An application of Richardson extrapolation on FEM solutions

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Abstract: The finite element metod (FEM) is widely used numerical method for numerical computation of different physical problems. The sequence of finite element method solutions with increasing the number of finite elements converge to analytical solutions. The idea for implementation of Richardson extrapolation is to get better solutions with less computation and without further increasing of number of finite elements what leads to larger systems of equations. The Richardson extrapolation is applied on finite element method solution in elastostatics. An algorithm for new solution calculated by Richardson method is given. The solutions calculated by applying Richardson extrapolation show more efficiency and accuracy with much less computation than solution with finite element method over more finite elements.

Key–Words: Richardson extrapolation, finite element method, elastostatics

1 Introduction

The finite element method (FEM) is widely used for numerical computation of different physical problems. The method is based on the discretization of the domain on the finite elements. After calculation of elementary stiffness matrices and the implementation of elementary matrices in the stiffness matrix of system (structure, beam, plate) and involving the boundary conditions, the method leads finally to large system of linear equations. The number of unknowns and equations depends on the number and type of finite elements and degrees of freedom at each knot of elements. The sequence of numerical solutions calculated by finite element method converges with discretization on more finite elements but followed with much more computation and much larger system of equations.

Richardson extrapolation, explained in [1], is recursive method that uses former solutions calculated with less computation to calculate improved numerical approximation. The metodology gives improved results with increasing order of accuracy of numerical solutions by using former solutions with less numwrical computations. Linear combination of former solutions would result with improved solution. The coefficients of linear combination depend on the convergence order of sequence of solutions. By combining the results from different mesh sizes, the leading order error terms would be removed. This extrapolation is widely used in numerical integration (Romberg method). Further application to partial differential equations is introduced in [2]. There is also some application in numerical analysis of nonlinear integral equations, as in [3]. In this paper, Richardson extrapolation is introduced to calculate improved solution as linear combination of finite element solutions.

2 Richardson extrapolation applied on numerical integration

We can approximate needed integral, I, with numerical result $I_m(h)$ calculated by numerical integration with step h and order of the convergence of the method k

$$I = I_m(h) + Ch^k \quad , \tag{1}$$

and with numerical result $I_m(h/2)$ calculated with step $h/2$ and same order of convergence

$$I = I_m(h/2) + C\left(\frac{h}{2}\right)^k \quad . \tag{2}$$

With widely used numerical integration algorithm, called Romberg integration, we have now procedure for getting improved numerical solution for given integral as

$$I_{m+1}(h) = \frac{2^k I_m(h/2) - I_m(h)}{2^k - 1} \tag{3}$$

with order of convergence equal to $k+2$. The procedure for improvement of numerical integration solution according Richardson extrapolation is described in Table 1.

Table 1: The procedure for Richardson extrapolation applied on numerical integration

m	$I_m(h)$	$I_m\left(\frac{h}{2}\right)$	$I_m\left(\frac{h}{4}\right)$	$I_m\left(\frac{h}{8}\right)$	$I_m\left(\frac{h}{16}\right)$
1	$I_1(h)$	$I_1\left(\frac{h}{2}\right)$	$I_1\left(\frac{h}{4}\right)$	$I_1\left(\frac{h}{8}\right)$	$I_1\left(\frac{h}{16}\right)$
	$\downarrow$	$\swarrow\downarrow$	$\swarrow\downarrow$	$\swarrow\downarrow$	$\swarrow$
2	$I_2(h)$	$I_2\left(\frac{h}{2}\right)$	$I_2\left(\frac{h}{4}\right)$	$I_2\left(\frac{h}{8}\right)$	
	$\downarrow$	$\swarrow\downarrow$	$\swarrow\downarrow$	$\swarrow$	
3	$I_3(h)$	$I_3\left(\frac{h}{2}\right)$	$I_3\left(\frac{h}{4}\right)$		
	$\downarrow$	$\swarrow\downarrow$	$\swarrow$		
4	$I_4(h)$	$I_4\left(\frac{h}{2}\right)$			
	$\downarrow$	$\swarrow$			
5	$I_5(h)$				

