A comparison between finite difference and asymptotic methods for solving a reaction-diffusion model in ecology

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Abstract--A reaction-diffusion system (RDS) as a mutualism model in ecology is studied using finite difference method and asymptotic methods. For nonlinear reaction term an explicit method is used and an implicit method for linear diffusion term. The numerical solutions are found as traveling wave solutions with no flux Neuman boundary conditions and for three different types of initial conditions which represent a common ecological cases. The asymptotic solutions are studied for this model when a small perturbed parameter $\lambda \ll 1$ appear from non dimensional of (RDS). The traveling wave solutions from the above two methods are compared and shown a good agreement.

Key-Words: -Reaction diffusion system, Mutualism, Finite difference methods, Asymptotic methods

1 Introduction

A theory of reaction-diffusion equations (RDE) is one of the important theorem that attracted different scientists from different fields, such as biology, physics, ecology, chemistry, etc. [2, 7]. Types of reaction-diffusion equations has been studied since 1930, by Fisher [8, 10], and the common topic that is attracted the research is a traveling wave solution [12]. This concept translated in elementary of ecology texts to represent the organism interaction, which has three fundamental ways, and generally define with names competition, predation, and mutualism. The only interaction that both species get benefit is mutualism. Mathematically, mutualism as interaction between two species in brief is defined $+/+$ interaction [5]. Finite difference is one of the oldest and simplest techniques used to find the numerical solutions for partial differential equations and specially (RDE). Numerical methods have been an active research area to solve reaction-diffusion equations (RDE), and the development methods are used widely to deal with this problem. Examples on that, Exponential time difference methods [11, 1], integrating factor methods [3, 9], and operator splitting methods [3]. A posterior error estimates added to operator splitting methods to solve this model adaptively [4]. It has been assumed that FDM is a numerical method that can be used to obtain precise solution but it works better when all parameters are of order one, whilst there are analytical (asymptotic) methods work only when there is a parameter in the equation is small or large. Asymptotic or perturbation methods give the physical explanation for the problem besides the boundary layer in the problem [6,7,8]. We study a reaction-diffusion model for a system of two species which exhibits mutualism population interactions, provided that the population is sufficiently small. The model we will study here is

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + P_u u(1 + Q_u u - R_u u^2 + S_u w)$$
$$\frac{\partial w}{\partial t} = D_w \frac{\partial^2 w}{\partial x^2} + P_w w(1 + Q_w w - R_w w^2 + S_w u) \qquad (1)$$

where the diffusion coefficients are described by D_u and D_w, in the reaction term both $P_u u(1 - R_u u^2)$and $P_w w(1 - R_w w^2)$ are generalized logistic growth rates for the species u and w. In this model, the intra specific cooperation has the cooperative parameters $S_u w$ and $S_w u$, whilst the terms $S_u$and $S_w$with positive sign in fronts describe that both species get benefits from interactions which also called mutualism.

In this paper, section two describes the model after non dimensional the RDS and we show the initial and boundary conditions that use to solve this model. In section three, the possible equilibrium solutions and their stability are studied. The finite difference method in section four is shown for finding the stable traveling wave solutions for (1). Asymptotic solutions are studied and compared in to the solutions of finite difference method in section 5, and conclusion is discussed in section 6.

2 Dimensionless of Reaction-diffusion system (1)

We define dimensionless variables,

$$u = U\bar{u} \quad , \quad w = W\bar{w} \quad , \quad x = (\frac{D_u}{P_u})^{\frac{1}{2}}\bar{x} \quad , \quad t = \frac{\bar{t}}{P_u}.$$

Substituted this value in equation (1), and using the dimensionless parameters
$\alpha_1 = Q_u U$, $\gamma_1 = S_u W$, $\alpha_2 = Q_w W$, $\gamma_2 = S_w W, \beta_1 = R_U U^2$, $\lambda = \frac{P_w}{P_u}$, $\frac{D}{\lambda} = \frac{D_w}{D_u}$, $\beta_2 = R_w W^2$,
we get after omitting the over bar for convenience,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial u^2} + u(1 + \alpha_1 u - \beta_1 u^2 + \gamma_1 w),$$
$$\frac{\partial w}{\partial t} = \frac{D}{\lambda}\frac{\partial^2 w}{\partial x^2} + \lambda w(1 + \alpha_2 w - \beta_2 w^2 + \gamma_2 u). \qquad (2)$$

