Iteratively computation the Nash equilibrium points in the two-player positive games

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Abstract: We consider the linear quadratic differential games for positive linear systems with the feedback information structure and two players. The accelerated Newton method to obtain the stabilizing solution of a corresponding set of Riccati equations is presented in [6], where the convergence properties are established. In addition, the Lyapunov iterative method to compute the Nash equilibrium point is presented in [7] (5th International Conference Applied and Computational Mathematics, WSEAS Conference at Mallorca, 2016). Based on these two methods we derive a new one - the accelerated Lyapunov method. Moreover, the convergence properties are proved. The performances of the proposed algorithm are illustrated on some numerical examples.

Key–Words: feedback Nash equilibrium, generalized Riccati equation, stabilizing solution, nonnegative solution.

1 Introduction

Recently, the theory of linear quadratic games and their application in economics and management science has focused the interests of many researchers [2, 3, 4]. Applications for positive systems can be found in [8]. Our investigation is motivated from the papers [1, 6], where the problem of finding a deterministic feedback Nash equilibrium for a two player infinite-horizon linear-quadratic differential game is studied. This equilibrium is defined as a pair of linear time-invariant state feedback strategies stabilizing the closed-loop system. However, the considered game is studied on positive systems and players's strategies are presented via the stabilizing solution of the associated coupled set of Riccati equations.

We introduce the following set of Riccati equations:

$$\begin{aligned} 0 &= \mathcal{R}_1(X_1, X_2) \\ 0 &= \mathcal{R}_2(X_1, X_2) \end{aligned} \tag{1}$$

where

$$\begin{aligned} \mathcal{R}_1(X_1, X_2) &:= -A^T X_1 - X_1 A - Q_1 \\ &\quad + X_1 S_1 X_1 - X_2 S_{12} X_2 \\ &\quad + X_1 S_2 X_2 + X_2 S_2 X_1 \,, \\ \mathcal{R}_2(X_1, X_2) &:= -A^T X_2 - X_2 A - Q_2 \\ &\quad + X_2 S_2 X_2 - X_1 S_{21} X_1 \\ &\quad + X_2 S_1 X_1 + X_1 S_1 X_2 \,, \end{aligned} \tag{2}$$

with $A, Q_1, Q_2 \in \mathcal{R}^{n\times n}$ and Q_1, Q_2 are symmetric nonnegative matrices, and $-A$ is a Z-matrix.

The accelerated Newton method , introduced in [6] , is given by the following set of recursive equations:

$$\begin{aligned} & -A^{(k)^T} X_1^{(k+1)} - X_1^{(k+1)} A^{(k)} \\ & +W_{12}^{(k)} X_2^{(k)} + X_2^{(k)} W_{12}^{(k)^T} = Q_1^{(k)} \,, \\ & -A^{(k)^T} X_2^{(k+1)} - X_2^{(k+1)} A^{(k)} \\ & W_{21}^{(k)} X_1^{(k+1)} + X_1^{(k+1)} W_{21}^{(k)^T} = Q_2^{(k)} \,, \end{aligned} \tag{3}$$

where

$$
\begin{aligned}
&A^{(k)} = A - S_1\, X_1^{(k)} - S_2\, X_2^{(k)}\,, \\
&W_{ij}^{(k)} = X_i^{(k)} S_j - X_j^{(k)} S_{ij}\,, \\
&i,j = 1,2; i \neq j\,, \\
&Q_i^{(k)} = Q_i + X_i^{(k)} S_i X_i^{(k)} - \textstyle\sum_{j\neq i} X_j^{(k)} S_{ij} X_j^{(k)} \\
&+\textstyle\sum_{j\neq i}[\, X_i^{(k)} S_j X_j^{(k)} + X_j^{(k)} S_j X_i^{(k)}\,]\,.
\end{aligned} \tag{4}
$$

The application of Newton iteration, investigated in [1, 6], involves solving of a high dimensional linear system at each iteration step. However, the accelerated Newton constructs sequences of Lyapunov algebraic equations, which solutions define matrix sequences converge to the stabilizing solution to set of Riccati equations (1) (Theorem 2.5 [6]). Hence, the execution of iteration (3) is required to solve two linear matrix Laypunov equations in each step.

The Lyapunov iterative method is considered in [7] and it is given by iteration equations:

$$
\begin{aligned}
-A^{(k)^T} X_1^{(k+1)} - X_1^{(k+1)}\, A^{(k)} = \tilde{Q}_1^{(k)} \\
-A^{(k)^T} X_2^{(k+1)} - X_2^{(k+1)}\, A^{(k)} = \tilde{Q}_2^{(k)}\,,
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
\tilde{Q}_1^{(k)} = Q_1 + X_1^{(k)} S_1 X_1^{(k)} + X_2^{(k)} S_{12} X_2^{(k)} \\
\tilde{Q}_2^{(k)} = Q_2 + X_2^{(k)} S_2 X_2^{(k)} + X_1^{(k)} S_{21} X_1^{(k)}\,.
\end{aligned} \tag{6}
$$

In this paper we will improve the Lyapunov iteration. We introduce a new iterative method for computing the nonnegative stabilizing solution to (1), where two sequences of Lyapunov algebraic equations are constructed. Numerical examples have been introduced so as to demonstrate the effectiveness of the proposed algorithms. We compare two iterative methods on some numerical examples.