3 Richardson extrapolation applied to improve finite element solutions

We can approximate displacement field, w, in elastostatics (bars, beams, walls, plates) with numerical result $w(h)$ calculated by finite element method over n equal finite elements of length h in 1D or $n \times n$ finite elements of dimension $h_x \times h_y$, $(h_x \approx h_y \approx h)$, in 2D problems. The order of the convergence of the sequence of solutions is equal to k and analytical solution could be expressed by using approximated solutions and order of convergence,

$$w = w(h) + Ch^k \ . \tag{4}$$

The order of convergence depends on the choice of finite element. The same displacement field could be also approximated with numerical result $w(h/2)$ calculated with finite elements of length $h/2$ and same order of convergence,

$$w = w(h/2) + C\left(\frac{h}{2}\right)^k \ . \tag{5}$$

By multiplying equation (5) with 2^k and distracting with equation (4), the leading error terms is removed. We get now solution with order of convergence equal to $k+1$. With this numerical procedure, called Richardson extrapolation, we have now the procedure for getting improved finite element method solution as linear combination of former finite element solutions,

$$w_{m+1}(h) = \frac{2^k w_m(h/2) - w_m(h)}{2^k - 1} \ , \tag{6}$$

with order of convergence equal to $k+1$. The procedure of Richardson extrapolation consists of successively eliminating terms in the error expansion to produce approximation of higher order. The application of described algorithm on FEM solution sequence is given in Table 2.

Table 2: The algorithm for Richardson extrapolation applied on FEM solutions

m	$w_m(h)$	$w_m\left(\frac{h}{2}\right)$	$w_m\left(\frac{h}{4}\right)$	$w_m\left(\frac{h}{8}\right)$	$w_m\left(\frac{h}{16}\right)$
1	$w_1(h)$	$w_1\left(\frac{h}{2}\right)$	$w_1\left(\frac{h}{4}\right)$	$w_1\left(\frac{h}{8}\right)$	$w_1\left(\frac{h}{16}\right)$
	$\downarrow$	$\swarrow\downarrow$	$\swarrow\downarrow$	$\swarrow\downarrow$	$\swarrow$
2	$w_2(h)$	$w_2\left(\frac{h}{2}\right)$	$w_2\left(\frac{h}{4}\right)$	$w_2\left(\frac{h}{8}\right)$	
	$\downarrow$	$\swarrow\downarrow$	$\swarrow\downarrow$	$\swarrow$	
3	$w_3(h)$	$w_3\left(\frac{h}{2}\right)$	$w_3\left(\frac{h}{4}\right)$		
	$\downarrow$	$\swarrow\downarrow$	$\swarrow$		
4	$w_4(h)$	$w_4\left(\frac{h}{2}\right)$			
	$\downarrow$	$\swarrow$			
5	$w_5(h)$				

The generalization of procedure could be introduced for the cases with uncorelated meshes, h/n_1 and h/n_2 and with order of convergence equal to k. Let we define w_{ap} as improved solution. We can define it by using both solutions respectively as

$$w_{ap} = w(h/n_1) + C\left(\frac{h}{n_1}\right)^k \ , \tag{7}$$

or

$$w_{ap} = w(h/n_2) + C\left(\frac{h}{n_2}\right)^k \ . \tag{8}$$

After some algebra we loose coefficient C,

$$\frac{w_{ap} - w_{h/n_1}}{w_{ap} - w_{h/n_2}} = \frac{n_2^k}{n_1^k} \ , \tag{9}$$

what leads to equation that gives expression for improved solution

$$w_{ap} = \frac{n_2^k w(h/n_2) - n_1^k w(h/n_1)}{n_2^k - n_1^k} \ , \tag{10}$$

which is generalization of equation (6).

4 Numerical examples

First example is fixed bar with concentrated force on its free edge. Cross-section area of the bar is defined with linear distribution of cross-sectional area $A(x) = A\frac{2L-x}{L}$, $(A(0) = 2A, A(L) = A)$.

Analytical solution of displacement at free edge is $w(L) = \frac{KL\ln 2}{EA} = 0.69314\frac{KL}{EA}$. After discretization on 1, 2 and 4 linear finite elements, FE solution are $0.66667\frac{KL}{EA}$, $0.68571\frac{KL}{EA}$, $0.69116\frac{KL}{EA}$ respectively with order of convergence equal to 2. Richardson extrapolation of first two solution (with 1 and 2 linear finite elements) gives new solution of displacement at free edge,

$$w(L) = 0.69206\frac{KL}{EA} \,. \tag{11}$$

Numerical value at free edge calculated by Richardson extrapolation of values with 1 and 2 finite elements with its $err = 0.16\%$ is more accurate than value with 4 finite elements with $err = 0.28\%$.

