EXISTENCE AND UNIQUENESS OF THE NON-POLYNOMIAL SPLINE SPACES WITH MAXIMAL SMOOTHNESS

Yu.K.Dem'yanovich
St. Petersburg State University
7/9 Universitetskaya nab., St.Petrsburg, 199034 Russia
y.demjanovich@spbu.ru

Abstract: The aims of the paper are to obtain necessary and sufficient conditions of existence and smoothness for non-polynomial spline spaces of order m, to establish the uniqueness of the B_φ-spline spaces in the class C^{m-1} among mentioned spaces (under condition of fixed grid and fixed generating vector function φ), and to prove the embedding of the B_φ-spline spaces corresponding to embedded grids.

Key–Words: non-polynomial splines, spline spaces, embedding, calibration relations, enlargement spline grid

1 Introduction

Smoothness of splines is important for the recovery of differentiable functions, for the smoothing of discrete data and so on (see [1]- [2], [4], [8]- [9], [14], [17, 20–22]). It is well known (see, for example, [2]) that maximal smoothness of polynomial splines of degree m (of which support of their basic splines consists of $m + 1$ grid interval) is $m - 1$. Maximal smoothness of non-polynomial splines have been discussed in rare cases. In particular, smooth non-polynomial splines were considered in [8]- [9].

Another question is the embedding of the function spaces. It is very important for construction wavelet decomposition (see [3], [5]- [7], [10]- [17]).

There are many difficulties for investigations of the embedding of non-polynomial spline spaces (see [8], [9]). The conditions of embedding have been investigated for some situations (see [14]). Specifically, embedding of spaces of smooth polynomial splines and the corresponding wavelet decompositions on infinite embedded grids were studied in many works (cf., for example, [10]- [15] and the references therein). For simplicity, it is important to investigate the embedding of spline spaces in the case of embedding of corresponding spline grids. Such grids can be enlarged by removing grid points one at a time (see [16]- [19]). Such considerations are based on approximate relations, owing to which it is possible to obtain wavelet decompositions of spline spaces with different smoothness and such that their approximate properties are asymptotically optimal with respect to the N-width of standard compact sets. The original numerical flow is regarded as a sequence of coefficients of decomposition with respect to the coordinate splines in the space constructed on the original (fine) grid (see [19]- [24]). This space is projected onto the embedded spline space (on an enlarged grid). As a result, we get a grid obtained by splitting the original numerical information flow into the basic flow (formed by the coefficients of decomposition relative to the coordinate splines of the embedded space) and the wavelet numerical flow which can be used to restore the original numerical flow.

The aims of the paper are to obtain necessary and sufficient conditions of existence and smoothness for non-polynomial spline spaces of order m, to establish the uniqueness of the B_φ-spline spaces in the class C^{m-1} among mentioned spaces (under condition of fixed grid and fixed generating vector function φ), and to prove the embedding of the B_φ-spline spaces corresponding to embedded grids. In this paper, the approximation relations with an initial grid and with a complete chain of vectors are applied to obtain the minimal spline spaces. Usage of locally orthogonal chains of vectors gives the opportunity to construct special approximation relations from which the initial space of B_φ-splines is constructed. Deletion of a knot from the initial grid gives a new grid, and as result, a new space of B_φ-splines is embedded in the initial space mentioned above. Consequent deletion of the knots (one by one) generates the sequence of the embedded spaces of B_φ-splines. Obtained results are successfully proved. They may be applied to spline-wavelet decompositions of the numerical flows.

2 Spaces $(X, \mathbf{A}, \varphi)$-splines of the order m

We discuss grid $X \stackrel{\text{def}}{=} \{x_j\}_{j\in\mathbb{Z}}$,

$$X: \ \ldots < x_{-1} < x_0 < x_1 < \ldots;$$

$$\alpha \stackrel{\text{def}}{=} \lim_{j\to-\infty} x_j, \ \beta \stackrel{\text{def}}{=} \lim_{j\to+\infty} x_j$$

in the interval (α, β) with finite or infinite values α β.