The dimensionless parameter $\lambda \ll 1$ is very small and therefore we will use asymptotic methods to solve (2).
Three ecologically important initial conditions which represent most important cases in ecologically population and assume that the initial conditions are symmetric about the origin, so we consider the problem for $x \geq 0$ and $t \geq 0$,

$$u(x,0) = u_0(x), \qquad w(x,0) = w_0(x),$$

which describe the

- initial condition A

$$A) \qquad u_0(x) = \begin{cases} 1, & x \le L_0 \\ 0, & x > L_0 \end{cases}$$
$$w_0(x) = 1,$$

where L_0 is a width of step function. The physical meaning for this case is that species w is native and species u is introduced.

- initial condition B

$$u_0(x) = 1.$$
$$B)\ w_0(x) = \begin{cases} 1, & x \le L_0 \\ 0, & x > L_0 \end{cases}$$

The physical meaning for this case is that species u is native and species w is introduced.

- initial condition C

$$C) \qquad u_0(x) = \begin{cases} 1, & x \le L_0 \\ 0, & x > L_0 \end{cases}$$
$$w_0(x) = \begin{cases} 1, & x \le L_0 \\ 0, & x > L_0 \end{cases}$$

and boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = 0, \qquad \frac{\partial w(0,t)}{\partial x} = 0.$$

We have chosen in the initial condition to have a value ($u_0 = 1$ *or* $w_0 = 1$) . The far field boundary conditions are therefore, $u \to 0$ and $w \to 0$ as $x \to \infty$. The physical meaning for this case is that species u and w are both introduced.

3. Equilibrium solutions:

We study the equilibrium solutions of (2) and analysis its stability in order to predict the type of traveling wave of solutions that can be constructed, since this type of solution connect stable equilibrium point to another one.

$$\frac{\partial u}{\partial t} = u(1 + \alpha_1 u - \beta_1 u^2 + \gamma_1 w) \equiv u(f(u) + \gamma_1 w)$$
$$\frac{\partial w}{\partial t} = \lambda w(1 + \alpha_2 w - \beta_2 w^2 + \gamma_2 u) \equiv \lambda w(g(w) + \gamma_2 u) \ldots.. (3)$$

Where $f(u) = 1 + +\alpha_1 u - \beta_1 u^2$ *and* $g(u) = 1 + \alpha_2 w - \beta_2 w^2 \ldots (4)$

The convenient way to analyses these solutions and get a qualitative picture of the dynamics of the ordinary differential equations (3) is using nullclines. The nullclines are

$$u(f(u) + \gamma_1 w) = 0$$
$$\lambda w(g(w) + \gamma_2 u) = 0 \qquad \ldots (5)$$

The intersection of $\gamma_1 w = f(u)$ and $\gamma_2 u = g(w)$ or the values of u and w for which the time derivatives in (3) are equal to zero are the spatially uniform solutions or the equilibrium solutions. Thus, the obvious equilibrium solutions are

- $(u_0, 0)$ *and* $(0, w_0)$, which are single species equilibrium points.
- $(u,w) = (0,0)$, which is an extinction of both species.

In addition, there is a coexistence equilibrium solution given by the intersections of the quadratic curves,

$$w = f(u)/\gamma_1 \ , u = g(w)/\gamma_2.$$

where $0 < \alpha_1, \alpha_2 < 1$, $\beta_1 > 1 - \alpha_1$, $\beta_2 > 1 - \alpha_2$
$\gamma_1 = \beta_1 - (1 + \alpha_1), \gamma_2 = \beta_2 - (1 + \alpha_2)$.
Thus there is a positive coexistence equilibrium solution namely (1,1). The stability analysis of the equilibrium points is shown that

- (0,0) : has two positive eigenvalues 1 and λ, therefore its unstable
- (1,1): The eigenvalues have the general form

$$\frac{\alpha_1 - 2\beta_1 + \lambda(\alpha_2 - 2\beta_2) + \sqrt{(\alpha_1 - 2\beta_1 + \lambda(\alpha_2 - 2\beta_2))^2 - 4\lambda(\beta_1\beta_2 + \beta_1 + \beta_2 - 1 - \alpha_1 - \alpha_2)}}{2},$$

$$\frac{\alpha_1 - 2\beta_1 + \lambda(\alpha_2 - 2\beta_2) - \sqrt{(\alpha_1 - 2\beta_1 + \lambda(\alpha_2 - 2\beta_2))^2 - 4\lambda(\beta_1\beta_2 + \beta_1 + \beta_2 - 1 - \alpha_1 - \alpha_2)}}{2},$$

therefore it is a stable node.