Second example is fixed bar with concentrated force on its free edge. Cross-section area of this bar is defined with exponential distribution of cross-sectional area $A(x) = A2^{\frac{L-x}{L}}$, $(A(0) = 2A, A(L) = A)$. Analytical solution of displacement of free edge is $w(L) = \frac{KL}{EA2\ln 2} = 0.721348\frac{KL}{EA}$. After discretization on 1, 2 and 4 linear finite elements, FE solution are $0.707107\frac{KL}{EA}$, $0.717750\frac{KL}{EA}$, $0.720344\frac{KL}{EA}$ respectively with order of convergence equal to 2. Richardson extrapolation of first two solution (with 1 and 2 linear finite elements) gives improved solution of displacement at free edge,

$$w(L) = 0.721298\frac{KL}{EA} \,. \tag{12}$$

Numerical value at free edge calculated by Richardson extrapolation of values with 1 and 2 finite elements with its $err = 0.007\%$ is more accurate than value with 4 finite elements with $err = 0.125\%$.

Third example is simply supported square plate under uniformly distributed load calculated with Hermite bicubic finite elements with four degrees of freedom at each knot. The presented algorithm, 2, is applied for improving numerical solution calculated by different mesh size. Table 3 shows numerical results of the displacement at mid-point calculated according the presented algorithm. The results after Richardson extrapolation are more accurate than results with more finite elements. Analytical solution for displacement at mid-point is $w = 0.004062353qL^4/D$, where D is flexural rigidity of the plate. We can find out that the calculated values by applying Richardson extrapolation ,$w_2(L/16)$ with $err = 0.09\%$ and $w_3(L/8)$ with $err = 0.07\%$, on finite element solution over less finite elements are much more accurate than finite element solution over more finite elements, $w_1(L/32) = 0.004069953qL^4/D$ with $err = 0.19\%$.

Table 3: Displacement at mid-point of square plate under uniform load

m	$w_m\left(\frac{L}{4}\right)$	$w_m\left(\frac{L}{8}\right)$	$w_m\left(\frac{L}{16}\right)$
1	0.004394822	0.004152497	0.004087584
2	0.004071722	0.004065967	
3	0.004065145		

Fourth example is simply supported square plate under concentrated force at mid-point calculated with Hermite bicubic finite elements with four degrees of freedom at each knot. The presented algorithm, 2, is applied for improving numerical solution calculated by different mesh size. Table 4 shows numerical results of the displacement at mid-point calculated according the presented algorithm. The results after Richardson extrapolation are more accurate than results with more finite elements. Analytical solution for displacement at mid-point is $w = 0.0116004KL^2/D$. The calculated value by applying Richardson extrapolation, $w_2(L/16)$ with error equal to 0.003%, on finite elements over 8×8 and 16×16 finite elements is much more accurate than finite element solution over more (32×32) finite elements, $w_1(L/32) = 0.0115988KL^2/D$ with error equal to 0.013%.

Table 4: Displacement at mid-point of square plate under concentrated force at mid-point

m	$w_m\left(\frac{L}{4}\right)$	$w_m\left(\frac{L}{8}\right)$	$w_m\left(\frac{L}{16}\right)$
1	0.0114870	0.0115692	0.0115928
2	0.0115967	0.0116007	

5 Conclusion

The extension of Richardson extrapolation to finite element method solution is developed. The proposed algorithm gives more accurate and efficient numerical results without discretization over more finite elements. The numerical solutions with increased accuracy are calculated according the algorithm based on recursive relations with less computation than solutions with discretization over more finite elements. Richardson extrapolation is used to improve order of accuracy of finite element solutions in elastostatics without further increasing of number of finite elements. The proposed method has been veryfied on 1D and 2D exaples through comparison to the known exact soliutions. As further research, the modified algorithm will be developed for nonlinear finite element solution and for problems in elastodynamics.

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