2 The accelerated Lyapunov method

We consider the matrix functions $\mathcal{R}_1(X_1, X_2)$ and $\mathcal{R}_2(X_1, X_2)$ introduced in (1). The following properties for $\mathcal{R}_i(X_1, X_2), i = 1, 2$ are very important in our investigation. We will present them without proofs.

Lemma 1 *For the matrix function $\mathcal{R}_i(X_1, X_2), i = 1, 2$ the following identities hold:*

(i)
$$
\begin{aligned}
\mathcal{R}_i(X_1, X_2) \;=\; & {A_X}^T X_i - X_i\, A_X \\
& -Q_i - X_i\, S_i\, X_i - \textstyle\sum_{j\neq i} X_j\, S_{ij}\, X_j\,,
\end{aligned} \tag{7}
$$

with $A_X = A - S_1\, X_1 - S_2\, X_2$, and

(ii)
$$
\begin{aligned}
&\mathcal{R}_i(X_1, X_2) = \mathcal{R}_i(Z_1, Z_2, X_1, X_2) := \\
&-Q_i - Z_i\, S_i\, Z_i + (X_i - Z_i) S_i (X_i - Z_i) \\
&+\textstyle\sum_{j\neq i}[(X_j - Z_j)\, S_j\, X_i + X_i\, S_j (X_j - Z_j)] \\
&-{A_Z}^T X_i - X_i\, A_Z - \textstyle\sum_{j\neq i} X_j\, S_{ij}\, X_j\,,
\end{aligned} \tag{8}
$$

where $A_Z = A - S_1\, Z_1 - S_2\, Z_2$ and $Z_i = Z_i^T$, $i = 1, 2$.

Proof: The statements of Lemma 1 are verified by direct manipulations. □

We denote $\mathcal{R}_i(\mathbf{Z}, \mathbf{X})$ the presentation of $\mathcal{R}_i(\mathbf{X})$ through a symmetric matrix $\mathbf{Z}$.

In Theorem 2.5 [6] the convergence properties of the accelerated Newton method for a two-player differential game are proved. It is typical that the information structure of each player is of a feedback patten. In order to improve the Lyapunov method (5) - (6) we change $\tilde{Q}_2^{(k)}$, i.e. we put $X_1^{(k+1)}$ instead of $X_1^{(k)}$ in the right hand of the second equation of (5) and obtain $\tilde{\tilde{Q}}_2^{(k)}$.

We obtain the accelerated Lyapunov iterative method:

$$
\begin{aligned}
-A^{(k)^T} X_1^{(k+1)} - X_1^{(k+1)}\, A^{(k)} = \tilde{Q}_1^{(k)} \\
-A^{(k)^T} X_2^{(k+1)} - X_2^{(k+1)}\, A^{(k)} = \tilde{\tilde{Q}}_2^{(k)}\,,
\end{aligned} \tag{9}
$$

where

$$
\begin{aligned}
\tilde{\tilde{Q}}_2^{(k)} &= Q_2 + X_2^{(k)} S_2 X_2^{(k)} \\
&+ X_1^{(k+1)} S_{21} X_1^{(k+1)}\,.
\end{aligned} \tag{10}
$$

In our investigation we exploit the fact that the following statements are equivalent for a Z-matrix (-A):

(a) $-A$ is a nonsingular M-matrix;

(b) $I_n \otimes (-A^T) + (-A^T) \otimes I_n$ is a nonsingular M-matrix;

(c) A is asymptotically stable.

The convergence properties of the accelerated Lyapunov iteration (9) are established in the following theorem:

Theorem 2 *Assume there exist symmetric nonnegative matrices $\hat{X}_1, \hat{X}_2$ and $X_1^{(0)} = 0, X_2^{(0)} = 0$ such that $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \geq 0$, and $-A$ is a nonsingular M-matrix. Then, the matrix sequences $\{X_1^{(k)}, X_2^{(k)}\}_{k=0}^{\infty}$ defined by (9) satisfies:*

(i) *$\hat{X}_i \geq X_i^{(k+1)} \geq X_i^{(k)}$ and $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \leq 0$ for $i = 1, 2\,,\ k = 0, 1, \ldots$;*

(ii) *The matrix* $-A^{(k)}$ *is an M-matrix for* $k = 0, 1, \ldots$*;*

(iii) *The matrix sequences* $\{X_1^{(k)}, X_2^{(k)}\}_{k=0}^{\infty}$ *converge to the nonpositive solution* $\tilde{X}_1, \tilde{X}_2$ *to the set of Riccati equations (1) with* $\tilde{X}_i \leq \hat{X}_i$ *and the matrix* $\tilde{A}$ *is asymptotically stable.*

Proof: The theorem is proved following the proof of theorem 2 in [7]. □

Note that the accelerated Lyapunov method preserves the convergence properties of the Lyapunov method (5).

