A Hybrid Method for Solving Some Particular Types of Fractional Differential Equations

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Abstract: This paper investigates the numerical solution of some particular types of fractional order differential equations which involve multi derivative terms. Fractional equations have advantages when compared with the integer order ones, since they describe some natural physical processes and dynamical systems much better. There are some suggested numerical algorithms for these types of equations but they usually involve a single term. Therefore, we present a hybrid method here. According to the method, the fractional order derivative is written in terms of Riemann Liouville integral and this integral is evaluated as Hadamard finite-part integral numerically as in [1]. On the other hand, the other ordinary derivatives are discretized in terms of standard finite difference approximation. In this study, error estimate has been dealt with and reliability and convergency of the method are tested on some illustrative examples.

Fractional Order Differential Equations, Hadamard Finite Part Integral and Finite Difference Method: Typing manuscripts, LATEX

1 Introduction

Nowadays, differential equations which involve fractional order terms are assumed to be suitable models for many physical, biological processes as it is cited in [2–4]. As a result, fractional differential equations may arise from the many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. The comprehensive details of the topic can be found in [2, 3, 5–7] and the references therein. Unfortunately, these model equations are usually difficult to solve analytically. Therefore, there is need to develop numerical or approximate techniques. Some numerical or approximate schemes have been developed so far. Among them, finite difference approximation methods [8–12], fractional linear multistep methods [13–15], the Adomian decomposition method [16–18], variational iteration method [18,19], differential transform method [20,21] can be accounted. But, most generally, these methods involve single derivative term and there is still need to develop more powerful techniques for solving fractional order equations with multi terms. In this study, we introduce a hybrid method which is the combination of Diethelm's method ([1]) and the finite difference method. We consider here a class of differential equations with the specified initial conditions of the form:

$$
\begin{aligned}
D^q[y(t) - y_0] &= \alpha y''(t) + \delta y'(t) + \beta y(t) + f(t) \\
y(0) &= y_0, y'(0) = y_0',
\end{aligned} \tag{1}
$$

where $0 < q < 1$, f is a function which is defined on [0,1], $\beta \leq 0$, α, δ are some constants. $D^q y$ defines the qth order Riemann-Liouville fractional derivative of the function y, and given by ([22]),

$$
(D^q y)(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dy} \int_0^t \frac{y(u)}{(t-u)^q} du. \tag{2}
$$

The initial condition $y(0)$ is incorporated into the left hand side of Eq. (1) by following the common practice in the theory of fractional equations. This study is organized as follows: in the next section, the Diethelm's quadrature method for the Riemann Liouville fractional derivative will be discussed. In the third section, the hybrid method will be introduced for a class of equations. The last section is the conclusion which includes the further work and the discussion.

2 The Hybrid Method

The algorithm which is used here to evaluate fractional derivative of a function is based on the obser-

vations in [23] therefore, the integral can be interchanged by differentiation in Eq.(2) as,

$$D^q y(t) = \frac{1}{\Gamma(-q)} \int_0^t \frac{y(u)}{(t-u)^{q+1}} du. \qquad (3)$$

The integral in Eq.(3) is analogous with the Hadamard finite-part integral. For a given n we define an equispaced discretization in the interval such as $t_j = j/n$. If we apply the discretization into Eq.(3), for $j = 1, 2, ..., n$, we obtain,

$$(D^q y)(t) = \frac{1}{\Gamma(-q)} \int_0^{t_j} \frac{y(u) - y(0)}{(t_j - u)^{q+1}} du. \qquad (4)$$

If we use the transformation, $t_j - t_j w = u$, then, we can write $t_j - u = t_j w$. Hence, arranging Eq.(4) gives,

$$\begin{aligned}
(D^q)y(t) &= \frac{1}{\Gamma(-q)} \int_0^1 \frac{y(t_j - t_j w) - y(0)}{(t_j^{q+1} w^{q+1})} t_j dw \\
&= \frac{t_j^{-q}}{\Gamma(-q)} \int_0^1 \frac{y(t_j - t_j w) - y(0)}{w^{q+1}} du \qquad (5)
\end{aligned}$$

