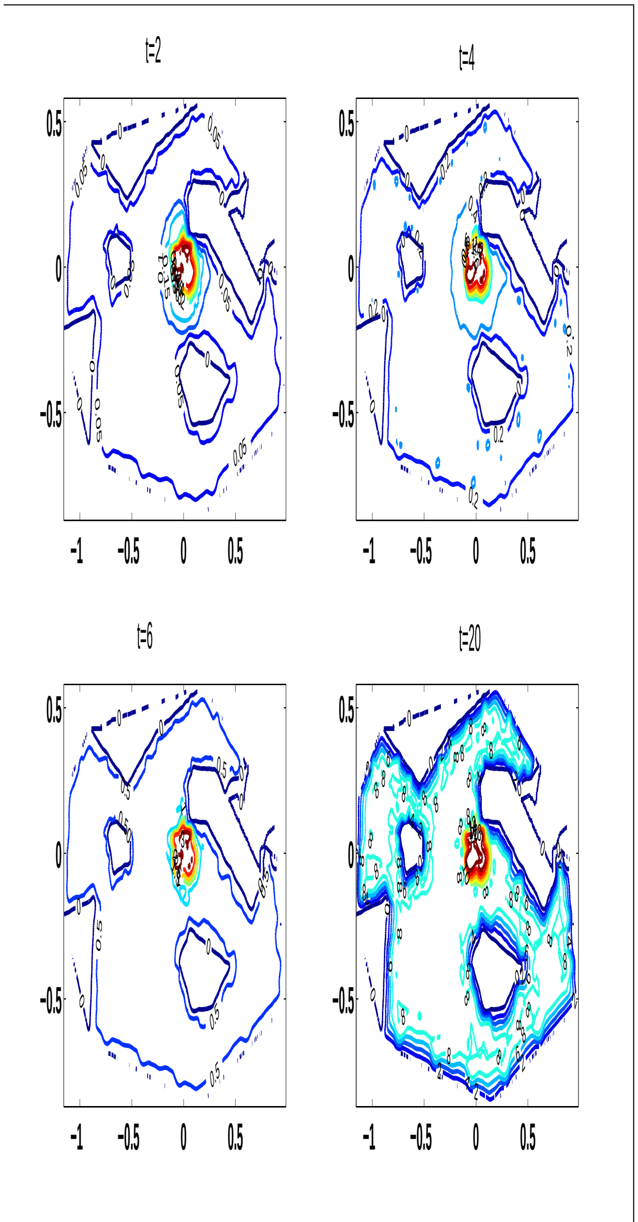

Figure 10: *Presentation of pollution zone at times* $t = 2$ *respectively* $t = 4$, $t = 6$ *and* $t = 20$.