Let $\varphi(t)$ be $m+1$-component vector function (column vector) with components belonging to $C^m[\alpha, \beta]$ such that the Wronskian can not equal zero:

$$|\det(\varphi, \varphi', \varphi'', \ldots, \varphi^{(m)})(t)| \geq c > 0 \qquad (1)$$

$$\forall t \in [\alpha, \beta].$$

We discuss sequence $\mathbf{A}$ of vector columns $\mathbf{a}_i \in \mathbb{R}^{m+1}$, $\mathbf{A} \stackrel{\text{def}}{=} \{\ldots, \mathbf{a}_{-1}, \mathbf{a}_0, \mathbf{a}_1, \ldots\}$, and matrix $A_j \stackrel{\text{def}}{=} (\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, \ldots, \mathbf{a}_j)$, $j \in \mathbb{Z}$. The sequence with property $\det A_j \neq 0 \forall j \in \mathbb{Z}$ is called *complete chain*.

Let $\mathcal{A}$ be the class $\{A \mid \det A_j \neq 0 \ \forall j \in \mathbb{Z}\}$.

By definition, put $\varphi_j \stackrel{\text{def}}{=} \varphi(x_j)$, $\varphi_j^{(s)} \stackrel{\text{def}}{=} \varphi^{(s)}(x_j)$, $\quad s = 1, 2, \ldots, m-1$,

$$h_X \stackrel{\text{def}}{=} \sup_{j\in\mathbb{Z}}(x_{j+1} - x_j), \quad M \stackrel{\text{def}}{=} \cup_{j\in\mathbb{Z}}(x_j, x_{j+1}),$$

$$S_j \stackrel{\text{def}}{=} [x_j, x_{j+m+1}], \quad J_k \stackrel{\text{def}}{=} \{k-m, \ldots, k\}.$$

If $\mathbf{A} \in \mathcal{A}$, then the approximation relations

$$\sum_{j'=i-m}^{i} \mathbf{a}_{j'} \omega_{j'}(t) \equiv \varphi(t) \qquad (2)$$

$$\forall t \in (x_i, x_{i+1}) \ \forall i \in \mathbb{Z}, \ \operatorname{supp} \omega_j \subset S_j \ \forall j \in \mathbb{Z},$$

define the functions $\omega_j(t)$, $t \in M$, $j \in \mathbb{Z}$, uniquely.

Because of (1) the functions ω_j, $j \in \mathbb{Z}$, are a linearly independent system. Consider the linear space $\mathbb{S}_m(X, \mathbf{A}, \varphi)$ of minimal $(X, \mathbf{A}, \varphi)$-splines of the order m, defined by relation

$$\mathbb{S}_m(X, \mathbf{A}, \varphi) \stackrel{\text{def}}{=} Cl_p \mathcal{L}\{\omega_j\}_{j\in\mathbb{Z}};$$

here $\mathcal{L}$ is the linear hull, and Cl_p is closure in the point topology.

3 Differentiability of the minimal splines

Theorem 1. *Suppose $\varphi^{(s)} \in C(\alpha, \beta)$, where s is a positive integer. It is a necessary and sufficient condition for the functions $\omega_j^{(s)}(t)$ ($\forall j \in \mathbb{Z}$, $t \in M$) to be prolonged to continuous functions on the interval (α, β) is that the relations*

$$\det\Big(\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, \ldots, \mathbf{a}_{j-1}, \varphi_j^{(s)}\Big) = 0 \quad \forall j \in \mathbb{Z}$$

be fulfilled.

Discuss a case of relations (2) selecting the vector chain $\mathbf{a}_j$ in a special way. By definition, put

$$\mathbf{d}_j^T \mathbf{x} \equiv \det\Big(\varphi_j, \varphi'_j, \varphi''_j, \ldots, \varphi_j^{(m-1)}, \mathbf{x}\Big) \quad \forall \mathbf{x} \in \mathbb{R}^{m+1}.$$