For a defined n, an equispaced mesh can be defined as $t_j = j/n$ on the interval and we replace the integral in Eq.(5) by the first order compound quadrature formula same as in [1]. Therefore, this integral can be evaluated as,

$$Q_j[g] := \sum_{k=0}^j \alpha_{kj} g(k/j) \approx \int_0^1 g(u) u^{-q-1} du \qquad (6)$$

where the residual term is equivalent to:

$$R_j[g] = \int_0^1 g(u) u^{-q-1} du - Q_j[g] \qquad (7)$$

If we ignore the quadrature error, then we have

$$\sum_{k=0}^j \alpha_{kj} \approx \int_0^1 u^{-q-1} du = -\frac{1}{q} \qquad (8)$$

Therefore, Eq.(5) becomes,

$$\begin{aligned}
&\int_0^1 \frac{y(t_j - t_j w) - y(0)}{w^{q+1}} dw \\
&= \sum_{k=0}^j \alpha_{kj} y_{j-k} - \frac{1}{q} y_0 \qquad (9) \\
&= \sum_{k=1}^j \alpha_{kj} y_{j-k} + \alpha_{0j} y_j - \frac{1}{q} y_0
\end{aligned}$$

Being use the both endpoints of the integration interval as a node, consequently, fractional derivative of y of order q are explicitly evaluated by the following formula:

$$\begin{aligned}
(D^q y)(t) &= \frac{t_j^{-q}}{\Gamma(-q)} \Big(\sum_{k=1}^j \alpha_{kj} y_{j-k} + \alpha_{0j} y_j + \frac{1}{q} y_0 \Big) \\
&= \frac{t_j^{-q} \alpha_{0j}}{\Gamma(-q)} y_j + \frac{t_j^{-q}}{\Gamma(-q)} \Big(\sum_{k=1}^j \alpha_{kj} y_{j-k} + \frac{1}{q} y_0 \Big)
\end{aligned}$$
$$(10)$$

where the weights, α_{kj} for $j \geq 1$, are calculated by the following Lemma;

Lemma 1 *In the quadrature formula Q_j, α_{kj}'s are evaluated by:*

$$q(1-q)j^{-q} \alpha_{kj} =$$
$$\begin{cases}
-1 & k = 0 \\
2k^{1-q} - (k-1)^{1-q} - (k+1)^{1-q} & k = 1, \ldots, j-1 \\
(q-1)k^{-q} - (k-1)^{1-q} + k^{1-q} & k = j
\end{cases}$$

Proof 1 *The Proof is straightforward from the definition of the quadrature formula.*

3 Solutions of Some Particular Types of Fractional Equations Involving Multi Derivative Terms

In this section, we will mention about the hybrid method for the solution of the Eq.(1) with the prescribed initial conditions. According to the method fractional derivative is evaluated by the quadrature method which is defined by the formula Eq. (10) and ordinary derivatives will be evaluated by finite difference method.

3.1 Case I: Solving Fractional Order Differential Equation Involving First Order Derivative Term

First of all, we will consider the case I, where we assume that $\alpha = 0$ and $\delta = 1$ in Eq. (1). Hence, we can write,

$$D^q[y - y_0](t) = y'(t) + \beta y(t) + f(t) \qquad (11)$$

with the initial condition;

$$y(t_0) = y_0. \qquad (12)$$

The numerical algorithm for the fractional order derivative in Eq.(11), which states on the left hand

side of equation, has already been defined by Eq.(10) in the case of equispaced grid for $j = 1, \ldots n$. Now, in Eq. (12), the term $y'(t)$ will be approximated by backward finite difference formulation again by using the same nodes. If we write backward finite difference formulation for the first derivative of y we approximately have

$$y'(t_j) \approx \frac{y(t_j) - y(t_{j-1})}{h}. \tag{13}$$

As it is well known the error for the backward finite difference formula is order $O(n^{-1})$ since $h = 1/n$. Substituting the last formula and the formula which is given by Eq. (10) into Eq. (11) we obtain,

$$\frac{y_j - y_{j-1}}{h} + f(t_j) + \beta y(t_j) = \frac{t_j^{-q}\alpha_{0j}}{\Gamma(-q)}y_j$$

$$+ \frac{t_j^{-q}}{\Gamma(-q)}\Big(\sum_{k=1}^{j}\alpha_{kj}y_{j-k} - \frac{1}{q}y_0\Big). \tag{14}$$