- $(u_0, 0)$ has the eigenvalues:

$$\frac{1+2\alpha_1 u_0-3\beta_1 u^2{}_0+\lambda\gamma_2 u_0+\sqrt{(1+2\alpha_1 u_0-3\beta_1 u^2{}_0+\lambda\gamma_2 u_0)^2-4\lambda(1+2\alpha_1 u_0-3\beta_1 u^2{}_0)(\gamma_2 u_0)}}{2},$$

$$\frac{1+2\alpha_1 u_0-3\beta_1 u^2{}_0+\lambda\gamma_2 u_0-\sqrt{(1+2\alpha_1 u_0-3\beta_1 u^2{}_0+\lambda\gamma_2 u_0)^2-4\lambda(1+2\alpha_1 u_0-3\beta_1 u^2{}_0)(\gamma_2 u_0)}}{2},$$

- $(0, w_0)$ has the eigenvalues:

$$\frac{1+2\alpha_2 \quad w_0-3\beta_2 w^2{}_0+\lambda\gamma_1 w_0+\sqrt{(1+2\alpha_2 w_0-3\beta_2 w^2{}_0+\lambda\gamma_1 w_0)^2-4\lambda(1+2\alpha_2 w_0-3\beta_2 w^2{}_0)(\gamma_2 u_0)}}{2},$$

$$\frac{1+2\alpha_2 \quad w_0-3\beta_2 w^2{}_0+\lambda\gamma_1 w_0-\sqrt{(1+2\alpha_2 w_0-3\beta_2 w^2{}_0+\lambda\gamma_1 w_0)^2-4\lambda(1+2\alpha_2 w_0-3\beta_2 w^2{}_0)(\gamma_2 u_0)}}{2}.$$

The expected type of traveling wave solutions can be classified to the following:

- Type (I_a), The traveling wave connects (1,1) to $(u_0, 0)$.
- Type (I_b), The traveling wave connects (1,1) to $(0, w_0)$.
- Type (I_c), The traveling wave connects (1,1) to (0,0).

4 Finite difference method

In this section, we solve (2) numerically and try to find the different types of travelling wave solutions that are generated by the different initial conditions. An implicit method is used to discretize the diffusion operator. For the nonlinear reaction part we use an explicit method. Finite difference method can be derived using a Taylor series expansion for $u(x_0 + \Delta x)$ and $u(x_0 - \Delta x)$, where Δx is the step size of x

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t} = \frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{(\Delta x)^2} + u_{i,j}\left(1 + \alpha_1 u_{i,j} - \beta_1 (u_{i,j})^2 + \gamma_1 w_{i,j}\right) \quad (6)$$

$$\frac{w_{i,j+1}-w_{i,j}}{\Delta t} = \left(\frac{D}{\lambda}\right)\frac{w_{i+1,j+1}-2w_{i,j+1}+w_{i-1,j+1}}{(\Delta x)^2} + \lambda w_{i,j}\left(1 + \alpha_2 w_{i,j} - \beta_2 (w_{i,j})^2 + \gamma_2 u_{i,j}\right). \quad (7)$$

These equations simplify to give us

$$u_{i,j+1} - u_{i,j} = \left(\frac{\Delta t}{(\Delta x)^2}\right)\left(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right) + \Delta t\, u_{i,j}\left(1 + \alpha_1 u_{i,j} - \beta_1 (u_{i,j})^2 + \gamma_1 w_{i,j}\right) \ldots\ldots (8)$$

$$w_{i,j+1} - w_{i,j} = \left(\frac{D\Delta t}{\lambda(\Delta x)^2}\right)\left(w_{i+1,j+1} - 2w_{i,j+1} + w_{i-1,j+1}\right) + \Delta t\, \lambda\, w_{i,j}\left(1 + \alpha_2 w_{i,j} - \beta_2 (w_{i,j})^2 + \gamma_2 u_{i,j}\right). \quad (9)$$