We introduce vectors $\mathbf{a}_j^*$ with symbolic determinant

$$\mathbf{a}_j^* \stackrel{\text{def}}{=} \det \begin{pmatrix} \varphi_{j+1} & \varphi'_{j+1} & \cdots & \varphi_{j+1}^{(m-1)} \\ \mathbf{d}_{j+2}^T \varphi_{j+1} & \mathbf{d}_{j+2}^T \varphi'_{j+1} & \cdots & \mathbf{d}_{j+2}^T \varphi_{j+1}^{(m-1)} \\ \mathbf{d}_{j+3}^T \varphi_{j+1} & \mathbf{d}_{j+3}^T \varphi'_{j+1} & \cdots & \mathbf{d}_{j+3}^T \varphi_{j+1}^{(m-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{d}_{j+m}^T \varphi_{j+1} & \mathbf{d}_{j+m}^T \varphi'_{j+1} & \cdots & \mathbf{d}_{j+m}^T \varphi_{j+1}^{(m-1)} \end{pmatrix}$$

and by $\mathbf{A}^*$ we denote $\{\mathbf{a}_j^*\}_{j\in\mathbb{Z}}$.

Theorem 2. *If the condition is true, then a number $\delta = \delta_\varphi > 0$ exists such that under condition $h_X < \delta$ the relation $\mathbf{A}^* \in \mathcal{A}$ is fulfilled.*

Suppose $\mathbf{A}^* \in \mathcal{A}$; now by ω_j^* denote splines obtained with (2), where $\mathbf{a}_j = \mathbf{a}_j^*$.

Theorem 3. *The formulas $\omega_j^* \in C^{m-1}(\alpha, \beta)$ $\forall j \in \mathbb{Z}$ are right.*

Corollary 1. *A linear space*

$$\mathbb{S}_m^*(X, \varphi) \stackrel{\text{def}}{=} Cl_p \, \mathcal{L}\{\omega_j^*\}_{j\in\mathbb{Z}}$$

belongs to $C^{m-1}(\alpha, \beta)$.

The space $\mathbb{S}_m^*(X, \varphi)$ is called the space of B_φ-splines.

Let $\mathfrak{S}_m(X, \varphi)$ be the set of spline spaces of the order m, that is

$$\mathfrak{S}_m(X, \varphi) \stackrel{\text{def}}{=} \{\mathbb{S}_m(X, \mathbf{A}, \varphi) \mid \forall \mathbf{A} \in \mathcal{A}\}.$$

Theorem 4. *In the set $\mathfrak{S}_m(X, \varphi)$ there exists the unique space of the class $C^{m-1}(\alpha, \beta)$; that is the space $\mathbb{S}_m^*(X, \varphi)$.*

Theorem 5. *Each function $\omega_j^*(t)$ is defined by values of the vector function $\varphi(t)$ on the set $\operatorname{supp} \omega_j^*$.*

4 Calibration relations

Here we discuss an enlargement of the grid X by removal of knot x_{k+1}: we put

$$\widetilde{x}_j = x_j \quad \text{for } j \le k, \qquad \widetilde{x}_j = x_{j+1} \quad \text{for } j \ge k+1.$$

Thus $\widetilde{X} \stackrel{\text{def}}{=} \{\widetilde{x}_j\}_{j\in\mathbb{Z}}$,

$$\widetilde{X}: \ \ldots < \widetilde{x}_{-1} < \widetilde{x}_0 < \widetilde{x}_1 < \ldots.$$

Let us denote

$$\widetilde{\varphi}_j \stackrel{\text{def}}{=} \varphi(\widetilde{x}_j), \quad \widetilde{\varphi}_j^{(s)} \stackrel{\text{def}}{=} \varphi^{(s)}(\widetilde{x}_j),$$

$$\widetilde{\mathbf{d}}_j^T \mathbf{x} \equiv \det\Big(\widetilde{\varphi}_j, \widetilde{\varphi}'_j, \widetilde{\varphi}''_j, \ldots, \widetilde{\varphi}_j^{(m-1)}, \mathbf{x}\Big) \quad \forall \mathbf{x} \in \mathbb{R}^{m+1}.$$