Arranging Eq. (14), that yields,

$$y_j\Big(\frac{1}{h} + \beta - \frac{t_j^{-q}\alpha_{0j}}{\Gamma(-q)}\Big) = \frac{y_{j-1}}{h}$$

$$- f(t_j) + \frac{t_j^{-q}}{\Gamma(-q)}\Big(\sum_{k=1}^{j}\alpha_{kj}y_{j-k} - \frac{1}{q}y_0\Big). \tag{15}$$

Hence, for $j = 1$ we have,

$$y_1\Big(\frac{1}{h} + \beta - \frac{t_1^{-q}\alpha_{01}}{\Gamma(-q)}\Big) = \frac{y_0}{h} - f(t_1)$$

$$+ \frac{t_1^{-q}}{\Gamma(-q)}\Big(\alpha_{11}y_0 - \frac{1}{q}y_0\Big) \tag{16}$$

Therefore, y_1 is obtained as,

$$y_1 = \frac{ht_1^q\Gamma(-q)}{\Gamma(-q)t_1^q + \beta h\Gamma(-q)t_1^q - h\alpha_{01}}\Big(\frac{y_0}{h} - f(t_1)$$

$$+ \frac{t_1^{-q}}{\Gamma(-q)}\Big(\alpha_{11}y_0 - \frac{1}{q}y_0\Big)\Big) \tag{17}$$

Subsequently, for $j = 2, 3, .., n$ all y_j's are evaluated from Eq. (15) easily. Now, for verifying the reliability of the new hybride method we can solve some prototype problems with the given initial condition.

Example 1 *Let us consider the following initial value problem;*

$$D^q[y](t) = y'(t) - y(t) + t^2 - 2t + \frac{2}{\Gamma(3-q)}t^{2-q},$$

$$y(0) = 0. \tag{18}$$

It can be taken out easily from the differential equation that the solution of the the problem is $y(t) = t^2$. By using the formula in Eq.(14), we evaluate the y_j's for $j = 1, 2, \ldots, n$, and Table 1 lists the absolute errors for $h = 0.01$ and $h = 0.001$ obtained from the hybrid method. The second column in the same Table shows the exact values of the function at particular points. For finer mesh it is clear that the results much closer to the exact solution.

$q = 1/2$

t	$y(t) = t^2$	$h = 0.01$	$h = 0.001$
0.1	0.01	0.000280	0.000001
0.2	0404	0.000150	0.000004
0.3	0.09	0.000570	0.000012
0.4	0.16	0.000186	0.000012
0.5	0.25	0.000319	0.000018
0.6	0.36	0.000494	0.000026
0.7	0.49	0.000724	0.000030
0.8	0.64	0.001025	0.000050
0.9	0.81	0.001437	0.000067
1.0	1.00	0.001929	0.000090

Table 1: The absolute errors for $h = 0.01$ and $h = 0.001$ at particular points t. The second column in Table, indicates the exact solutions of the equation.
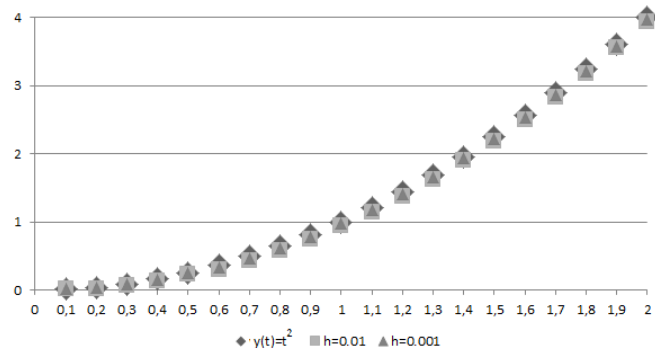

Figure 1: The comparison between the exact solution and numerical solutions for $h = 0.01$ and $h = 0.001$ for different mesh size.