Put $r_1 = \left(\frac{\Delta t}{(\Delta x)^2}\right)$ and $r_2 = \left(\frac{D\Delta t}{\lambda(\Delta x)^2}\right)$

$$-r_1 u_{i+1,j+1} + (1+2r_1)u_{i,j+1} - r_1 u_{i-1,j+1} = \Delta t\, u_{i,j}\left(1 + \alpha_1 u_{i,j} - \beta_1 (u_{i,j})^2 + \gamma_1 w_{i,j}\right) + u_{i,j},$$

$$-r_2 w_{i+1,j+1} + (1+2r_2)w_{i,j+1} - r_2 w_{i-1,j+1} = \Delta t\, \lambda\, w_{i,j}\left(1 + \alpha_2 w_{i,j} - \beta_2 (w_{i,j})^2 + \gamma_2 u_{i,j}\right) + w_{i,j}.$$

A three point formula boundary conditions are used

$$u_x = \frac{-3u_n^{t+\Delta t} + 4u_{n+\Delta x}^{t+\Delta t} - u_{n+2\Delta x}^{t+\Delta t}}{2\Delta x} = 0,$$

$$w_x = \frac{-3w_n^{t+\Delta t} + 4w_{n+\Delta x}^{t+\Delta t} - w_{n+2\Delta x}^{t+\Delta t}}{2\Delta x} = 0.$$

From discretization we get a system of algebraic equations which can be written in the form

$$\begin{bmatrix} 3 & -4 & 1 & & \\ -r_1 & (1+2r_1) & -r_1 & & \\ \ddots & \ddots & \ddots & & \\ & & -r_1 & (1+2r_1) & -r_1 \\ & & 1 & -4 & 3 \end{bmatrix}\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{N-1,j+1} \\ u_{N-1,j+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta t\, u_{2,j}\left(1 + \alpha_1 u_{2,j} - \beta_1 (u_{2,j})^2 + \gamma_1 w_{2,j}\right) + u_{2,j} \\ \vdots \\ \Delta t\, u_{N-1,j}\left(1 + \alpha_1 u_{N-1,j} - \beta_1 (u_{N-1,j})^2 + \gamma_1 w_{N-1,j}\right) + u_{N-1,j} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 & 1 & & \\ -r_2 & (1+2r_2) & -r_2 & & \\ \ddots & \ddots & \ddots & & \\ & & -r_2 & (1+2r_2) & -r_2 \\ & & 1 & -4 & 3 \end{bmatrix}\begin{bmatrix} w_{1,j+1} \\ w_{2,j+1} \\ \vdots \\ w_{N-1,j+1} \\ w_{N-1,j+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta t\, w_{2,j}\left(1 + \alpha_1 w_{2,j} - \beta_2 (w_{2,j})^2 + \gamma_1 u_{2,j}\right) + w_{2,j} \\ \vdots \\ \Delta t\, w_{N-1,j}\left(1 + \alpha_1 w_{N-1,j} - \beta_2 (w_{N-1,j})^2 + \gamma_1 u_{N-1,j}\right) + w_{N-1,j} \\ 0 \end{bmatrix}$$

(10)

We solve the linear system (10) at each time step using the backslash operator in MATLAB. Three types of traveling wave solution are shown in figure (1-3) for specific type of initial condition and specific values of parameters.

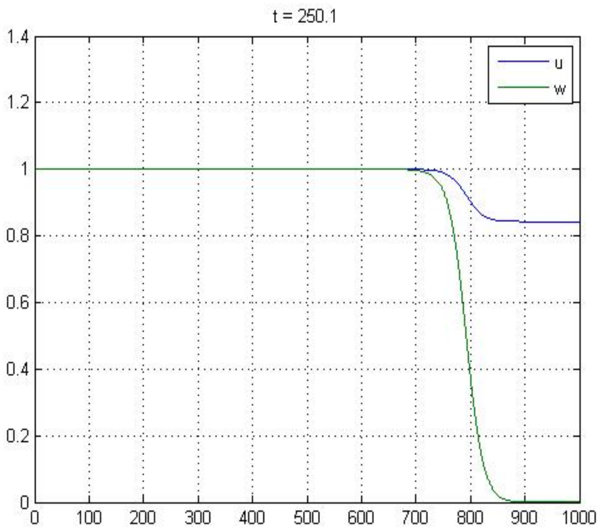

Fig. 1: With initial condition A .The travelling wave that develops (I_a) when $\alpha_1 = 0.5$, $\beta_1 = 2$, $\alpha_2 = 0.5$, $\beta_2 = 3$, $\lambda = 0.05$ $and\ D = 1$

