Exact Consensus Controllability of Multi-agent Linear Systems

M. ISABEL GARCÍA-PLANAS
Universitat Politècnica de Catalunya
Departament de Matèmatiques
Minería 1, Esc. C, 1-3, 08038 Barcelona
SPAIN
maria.isabel.garcia@upc.edu

Abstract: In this paper we study the exact controllability of multi-agent linear systems, in which all agents have an identical linear dynamic mode that can be in any order.

Key–Words: Multi-agent systems, consensus, controllability, exact consensus controllability.

1 Introduction

In the last years, the study of dynamic control multi-agents systems have attracted considerable interest, because they arise in a great number of engineering situations as for example in distributed control and coordination of networks consisting of multiple autonomous agents. There are many publications as for example ([4], [10], [12], [14]). It is due to the multi-agents appear in different fields as for example in consensus problem of communication networks ([10]), or formation control of mobile robots ([2]).

The consensus problem has been studied under different points of view, for example Jinhuan Wang, Daizhan Cheng and Xiaoming Hu in [12], analyze the case of multiagent systems in which all agents have an identical stable linear dynamics system, M.I. García-Planas in [4], generalize this result to the case where the dynamic of the agents are controllable.

Controllability is a fundamental topic in dynamic systems and it is studied under different approaches (see [1],[3],[7], for example). Given a linear system $\dot{x} = Ax$, there are many possible control matrices B making the system $\dot{x} = Ax + Bu$ controllable. The goal is to find the set of all possible matrices B, having the minimum number of columns corresponding to the minimum number $n_D(A)$ of independent controllers required to control the whole network. This minimum number is called exact controllability, that in a more formal manner is defined as follows.

Definition 1 *Let A be a matrix. The exact controllability $n_D(A)$ is the minimum of the rank of all possible matrices B making the system $\dot{x} = Ax + Bu$ controllable.*

$$n_D(A) = \min \{\text{rank}\, B, \forall B \in M_{n\times i}\, 1 \leq i \leq n \mid (A, B) \text{ controllable}\}.$$

In this paper, we investigate the exact controllability of a class of multiagent systems consisting of k agents with dynamics

$$\begin{aligned} \dot{x}^1 &= Ax^1 + Bu^1 \\ &\vdots \\ \dot{x}^k &= Ax^k + Bu^k \end{aligned}$$

where $A \in M_n(\mathbb{C})$, and B an unknown matrix having n rows and an indeterminate number $1 \leq \ell \leq n$ of columns.

For this study, we need to introduce some basic concepts on Graph theory and matritial algebra.

We consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of order k with the set of vertices $\mathcal{V} = \{1, \ldots, k\}$ and the set of edges $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$.

Given an edge (i, j) i is called the parent node and j is called the child node and j is in the neighbor of i, concretely we define the neighbor of i and we denote it by $\mathcal{N}_i$ to the set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$.

The graph is called undirected if verifies that $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. The graph is called connected if there exists a path between any two vertices, otherwise is called disconnected.

Associated to the graph we consider a matrix $G = (g_{ij})$ called (unweighted) adjacency matrix defined as follows $g_{ii} = 0$, $g_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise.

In a more general case we can consider that a weighted adjacency matrix is $G = (g_{ij})$ with $g_{ii} = 0$, $g_{ij} > 0$ if $(i, j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise).

The Laplacian matrix of the graph is

$$\mathcal{L} = (l_{ij}) = \begin{cases} |\mathcal{N}_i| & \text{if } i = j \\ -1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$$

Remark 2 *i) If the graph is undirected then the* *matrix $\mathcal{L}$ is symmetric, then there exist an orthogonal matrix P such that $P\mathcal{L}P^t = \mathcal{D}$.*

ii) If the graph is undirected then 0 is an eigenvalue of $\mathcal{L}$ and $\mathbf{1}_k = (1,\dots,1)^t$ is the associated eigenvector.

iii) If the graph is undirected and connected the eigenvalue 0 is simple.

