Suppose that $supp\,\omega_{k,0} = [x_{k-1}, x_{k+2}]$, $supp\,\omega_{k,1} = [x_{k-1}, x_{k+1}]$. It is easy to see that $\omega_{k,0}, \omega_{s,1}, \in C^1(R^1)$, $k = j-1, j, j+1$, $s = j, j+1$.

We have for $x = x_j + th$, $t \in [0,1]$, the next formulae:

$$\omega_{j,0}(x_j + th) = (-1+t)^2(t+1)^2, \quad (12)$$

$$\omega_{j+1,0}(x_j + th) = -(1/4)t^2(t+1)(5t-7), \quad (13)$$

$$\omega_{j,1}(x_j + th) = th(t+1)(-1+t)^2, \quad (14)$$

$$\omega_{j+1,1}(x_j + th) = (1/2)ht^2(-1+t)(t+1), \quad (15)$$

$$\omega_{j-1,0}(x_j + th) = (1/4)t^2(-1+t)^2. \quad (16)$$

Table 1 shows the errors $R^I = \max_{x\in[a,b]} |\tilde{u} - u|$, $R^{II} = \max_{x\in[a,b]} |\tilde{u}^H - u|$ when $[a,b] = [-1,1]$, $h = 0.1$. Calculations were done in Maple, Digits=15.

Table 1.

$u(x)$	R^I	R^{II}
x^4	0.0	0.0
$1/((1+25x^2)$	$0.1417e-2$	$0.1531e-2$
$\sin(5x) - \cos(5x)$	$0.2913e-4$	$0.3466e-4$

4 About approximations with two variables

Suppose that n, m are natural numbers, while a, b, c, d are real numbers, $h_x = (b-a)/n$, $h_y = (d-c)/m$. Let us build the grid of interpolation nodes $x_j = a + jh_x$, $j = 0,1,\ldots,n$, $y_k = c + kh_y$, $k = 0,1,\ldots,m$. On every line parallel to axis y, we can construct the approximation in the form:

$$\widetilde{u}(y) = u(y_k)\omega_{k,0}(y) + u(y_{k+1})\omega_{k+1,0}(y) +$$
$$u'(y_k)\omega_{k,1}(y) + u'(y_{k+1})\omega_{k+1,1}(y) +$$
$$+V_k\,\omega_k^{<0>}(y), \;\; y \in [y_k, y_{k+1}]. \quad (17)$$

Now the formulae for $\omega_{k,0}(y)$, $\omega_{k+1,0}(y)$, $\omega_{k,1}(y)$, $\omega_{k+1,1}(y)$, $\omega_k^{<0>}(y)$ are similar to the previous ones.

If $(x,y) \in \Omega_{j,k}$ then we get the next expression using the tensor product:

$$\tilde{u}(x,y) = \sum_{i=0}^{1}\sum_{p=0}^{1} u(x_{j+i}, y_{k+p})\omega_{j+i,0}(x)\omega_{k+p,0}(y) +$$

$$\sum_{i=0}^{1}\sum_{p=0}^{1} u'_y(x_{j+i}, y_{k+p})\omega_{j+i,0}(x)\omega_{k+p,1}(y) +$$

$$\sum_{i=0}^{1} V_{j+i,k}(x)\omega_{j+i,0}(x)\omega_k^{<0>}(y) +$$

$$\sum_{i=0}^{1} V_{j,k+i}\omega_j^{<0>}(x)\omega_{k+i,0}(y) +$$

$$\sum_{i=0}^{1} S_{j,k+i}\omega_j^{<0>}(x)\omega_{k+i,1}(y) +$$

$$W_{j,k}\omega_k^{<0>}(y)\omega_j^{<0>}(x) +$$

$$\sum_{i=0}^{1} u'_x(x_j, y_{k+i})dt\omega_{j,0}(x)\omega_{k+i,0}(y) +$$

$$+\sum_{i=0}^{1} u''_{xy}(x_j, y_{k+i})dt\omega_{j,0}(x)\omega_{k+i,1}(y) +$$

$$\sum_{i=0}^{1} P_{j+i,k}\omega_{j+i,1}(x)\omega_k^{<0>}(y), \quad (18)$$

where

$$V_{j+i,k} = \frac{(y_{k+1}-y_{k-1})}{30}(7u(x_{j+i}, y_{k-1}) +$$
$$7u(x_{j+i}, y_{k+1}) + 16u(x_{j+i}, y_k)) -$$
$$\frac{(y_{k+1}-y_{k-1})^2}{60}(u'_y(x_{j+i}, y_{k+1}) - u'_y(x_{j+i}, y_{k-1})),$$

$$V_{j,k+i} = \frac{(x_{j+1}-x_{j-1})}{30}(7u(x_{j-1}, y_{k+i}) +$$
$$7u(x_{j+1}, y_{k+i}) + 16u(x_j, y_{k+i})) -$$
$$\frac{(x_{j+1}-x_{j-1})^2}{60}(u'_x(x_{j+1}, y_{k+i}) - u'_x(x_{j-1}, y_{k+i})),$$

$$S_{j,k+i} = \frac{(x_{j+1}-x_{j-1})}{30}(7u'_y(x_{j-1}, y_{k+i}) +$$
$$7u'_y(x_{j+1}, y_{k+i}) + 16u_y(x_j, y_{k+i})) -$$
$$\frac{(x_{j+1}-x_{j-1})^2}{60}(u''_{xy}(x_{j+1}, y_{k+i}) - u''_{xy}(x_{j-1}, y_{k+i})),$$