Example 2 *Now, we consider another initial value problem which involves both fractional derivative and first order derivative as follows;*

$$
\begin{aligned}
D^q(y)(t) &= y'(t) - y(t) + t^4 - \frac{1}{2}t^3 - 4t^3 + \frac{3}{2}t^2 \\
&\quad - \frac{3}{\Gamma(4-q)}t^{3-q} + \frac{24}{\Gamma(5-q)}t^{4-q}, \\
y(0) &= 0 \qquad\qquad\qquad (19)
\end{aligned}
$$

Analytical solution of this problem is defined by the function:

$$
y(t) = t^4 - \frac{1}{2}t^3 \qquad\qquad (20)
$$

The results which are obtained from Eq.(15) for y_j's are listed in Table 2.

$q = 1/2$

t	$y(t) = t^4 - \frac{1}{2}t^3$	$h = 0,01$	$h = 0,001$
0.1	-0.0004	-0.000394	-0.000400
0.2	-0.0024	-0.002388	-0.002400
0.3	-0.0054	-0.005389	-0.005400
0.4	-0.0064	-0.006409	-0.006400
0.5	0.0	-0.000063	-0.000001
0.6	0.0216	0.021430	0.021596
0.7	0.0686	0.068244	0.068559
0.8	0.01536	0.152945	0.153583
0.9	0.2916	0.290491	0.291570
1.0	0.5	0.498223	0.499952

Table 2: Comparing the numerical results, which are obtained from the new method for different mesh sizes, $h = 0.01$ and $h = 0.001$, with exact solutions at particular points t.

According to the table, the numerical method implies that the results are in good agreement with the exact solutions. It is clear that finer mesh gives more accurate results.

3.2 Case II: Solving Fractional Order Differential Equation, Involving Second Order Derivative Term

As a second case, we will consider an another fractional differential equations involving second order derivative term. The method is straightforward and similar steps will be applied to the Case.II. Here, for simplicity, we assume that $\alpha = 1$ and $\delta = 0$ and $\beta < 0$ in Eq. (1). Hence, we can write,

$$
\begin{aligned}
D^q[y - y_0](t) &= y''(t) + \beta y(t) + f(t) \\
y(0) &= y_0, y'(0) = 0 \qquad (21)
\end{aligned}
$$

Again, we consider, $0 < q < 1$, and f is a known function. We will compute the y values on the equispaced grid as it is mentioned in case I. For $y''(t)$ we will use central finite difference formula ;

$$
f''(t_i) \approx \frac{f(t_{i+1}) - 2f(t_j) + f(t_{i-1})}{h^2} \qquad (22)
$$

where the error of the formula is $O(n^{-2})$ as known. By substituting Eq. (15) and Eq. (22) into Eq. (21), we have;

$$
\begin{aligned}
&\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + f(t_j) + \beta y(t_j) = \frac{t_j^{-q}\alpha_{0j}}{\Gamma(-q)}y_j \\
&+ \frac{t_j^{-q}}{\Gamma(-q)}\left(\sum_{k=1}^{j}\alpha_{kj}y_{j-k} - \frac{1}{q}y_0\right) \qquad (23)
\end{aligned}
$$

Arranging the last equation,

$$
\begin{aligned}
&\frac{y_{j+1}}{h^2} = \frac{2y_j}{h^2} - \frac{y_{j-1}}{h^2} - f(t_j) - \beta y(t_j) \\
&+ \frac{t_j^{-q}\alpha_{0j}}{\Gamma(-q)}y_j + \frac{t_j^{-q}}{\Gamma(-q)}\left(\sum_{k=1}^{j}\alpha_{kj}y_{j-k} - \frac{1}{q}y_0\right) \qquad (24)
\end{aligned}
$$

and solving in terms of y_{j+1} we obtain,

$$
\begin{aligned}
y_{j+1} &= h^2\left[-\frac{y_{j-1}}{h^2} - f(t_j) + y(t_j)\left(\frac{2}{h^2} + \frac{t_j^{-q}\alpha_{0j}}{\Gamma(-q)} - \beta\right) \right. \\
&\quad \left. + \frac{t_j^{-q}}{\Gamma(-q)}\left(\sum_{k=1}^{j}\alpha_{kj}y_{j-k} - \frac{1}{q}y_0\right) \right] \qquad (25)
\end{aligned}
$$

Here, we will give an example related to Case II.