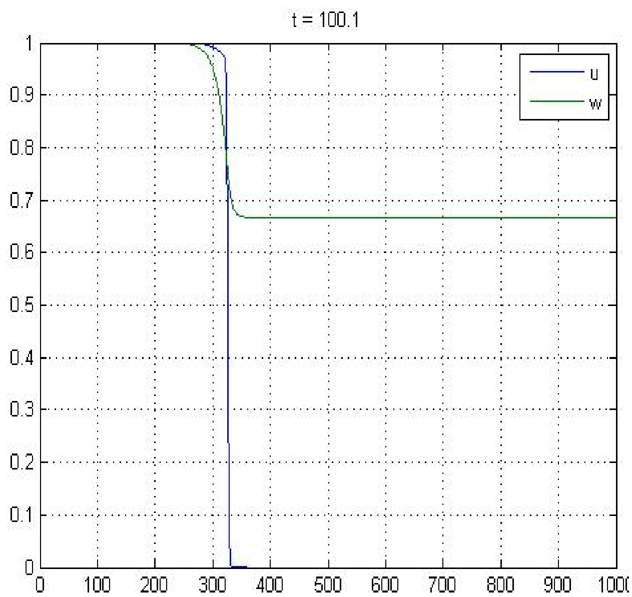

Fig. 2: With initial condition B .The travelling wave that develops (I_b) when $\alpha_1 = 0.5$, $\beta_1 = 2$, $\alpha_2 = 0.5$, $\beta_2 = 3$, $\lambda = 0.05$ $and\ D = 1$

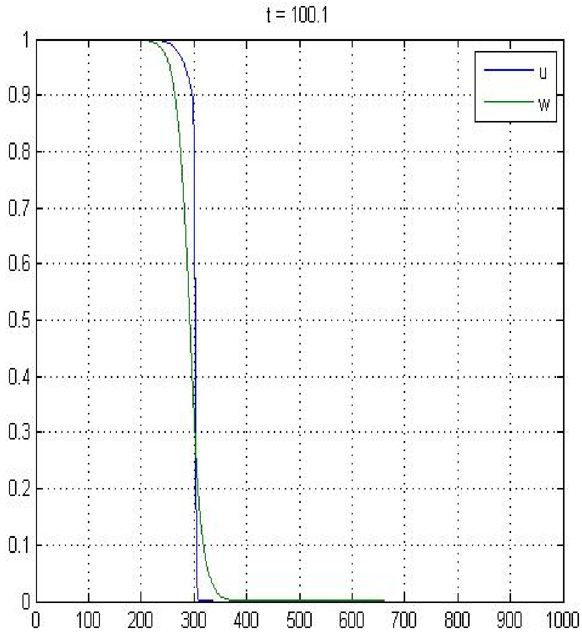

Fig. 3: With initial condition C .The travelling wave that develops (I_c) when $\alpha_1 = 0.5$, $\beta_1 = 2$, $\alpha_2 = 0.5$, $\beta_2 = 3$, $\lambda = 0.05$ $and\ D = 1$.

5 Asymptotic solutions for $\lambda \ll 1$

We have seen that a variety of travelling waves develops as solutions of the initial value problem. We define $z = x - ct$, and seek permanent form travelling wave solutions $\bar{u} = u(z) and\ \bar{w} = w(z)$ with wave speed $c > 0$.

Substituting the new variable z in the reaction diffusion system (2), we get

$$\frac{d^2u}{dZ^2} + c\frac{du}{dZ} + u(1 + \alpha_1 u - \beta_1 u^2 + \gamma_1 w) = 0$$
$$\frac{D}{\lambda}\frac{d^2w}{dZ^2} + c\frac{dw}{dZ} + \lambda w(1 + \alpha_2 w - \beta_2 w^2 + \gamma_2 u) = 0$$
….(11)

5.1 Regular perturbation solutions for type (I_a)

At leading order, provided that $U \nrightarrow 0$ as $Z \to \pm\infty$, this is a regular perturbation problem, with the leading order equations

$$D\frac{d^2W}{dZ^2} + c\frac{dW}{dZ} + W(1 + \alpha_2 W - \beta_2 W^2 + \gamma_2 U) = 0 \quad \ldots (12)$$

$$U(1 + \alpha_1 U - \beta_1 U^2 + \gamma_1 W) = 0, \ldots (13)$$

or equivalently

Let $\frac{dW}{dz} = V,$

$$\frac{dV}{dZ} = \frac{c}{D}V + \frac{1}{D}W(1 + \alpha_2 W - \beta_2 W^2 + \gamma_2 U) = 0 \qquad (14)$$

$\gamma_1 w = f(u)$.