Now we introduce the vectors with symbolic determinant

$$\widetilde{\mathbf{a}}_j^* \stackrel{\text{def}}{=} \det \begin{pmatrix} \widetilde{\varphi}_{j+1} & \widetilde{\varphi}'_{j+1} & \ldots & \widetilde{\varphi}_{j+1}^{(m-1)} \\ \widetilde{\mathbf{d}}_{j+2}^T \widetilde{\varphi}_{j+1} & \widetilde{\mathbf{d}}_{j+2}^T \widetilde{\varphi}'_{j+1} & \ldots & \widetilde{\mathbf{d}}_{j+2}^T \widetilde{\varphi}_{j+1}^{(m-1)} \\ \widetilde{\mathbf{d}}_{j+3}^T \widetilde{\varphi}_{j+1} & \widetilde{\mathbf{d}}_{j+3}^T \widetilde{\varphi}'_{j+1} & \ldots & \widetilde{\mathbf{d}}_{j+3}^T \widetilde{\varphi}_{j+1}^{(m-1)} \\ \ldots & \ldots & \ldots & \ldots \\ \widetilde{\mathbf{d}}_{j+m}^T \widetilde{\varphi}_{j+1} & \widetilde{\mathbf{d}}_{j+m}^T \widetilde{\varphi}'_{j+1} & \ldots & \widetilde{\mathbf{d}}_{j+m}^T \widetilde{\varphi}_{j+1}^{(m-1)} \end{pmatrix}.$$

Consider the approximation relations

$$\sum_{j'=i-m}^{i} \widetilde{\mathbf{a}}_{j'}^* \, \widetilde{\omega}_{j'}^*(t) \equiv \varphi(t) \quad \forall t \in (\widetilde{x}_i, \widetilde{x}_{i+1}) \ \forall i \in \mathbb{Z},$$

$$\operatorname{supp} \widetilde{\omega}_j^* = [\widetilde{x}_j, \widetilde{x}_{j+m+1}] \ \forall j \in \mathbb{Z}.$$

Discuss the space of B_φ-splines according to the new grid

$$\mathbb{S}_m^*(\widetilde{X}, \varphi) \stackrel{\text{def}}{=} Cl_p \, \mathcal{L}\{\widetilde{\omega}_j^*\}_{j\in\mathbb{Z}}.$$

Theorem 6. *If the grid $\widetilde{X}$ is so fine that the chain $\widetilde{\mathbf{A}}^* \stackrel{\text{def}}{=} \{\widetilde{\mathbf{a}}_j^*\}_{j\in\mathbb{Z}}$ belongs to $\mathcal{A}$, then the space of B_φ-splines constructed for the grid X contains the space of B_φ-splines for the grid $\widetilde{X}$:*

$$\mathbb{S}_m^*(\widetilde{X}, \varphi) \subset \mathbb{S}_m^*(X, \varphi).$$

The calibration relations can be presented in the next form

$$\widetilde{\mathbf{a}}_{k-m}^* \widetilde{\omega}_{k-m}^*(t) + \widetilde{\mathbf{a}}_{k-m+1}^* \widetilde{\omega}_{k-m+1}^*(t) + \ldots + \widetilde{\mathbf{a}}_k^* \widetilde{\omega}_k^*(t) \equiv$$

$$\equiv \mathbf{a}_{k-m}^* \omega_{k-m}^*(t) + \mathbf{a}_{k-m+1}^* \omega_{k-m+1}^*(t) + \ldots +$$
$$+ \mathbf{a}_k^* \omega_k^*(t) + \mathbf{a}_{k+1}^* \omega_{k+1}^*(t) \quad \forall t \in (\alpha, \beta).$$

Corollary 2. *Under conditions of Theorem 2 for arbitrary grid X' with properties $X' \subset X$, $h_{X'} \le \delta_\varphi$, the inclusion*

$$\mathbb{S}_m^*(X', \varphi) \subset \mathbb{S}_m^*(X, \varphi)$$

is correct.

Acknowledgements

This research was partly supported by Grant of RFFI No. 15-01-008847.

References:

[1] J.J.Goel. Construction of basis functions for numerical utilization of Ritzs method. Numer. Math. 12, 435447 (1968).

[2] L.L.Schumaker. On super splines and finite elements. SIAM J. Numer. Anal. 26, 9971005 (1989).

[3] S.Mallat. Multiresolution approximation and wavelets. Trans. Amer. Math. Soc., Vol.315, 69-88 (1989).