Example 3 *Consider the following initial value problem;*

$$
\begin{aligned}
D^q[y](t) &= y''(t) - y(t) + t^2 - 2 + \frac{2}{\Gamma(3-q)}t^{2-q}, \\
y(0) &= 0, y'(0) = 0. \qquad (26)
\end{aligned}
$$

Solution of the differential equation is easily verified that $y(t) = t^2$. Now letting $q = 1/2$, the numerical values of y_j's can be evaluated from Eq. (25). These results are listed in Table 3. Column 2 indicates the exact results at particular values of t and columns 2 and 3 shows absolute errors for $h = 0.01$ and $h = 0.001$ respectively. For finer mesh, it is clear that the results much more reliable.

Now, we will consider another example for Case II, but this time we will evaluate the numerical results for different q values.

$q = 1/2$

t	$y(t) = t^2$	$h = 0.001$	$h = 0.0001$
0.1	0.01	0.000101	0.000010
0.2	0.04	0.000207	0.000021
0.3	0.09	0.003210	0.000032
0.4	0.16	0.000444	0.000044
0.5	0.25	0.000581	0.000058
0.6	0.36	0.000733	0.000073
0.7	0.49	0,489095	0.000090
0.8	0.64	0.001100	0.000110
0.9	0.81	0.001321	0.000132
1.0	1.0	0.001575	0.000157

Table 3: The absolute errors for $h = 0.01$ and $h = 0.001$ at particular points t in Example 3. The second columns indicates the exact solutions of the equation

Example 4 *Consider the following initial value problem,*

$$
\begin{aligned}
D^q[y](t) &= y''(t) - y(t) + t^4 - \frac{1}{2}t^3 - 12t^2 + 3t \\
&\quad - \frac{3}{\Gamma(3,5)}t^{2,5} + \frac{24}{\Gamma(4,5)}t^{3,5} \quad (27) \\
y(0) &= 0, y'(0) = 0,
\end{aligned}
$$

For $q = 0.5$ we have,

$$
\begin{aligned}
D^{0.5}[y](t) &= y''(t) - y(t) + t^4 - \frac{1}{2}t^3 - 12t^2 + 3t \\
&\quad - \frac{3}{\Gamma(3,5)}t^{2,5} + \frac{24}{\Gamma(4,5)}t^{3,5} \quad (28) \\
y(0) &= 0, y'(0) = 0,
\end{aligned}
$$

and the analytical solution of Eq.(28) is

$$
y(t) = t^4 - \frac{1}{2}t^3 \quad (29)
$$

For different values of q, as far as we know, analytical solutions do not exist. Therefore, we have used the hybrid method for evaluating solution of the problem for different q values with step size $h = 0.01$ and the related plots are shown in Figure 2. Table 4 also shows numerical results for the same problem.

3.3 Error Estimation

Now, we will give the error estimation for the hybrid method given in Section 2. Based on residual error estimation, the error of the method is negligable. Let

t	$q = 0.25$	$q = 0.75$	$q = 0.99$	$q = 1$
0.1	-0.000395	-0.000398	-0.000403	-0.000409
0.2	-0.002378	-0.002426	-0.002489	-0.002512
0.3	-0,005322	-0.005526	-0.005762	-0.005814
0.4	-0.006221	-0.006695	-0.007188	-0.007264
0.5	0.002730	-0.000419	-0.001035	-0.001089
0.6	0.021797	0.021390	0.021270	0.021363
0.7	0.068236	0.069455	0.071285	0.071763
0.8	0,151672	0.157197	0.163340	0.164957
0.9	0.286335	0.300796	0.316437	0.319108
1.0	0.488557	0.519255	0.550879	0.555859