In the (W, V) phase plane, this system has equilibrium points at $(0,0)$, which corresponds to the equilibrium solutions $(u_0, 0)$, We can use the Matlab routine 'ode45' to find traveling wave solutions of (14), which connect two equilibrium points of the system. In the (W, V) phase plane, this system has equilibrium points at (0,0), which corresponds to the steady state $U = u_0$, $W = 0$, and (1,0) where is such that (1,1) is a coexistence equilibrium state. Possible traveling wave solutions with this structure therefore connect these two equilibria. We will focus on traveling wave solutions that satisfy $(W, V) \to (1,0)$ as $Z \to -\infty$ and $(W, V) \to (0,0)$ as $Z \to \infty$. By linearizing about (1,0) we find that the stable coexistence equilibrium point corresponds to a saddle point in (14). If a traveling wave solution exists it is therefore represented by the unstable separatrix of (1,0) those points into $Z < 0$. The other equilibrium point (0,0) is a stable node provided that $c^2 > 4D(1 + \gamma_2 U_0)$, and a stable focus for $c^2 < 4D\left(1 + \gamma_2 U_0\right)$. Since we require $W > 0$, this provides a lower bound, $c \geq c_{lb} \equiv 2\sqrt{D\left(1 + \gamma_2\right)}$, on the wavespeed. In figure (4), we see traveling wave develops and connect (1,1) to $(u_0, 0)$. In figure (5), we have compared the traveling wave solution which are found by two methods, finite difference and regular perturbation methods, and shown a good agreement between both.

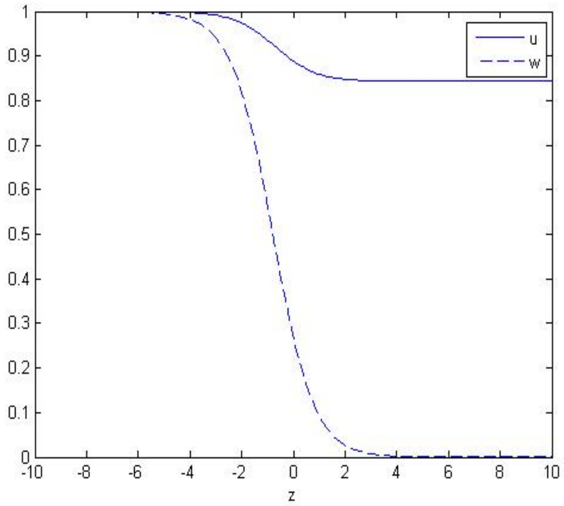

Fig. 4: The travelling wave that develops, when $\alpha_1 = 0.5$, $\beta_1 = 2$, $\alpha_2 = 0.5$, $\beta_2 = 3$, $\lambda = 0.05$ $and\ D = 1$.

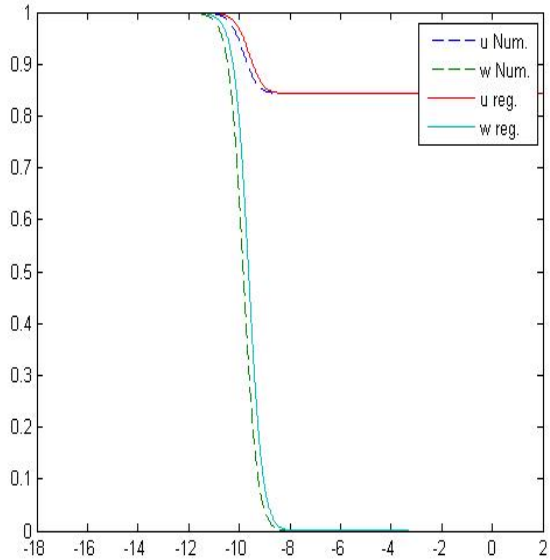

Fig. 5: Comparison between travelling wave solutions that develops using finite difference method and regular perturbation method. The solutions are for type (I_a), and when $\alpha_1 = 0.5$, $\beta_1 = 2$, $\alpha_2 = 0.5$, $\beta_2 = 3$, $\lambda = 0.05$ and $D = 1$.