[4] S.G.Mikhlin. Error analysis in numerical processes. Chichester; New York: Wiley (1991).

[5] Y.Meyer. Wavelets and operators. Cambridge Universty Press, Cambridge (1992).

[6] I.Daubeshies. Ten lectures on wavelets. CBMS-NSR Series in Appl. Math., SIAM (1992).

[7] C.K.Chui. An Introduction to Wavelets, Academic Press (1992).

[8] M.D.Buhmann. Multiquadric prewavelets on nonequally spaced knots in one dimension. Math. Comput. 64(212), 16111625 (1995).

[9] W.Lawton, S.L.Lee, Z.Chen. Characterization of compactly supported refinable splines. Adv. Comput. Math., Vol.3, No.1-2, 137-145 (1995).

[10] G.Gripenberg. A necessary and sufficient condition for the existence of father wavelet. Studia Mathematica. Vol.114, 3, 207-226 (1995).

[11] I.Ya.Novikov, S.B.Stechkin, Basic wavelet theory. Usp. Mat. Nauk 53 (1998), No. 6, 53128 (Russian); English transl.: Russ. Math. Surv. 53 (1998), No. 6, 11591231.

[12] Yu. K. Demyanovich. Wavelet expansions in spline spaces on an irregular grid, Dokl. Akad. Nauk, Ross. Akad. Nauk 382 (2002), No. 3, 313316 (in Russian); English transl.: Dokl. Math. 65 (2002), No. 1, 4750.

[13] I.Ya.Novikov, V.Yu.Protasov, M.A.Scopina. Wavelet theory. AMS, v.239 (2011).

[14] Yu.K.Demyanovich. Embedded spaces of trigonometric splines and their wavelet expansion. Math Notes 78(5), 615630 (2005).

[15] Yu.K.Demyanovich. Wavelet decompositions on nonuniform grid. In: N.N.Uraltseva (ed.) Proceedings of the St. Petersburg Mathematical Society XIII, 2007. American Mathematical Society Translations, series 2, vol. 222, pp. 2343. American Mathematical Society, RI, USA (2008).

[16] Yu.K.Demyanovich. Minimal splines and wavelets, Vestn. St.-Petersbg. Univ., Ser. I, No. 2, 822 (2008) (in Russian); English transl.: Vestn. St.-Petersbg. Univ., Math. 41, No. 2, 88101 (2008).

[17] Yu.K.Demyanovich, Minimal splines of Lagrange type, Probl. Mat. Anal. 50, 2164 (2010) (in Russian); English transl.: J. Math. Sci., New York 170 No. 4, 444495 (2010) .

[18] Yu.K.Demyanovich, O.M.Kosogorov. Calibration relations for non-polynomial splines. J. Math Sci. 164(3), 364382 (2010).

[19] Yu.K.Dem'yanovich, I.D.Miroshnichenko. Calibration relations to splines of the fourth order. J.Math.Sci. Vol.178, No.6. (2011). P.576-588.

[20] Burova Irina. On Integro- Differential Splines Construction. *Advances in Applied and Pure Mathematics. Proceedinngs of the 7-th International Conference on Finite Differences, Finite Elements, Finite Volumes, Boundary Elements (F-and-B'14)*. Gdansk. Poland. May 15–17, 2014, pp.57–61.

[21] I. G. Burova, On left integro-differential splines and Cauchy problem, International Journal of Mathematical Models and Methods in Applied Sciences, 9, 683-690, 2015.

[22] I.G. Burova, A.G.Doronina, Nonpolynomial Integro-Differential Splines. Applied Mathematical Sciences, Vol. 10, 735 - 745, 2016.

[23] Yu.K.Dem'yanovich, I.G.Burova. On Adaptive Processing of Discrete Flow. WSEAS TRANSACTIONS on MATHEMATICS, Vol.14, 226-236 (2015).

[24] Yu.K.Dem'yanovich. Adaptive properties of Hermite Splines. The 1st IFAC Conference on Modelling. Identification and Control of Nonlinear Systems. Final Program and Book of Abstracts(MICNON 2015, Saint-Petersburg, June,24-26).