Table 4: Comparing the numerical results, which are obtained from the new method for $h = 0.01$ and $q = 0.25$, $q = 0.75$, $q = 0,99$, $q = 1$.
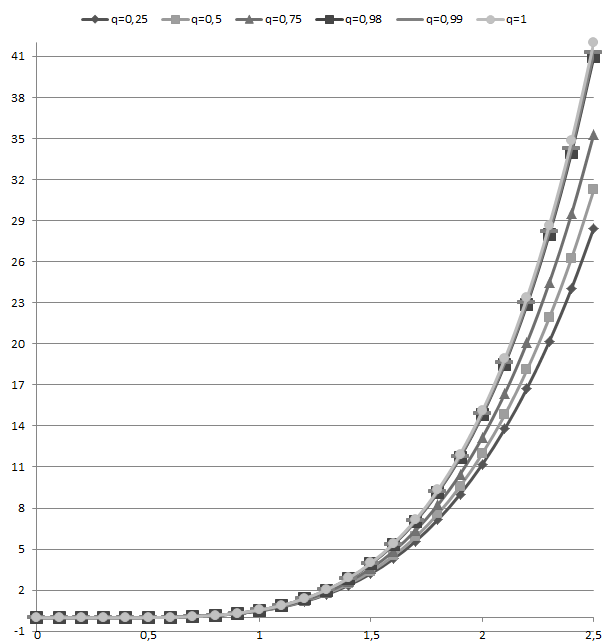

Figure 2: Numerical Solutions of Example 4 , for $h = 0.01$ and comparison of the solutions for $q = 0.25$, $q = 0.75$, $q = 0.99$ and for $q = 1$, results are obtained via Maple 18.

$y(t_j)$ and y_j denote the exact and numerical solutions respectively. Hence, we can write,

$$
\begin{aligned}
D^q[y(t_j)] &- \alpha y''(t_j) - \delta y'(t_j) - \beta y(t_j) \\
&- f(t_j) = 0 \quad (30)
\end{aligned}
$$

and

$$
D^q[y_j] - \alpha y_j'' - \delta y_j' - \beta y_j - f_j = Res \quad (31)
$$

where Res is the residual function. Subtracting Eq. (31) from (30), we have

$$
\begin{aligned}
D^q[y(t_j) - y_j] \quad & - \quad \alpha(y''(t_j) - y_j'') - \delta(y'(tj) - y_j') \\
& - \quad \beta(y(t_j) - y_j) = Res \qquad (32)
\end{aligned}
$$

If we denote $e_j = y(t_j) - y_j$ as an error function then, by Eq. (32), we get,

$$
D^q[e_j] - \alpha e_j'' - \delta e_j' - \beta e_j = Res \qquad (33)
$$

Hence, Eq. (33) with the initial conditions $e_j(0) = 0$ and $e_j' = 0$ can be solved same as in Section 2. As a result, we define a measure for the error estimation as follows:

$$
E = max\{|e_j| : 0 < t < 1\}. \qquad (34)
$$

4 Conclusion

So far, many numerical or approximate methods have been introduced for solving fractional order differential equations but many of them involve only one derivative term. Here, we aimed to solve fractional equations involving multi derivative terms. The new method, which is called the hybrid method, is the combination of quadrature and finite difference methods. The method is very easy to apply and the calculations are straightforward. In the given examples above, we have considered two cases. The first one involves first order derivative term together with the fractional order one and in the second case, we have investigated an equation which involves a second order derivative term and a fractional order term. These examples indicate that the relaibility and applicibility of the method is quite well when are compared with the analytical solutions and for finer mesh one can obtain more relaible results. In this work, we also consider the solution of fractional order differential equation for different q values. If we examine the Figure 4, we see that the solution of ordinary differential equation for $q = 1$ overlaps with the cases $q \neq 1$ up to some particular value of $t \approx 1$. When t is greater than 1, solutions tend to diverge from the solution of ordinary differential equation. When $q = 0.99$ there is a small difference between the case $q = 1$. The considerable difference occurs when $q = 0.25$ and $q = 1$. This reveals that the fractional order equations presumably describe more realistic models. As a further work, this method can be applied to much wider interval for q.

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