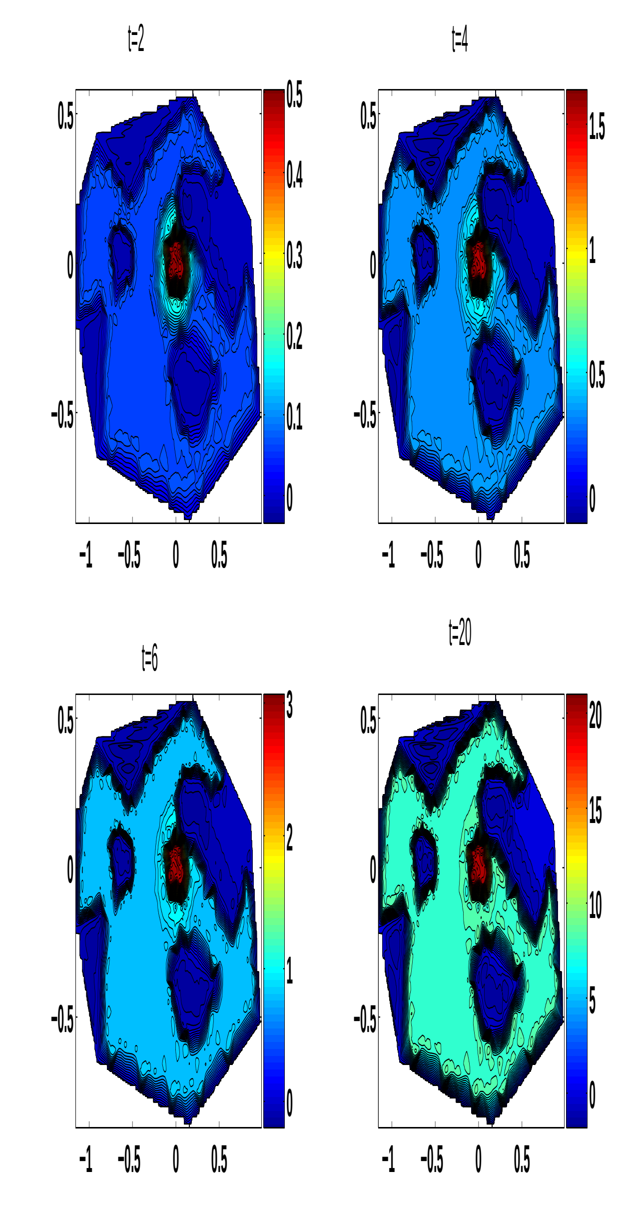

Figure 11: *Presentation of pollution zone at times* $t = 2$ *respectively* $t = 4$, $t = 6$ *and* $t = 20$.

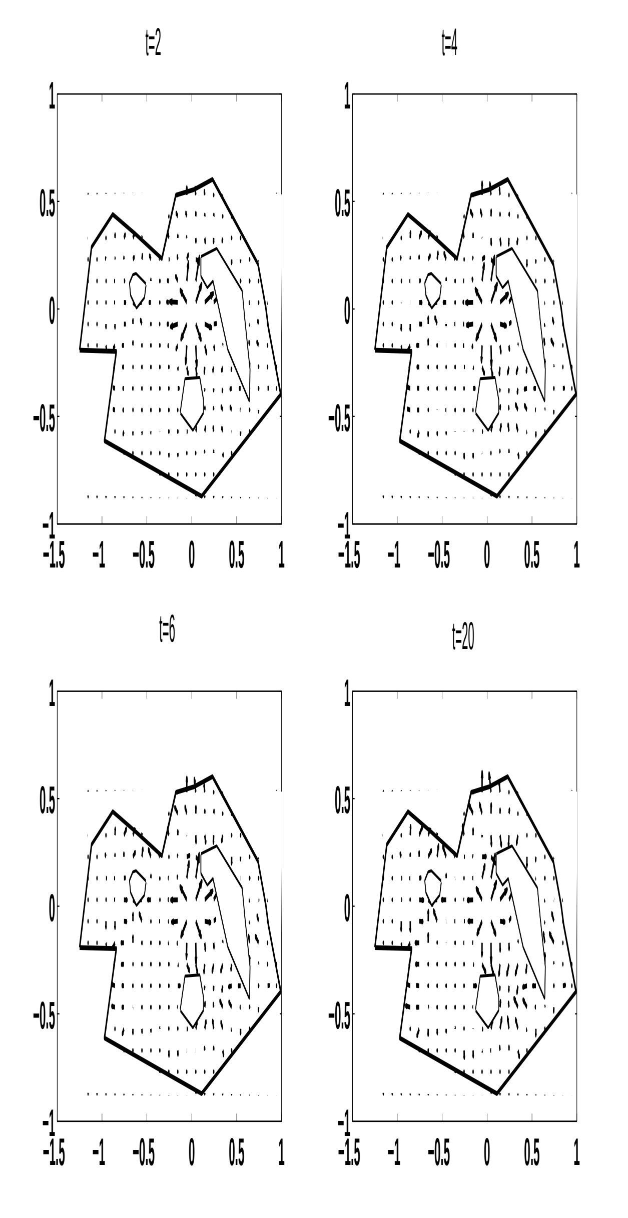

Figure 12: *Presentation of pollution movement sens at times* $t = 2$ *respectively* $t = 4$, $t = 6$ *and* $t = 20$.