For more details about graph theory see (D. West, 2007).

With respcet Kronecker product, remember that $A = (a_{ij}) \in M_{n\times m}(\mathbb{C})$ and $B = (b_{ij}) \in M_{p\times q}(\mathbb{C})$ the Kronecker product is defined as follows.

Definition 3 *Let $A = (a^i_j) \in M_{n\times m}(\mathbb{C})$ and $B \in M_{p\times q}(\mathbb{C})$ be two matrices, the Kronecker product of A and B, write $A \otimes B$, is the matrix*

$$A\otimes B = \begin{pmatrix} a^1_1B & a^1_2B & \dots & a^1_mB \\ a^2_1B & a^2_2B & \dots & a^2_mB \\ \vdots & \vdots & & \vdots \\ a^n_1B & a^n_2B & \dots & a^n_mB \end{pmatrix} \in M_{np\times mq}(\mathbb{C})$$

Among the properties that verifies the product of Kronecker we will make use of the following

1) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$

2) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$

3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

4) If $A \in Gl(n;\mathbb{C})$ and $B \in Gl(p;\mathbb{C}))$, then $A \otimes B \in Gl(np;\mathbb{C}))$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

5) If the products AC and BD are possible, then $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

See [9] for more information and properties.

Given a square matrix $A \in M_n(\mathbb{C})$, it can be reduced to a canonical reduced form (Jordan form):

$$J = \begin{pmatrix} J(\lambda_1) & & \\ & \ddots & \\ & & J(\lambda_r) \end{pmatrix}, \; J(\lambda_i) = \begin{pmatrix} J_1(\lambda_i) & & \\ & \ddots & \\ & & J_{n_i}(\lambda_i) \end{pmatrix},$$

$$J_j(\lambda_i) = \begin{pmatrix} \lambda_i & & & \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_i \end{pmatrix}. \qquad (1)$$

See [5] for more information and properties.

2 Consensus

The consensus problem can be introduced as a collection of processes such that each process starts with an initial value, where each one is supposed to output the same value and there is a validity condition that relates outputs to inputs. It is a canonical problem that appears in the coordination of multi-agent systems. The objective is that Given initial values (scalar or vector) of agents, establish conditions under which through local interactions and computations, agents asymptotically agree upon a common value, that is to say: to reach a consensus.

The dynamic of each agent defining the system considered, is given by the following manner.

$$\begin{aligned} \dot{x}^1 &= Ax^1 + Bu^1 \\ &\vdots \\ \dot{x}^k &= Ax^k + Bu^k \end{aligned} \qquad (2)$$

$x^i \in \mathbb{R}^n$, $u^i \in \mathbb{R}^\ell$, $1 \le i \le k$. Where matrices $A \in M_n(\mathbb{R})$ and $B \in M_{n\times\ell}(\mathbb{R})$, $1 \le \ell \le n$.

The communication topology among agents is defined by means the undirected graph $\mathcal{G}$ with

i) Vertex set: $\mathcal{V} = \{1,\dots,k\}$

ii) Edge set: $\mathcal{E} = \{(i,j) \mid i,j \in \mathcal{V}\} \subset \mathcal{V}\times\mathcal{V}$.

an in a more specific form, we have the following definition.

Definition 4 *Consider the system 2. We say that the consensus is achieved using local information if there exists a state feedback*

$$u^i = K_i \sum_{j\in\mathcal{N}_i}(x^i - x^j), \; 1 \le i \le k$$

such that

$$\lim_{t\to\infty} \|x^i - x^j\| = 0, \; 1 \le i,j \le k.$$

$z^i = \sum_{j\in\mathcal{N}_i}(x^i - x^j), \; 1 \le i \le k.$

$$\begin{aligned} \dot{\mathcal{X}} &= (I_k \otimes A)\mathcal{X} + (I_k \otimes B)\mathcal{U} \\ \mathcal{Z} &= (\mathcal{L} \otimes I)\mathcal{X} \\ \mathcal{U} &= (I_k \otimes K)\mathcal{Z} \end{aligned}$$