5.2 Singular perturbation solutions

When one of the equilibrium solutions connected by the travelling wave solution has $U = 0$, we most solve a singular perturbation problem. The leading order problem in the outer region has, from as in (13):

$U(1 + \alpha_1 U - \beta_1 U^2 + \gamma_1 W) = 0.$

The solution must smoothly connect a state with $U = 0$ to one with $(1 + \alpha_1 U - \beta_1 U^2 + \gamma_1 W) = 0$, so an inner asymptotic region is required. Both types I_b and I_c

Are singular perturbation problems and for convenience we solve only type I_c to avoid repetition of process.

5.2.1 Outer solutions for type (I_c)

For this type of travelling wave, $U \to U_0$ as $Z \to -\infty$ and $U \to 0$ as $Z \to \infty$, so for $Z < 0$,

solution must satisfy (12) and (13), whilst for $Z > 0$, and W satisfies

$\frac{dw}{dz} = V,$

$$\frac{dV}{dz} = \frac{-c}{D} V - \frac{1}{D} g(W), \qquad \dots (15)$$

where $g(W) = W(1 + \alpha_2 W - \beta_2 W^2)$

In the (W, V) phase plane, this system has equilibrium points at (1,0), which corresponds to the equilibrium solution $U = 0, W = 1$, and (0,0), which correspond to theequilibrium solution $U = 0, W = 0$. The stability of equilibrium points are shown as follows;

- (0,0) is a stable (If $c > 2$ then (0,0) is a stable node and if $c > 2$ then (0,0) is a stable spiral).

- (1,0) is a saddle point.

In figure (6), two outer regions are plots by solving eq. (12, 13, 15) using Matlab codes

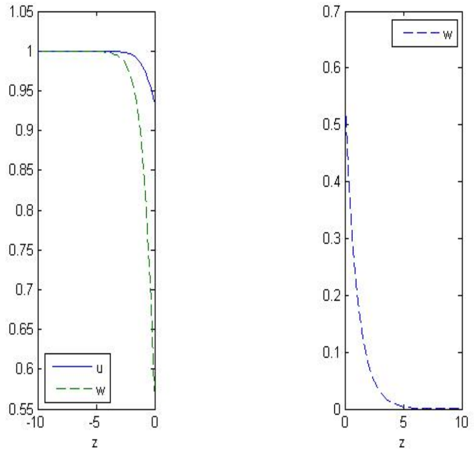

Fig. 6: Outer solutions of type (I_c), when $\alpha_1 = 0.5$, $\beta_1 = 2$, $\alpha_2 = 0.5$, $\beta_2 = 3$, $\lambda = 0.05$ $and\ D = 1$

5.2.2 Inner solution for (I_c)

In the inner region, $z = O(1)$ and $w(z) = W(z)$ is constant at leading order, with value w_0 determined by matching with the outer solution. At leading order, (11) is reduced to an ordinary differential equation for $u(z) = W(z)$.

At o(1),

$$\frac{d^2U}{dz^2} + c\frac{dU}{dz} + U(L + \alpha_1 U - \beta_1 U^2) = 0\,, \qquad ....(16)$$

$$\frac{d^2W}{dz^2} = 0,$$

where $L = 1 + \gamma_1 W_0$. By integration W with respect to z, we get

$W = az + b.$ (17)

Applying the principle of matching between inner and outer regions, and since z in the outer region is of order $\frac{1}{\lambda}$, thus,

$$W = a\frac{1}{\lambda} + b$$

As $\lambda \to 0,\ W \to \infty$. Therefore the only accept form for (17) is W being a constant.