$$P_{j+i,k} = \frac{(y_{k+1}-y_{k-1})}{30}(7u'_x(x_{j+i}, y_{k-1}) +$$
$$7u'_x(x_{j+i}, y_{k+1}) + 16u'_x(x_{j+i}, y_k)) -$$
$$\frac{(y_{k+1}-y_{k-1})^2}{60}(u''_{yx}(x_{j+i}, y_{k+1}) - u''_{yx}(x_{j+i}, y_{k-1})),$$

$$W_{jk} = \frac{(y_{k+1}-y_{k-1})}{30}(7G(x_j, y_{k-1}) +$$
$$7G(x_j, y_{k+1}) + 16G(x_j, y_k)) -$$
$$\frac{(y_{k+1}-y_{k-1})^2}{60}(G'y(x_j, y_{k+1}) - G'y(x_j, y_{k-1})),$$

$$G(x_j, y) = \frac{(x_{j+1}-x_{j-1})}{30}(7u(x_{j-1}, y) +$$

$$7u(x_{j+1},y)+16u(x_j,y))-$$

$$\frac{(x_{j+1}-x_{j-1})^2}{60}(u'_x(x_{j+1},y)-u'_x(x_{j-1},y)).$$

Graphs 4, 5 shows approximations and the errors of approximations $\tilde{u}(x,y)-u(x,y)$ by (18), (3)–(7), (12)–(16) of functions $u_1(x,y)=\sin(3x-3y)\cos(3x-3y)$, $u_2(x,y)=(x-y)^2(x+y)^2$, when $[a,b]=[-1,1]$, $[c,d]=[-1,1]$, $h_x=h_y=h=0.2$. Calculations were done in Maple, Digits=15.

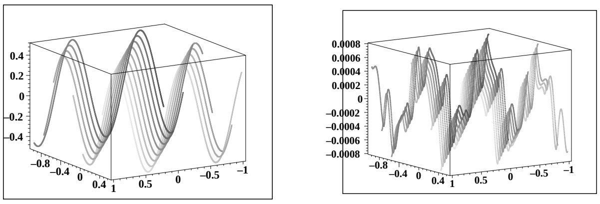

Figure 4: Plots of the functions: $\tilde{u}(x,y)=\sin(3x-3y)\cos(3x-3y)$ (left) and $\tilde{u}(x,y)-u(x,y)$ (right) when $h=0.2$, $[-1,1]\times[-1,1]$

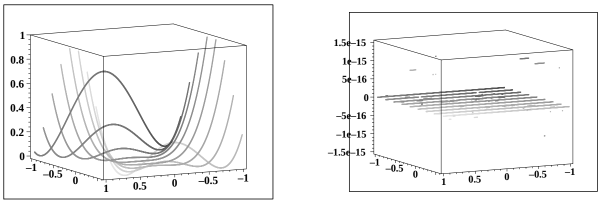

Figure 5: Plots of the functions: $\tilde{u}(x,y)=(x-y)^2(x+y)^2$ (left) and $\tilde{u}(x,y)-u(x,y)$ (right) when $h=0.2$, $[-1,1]\times[-1,1]$

5 Conclusion

Basic spines can be applied for solving various mathematical problems. We can obtain the formulae of our basic splines in the following way. In the interval $[x_{j-1},x_j]$ we obtain basic splines from the system:

$$\widetilde{u}(x)\equiv u(x),\;\; u(x)=x^{i-1}, i=1,2,3,4,5,$$

where

$$\widetilde{u}(x)=u(x_{j-1})\omega_{j-1,0}(x)+u(x_j)\omega_{j,0}(x)+$$

$$u'(x_{j-1})\omega_{j-1,1}(x)+u'(x_j)\omega_{j,1}(x)+V_{j-1}\,\omega_{j-1}^{<0>}(x).$$

If we take the basic splines with the same numbers from $[x_{j-1},x_j]$ and $[x_j,x_{j+1}]$ then we have:

$$\omega_{j,0}(x_j+th)=\begin{cases}-\frac{15}{8}t^4-\frac{23}{4}t^3-\frac{39}{8}t^2+1, t\in[-1,0],\\ 2t^3-3t^2+1, \qquad t\in[0,1],\\ 0, \qquad t\notin[-1,1],\end{cases}$$

$$\omega_{j,1}(x_j+th)=\begin{cases}\frac{5h}{8}t^4+\frac{9h}{4}t^3+\frac{21h}{8}t^2+th,\\ \qquad t\in[-1,0],\\ \frac{5h}{4}t^4-\frac{3h}{2}t^3-\frac{3h}{4}t^2+th, t\in[0,1],\\ 0, \qquad t\notin[-1,1],\end{cases}$$

$$\omega_j^{<0>}(x_j+th)=\begin{cases}\frac{15}{16h}t^2(t-1)^2,\\ \qquad t\in[0,1]\\ 0, \qquad t\notin[0,1].\end{cases}$$

Figure 6 shows the plots of the basic splines $\omega_{j,0}$, $\omega_{j,1}$. The plot of the basic spline $\omega_j^{<0>}$ is shown in Figure 3.

The construction of the nonpolynomial splines with the same properties and their application for the solving of different problems will be regarded in further papers.

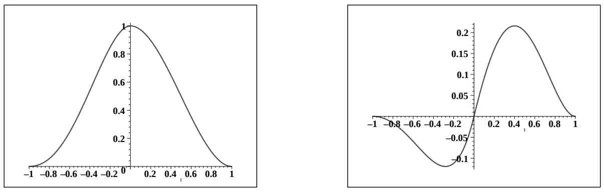

Figure 6: Plots of the basic functions: $\omega_{j,0}(jh+th)$ (left), and $\omega_{j,1}(jh+th)$, when $h=1$ (right)

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