3 Numerical examples

We carry out some numerical experiments for computing the stabilizing solution to the set of generalized Riccati equations (1). The accelerated Newton method (ANM) (3) and accelerated Lyapunov method (ALM) (9) are applied and compared on some examples.

We consider a two-player game where the matrix coefficients: A, B_i, Q_i and R_{ij} for $i, j = 1, 2$ are the following. We define them using the Matlab description.

B_1=full(abs(sprandn(n,4,0.7))/10);

B_2=full(abs(sprandn(n,3,0.7))/10);

R_{11} = [-400 0 0 -40; 0 -150 0 0; 0 0 -300 0; -40 0 0 -300];

R_{22} = [-90 0 0; 0 -120 -5; 0 -5 -120];

R_{12} = [220 190 190; 190 180 22; 190 22 190];

R_{21} = [100 88 0 99; 88 250 190 0; 0 190 240 130; 99 0 130 300];

Q_1=0.375*eye(n,n); Q_1(1,n)=0.45; Q_1(n,1)=0.45;

Q_2=0.285*eye(n,n); Q_2(1,n)=1.5; Q_2(n,1)=1.5;

Test 1: A=(abs(rand(n,n))/1-7*eye(n,n))/10;

Test 2: A=(abs(rand(n,n))/1-15*eye(n,n))/10;

The latter example is executed for different values of n, also 100 runs are completed for each values of n.

The latter tests are executed Test 1 for n=9,10,11,12; Test 2 for n=27. All tests are executed for 100 runs. We take $X_1^{(0)} = X_2^{(0)} = 0$ and thus $\mathcal{R}_i(\mathbf{X}^{(0)}) = -Q_i \leq 0$ (i.e. the matrix is nonpositive). Regrading the outlined choice, we might note that the conditions of theorem 2 are fulfilled, i.e. $\mathbf{X}^{(0)} \leq \hat{\mathbf{X}}$, $\mathcal{R}_i(\mathbf{X}^{(0)}) \leq 0$ and $\mathcal{R}_i(\hat{\mathbf{X}}) \geq 0$, $i = 1, 2$.

On the basis of the experiments, performed for $n = 12$, the following summary of results might be

Table 1: Results from 100 runs for each value of n.

	ANM (3)			ALM (9)		
n	It_M	It_S	CPU	It_M	It_S	CPU
Test 1						
9	6	5.1	0.3s	6	5.1	0.3s
10	7	5.4	0.4s	7	5.4	0.4s
11	9	6.2	0.5s	9	6.2	0.4s
12	45	8.3	0.63s	45	8.3	0.45s
Test 2						
25	6	5.7	1.61s	6	5.7	1.47s
26	8	6.3	1.82s	8	6.3	1.49s
27	10	7.2	2.07s	10	7.2	1.95s

outlined (see Table 1). The average number of iteration steps is 8.3 for both methods - accelerated Newton method and accelerated Lyapunov method. However, the CPU time is 0.63s and and 0.45s respectively for executing the Newton iteration with 100 runs and the Lyapunov iteration with 100 runs. And in the case of Test 2 for n=27 the CPU time is 2.07s for ANM and 1.95s for ALM.

4 Conclusion

We study a new modification of Lyapunov iterative process for finding the nonnegative stabilizing solution to a set of Riccati equations (1). The convergence properties of the Lyapunov method is derived in Theorem 2. Numerical experiments are carried out and the obtained results are used to compare two accelerated methods. The effectiveness of the both accelerated iterative methods are confirmed.

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References:

[1] T. Azevedo-Perdicoulis and G. Jank, Linear Quadratic Nash Games on Positive Linear Systems, *European Journal of Control*, 11, 2005, 1–13.

[2] B. Bazar and G. J. Olsder, Dynamic Noncooperative Game Theory, SIAM, (1999).

[3] S. Jorgensen, and G. Zaccour, Differential Games in Marketing. International Series in Quantitative Marketing, Kluwer Academic Publisher in 2004.

[4] E. Dockner, S. Jorgensen, N. V. Long, G. Sorger, Differential games in economics and management science, Cambridge University Press, (2000).

[5] L. Imsland, I. G.Ivanov and S.Kostova, Linear Quadratic Differential Games and Applications, *Biomath Communications*, 3, 2016, N 2,

[6] I. G.Ivanov, L. Imsland and B. C. Bogdanova, Iterative Algorithms for Computing the Feedback Nash Equilibrium Point for Positive Systems, accepted in *International Journal of System Science*,

[7] I. G.Ivanov and N. Netov, *The Nash Equilibrium Point in the LQ Game on Positive Systems with Two Players*, Mathematical and Computational Methods, 1, 2016, 242–246,

[8] L. Metzler, Stability of multiple markets: the Hicks condition, *Econometrica*, 13(4), 1945, pp. 277–292.