This system (16) will be solved subject to the boundary conditions

$U \to 0 \quad as \quad z \to \infty,$

$U \to \frac{\alpha_1 + \sqrt{\alpha^2{}_1 + 4\beta_1 L}}{2\beta_1} \qquad as \quad z \to -\infty.$

Figure (7) shows the inner and outer region than can be formulated for the type (I_c),

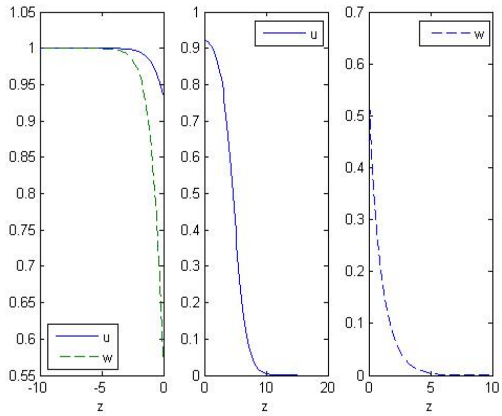

Fig. 7: Travelling wave from the inner and outer solutions (I_c), when $\alpha_1 = 0.5$, $\beta_1 = 2$, $\alpha_2 = 0.5$, $\beta_2 = 3$, $\lambda = 0.05$ $and\ D = 1$

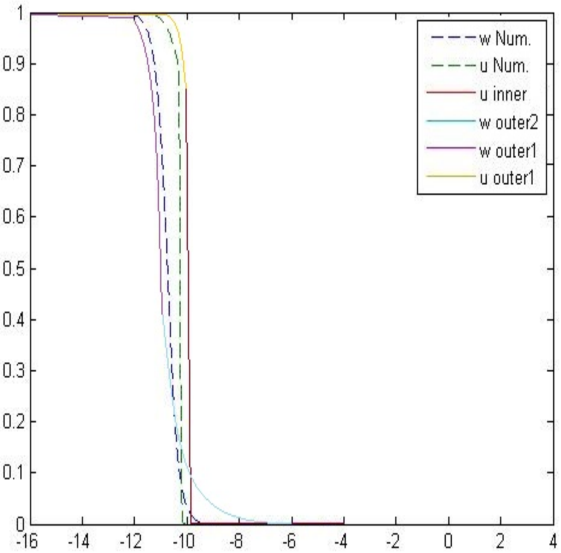

Fig. 8: Travelling wave from Matching principle inner with outer and Numerical of(I_c), when $\alpha_1 = 0.5$, $\beta_1 = 2$, $\alpha_2 = 0.5$, $\beta_2 = 3$, $\lambda = 0.05$ $and\ D = 1$

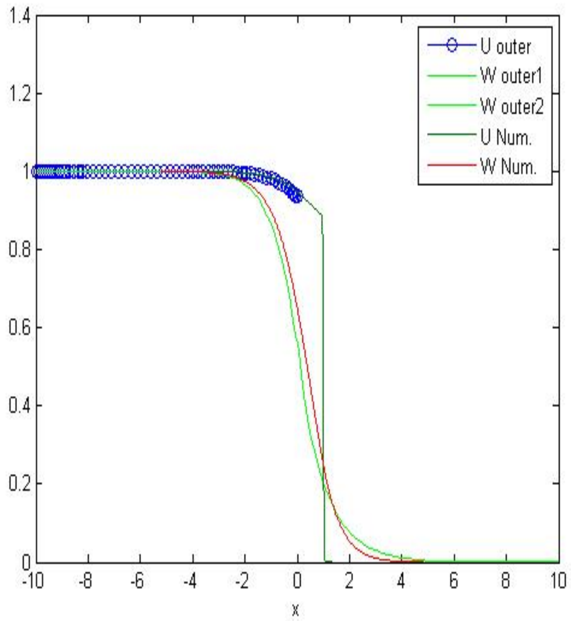

Fig. 9: Comparison between the wave of the asymptotic and numerical solutions for different values of (I_c), when $\alpha_1 = 0.5$, $\beta_1 = 2$, $\alpha_2 = 0.5$, $\beta_2 = 3$, $\lambda = 0.05$ and $D = 1$.

6. Conclusion

It can be seen that three types of traveling wave solutions can be developed in the RDS (2). There is only one regular type of solutions and two types of singular perturbation solutions can be found from asymptotic methods. A semi-implicit method is used as a numerical method for solving (2) and we find the traveling wave solutions. The traveling wave solutions are always connecting a stable coexistence equilibrium solution namely (1, 1) to another equilibrium solution. The comparison between numerical and asymptotic methods shows a good agreement.

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