Then, and taking into account that

$$(I_k \otimes B)(I_k \otimes K)(\mathcal{L} \otimes I_n)\mathcal{X} = (\mathcal{L} \otimes BK)\mathcal{X} = (\mathcal{L} \otimes B)(I_k \otimes K)\mathcal{X}$$

The system is equivalent to

$$\begin{aligned} \dot{\mathcal{X}} &= (I_k \otimes A)\mathcal{X} + (\mathcal{L} \otimes B)\bar{\mathcal{U}} \\ \bar{\mathcal{U}} &= (I_k \otimes K)\mathcal{X} \end{aligned} \qquad (3)$$

3 Exact Consensus Controllability

We are interested in study the exact controllability of the obtained system 3. In our particular setup

Definition 5 *Let A be a matrix. The exact controllability $n_D(I_k \otimes A)$ is the minimum of the rank of all possible matrices B making the system 3 controllable.*

$$\begin{array}{l} n_D(I_k \otimes A) = \\ \min\{\text{rank}\, B, \forall B \in M_{n\times i}\, 1 \leq i \leq n \mid \\ \qquad (I_k \otimes A, \mathcal{L} \otimes B) \text{ controllable}\}. \end{array}$$

The controllability character can be analyzed using the Hautus criteria

Proposition 6 *The system is controllable if and only if*

$$\text{rank}\begin{pmatrix} sI_{nk} - (I_k \otimes A) & \mathcal{L} \otimes B \end{pmatrix} = kn$$

The controllability condition depends directly on the structure of the matrix L.

Proposition 7 *Let J be the Jordan reduced of the matrix $\mathcal{L}$ and P such that $\mathcal{L} = P^{-1}JP$. Then, the system 3 is controllable if and only if*

$$\text{rank}\begin{pmatrix} sI_{nk} - (I_k \otimes A) & J \otimes B \end{pmatrix} = kn$$

Proof. Suppose that there exist S such that $P^{-1}JP = \mathcal{L}$ and

$$\begin{array}{l} \text{rank}\begin{pmatrix} sI_{kn} - (I_k \otimes A) & \mathcal{L} \otimes B \end{pmatrix} = \\ \text{rank}\,(P^{-1} \otimes I_n)\begin{pmatrix} sI_k \otimes I_n) - (I_k \otimes A) & J \otimes B \end{pmatrix} \\ \qquad \begin{pmatrix} P \otimes I_n & \\ & P \otimes I_n \end{pmatrix} = \\ \text{rank}\begin{pmatrix} sI_{kn} - (I_k \otimes A) & J \otimes B \end{pmatrix} \end{array}$$

□

Corollary 8 *Suppose that the matrix $\mathcal{L}$ can be reduced to the Jordan form (1), with non-zero eigenvalues $\lambda_1, \ldots, \lambda_r$. Then, the system 3 is controllable if and only if each agent is controllable.*

Proof. Let $\lambda_i \neq 0$, $i = 1, \ldots r$ be the eigenvalues of $\mathcal{L}$.

$$\text{rank}\begin{pmatrix} s(I_{k_{ij}} \otimes I_n) - (I_{k_{ij}} \otimes A) & J_j(\lambda_i) \otimes B \end{pmatrix} =$$

$$\text{rank}\begin{pmatrix} sI_n - A & & & & \lambda_i B & & & \\ & sI_n - A & & & B & \lambda_i B & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & sI_n - A & & & B & \lambda_i B \end{pmatrix} =$$

$$\text{rank}\begin{pmatrix} sI_n - A & & & & B & & & \\ & sI_n - A & & & & B & & \\ & & \ddots & & & & \ddots & \\ & & & sI_n - A & & & & B \end{pmatrix} =$$

$$k \cdot \text{rank}\begin{pmatrix} sI_n - A & B \end{pmatrix}$$

with $k_1 + \ldots + k_r = k$, $k_{i_1} + \ldots k_{i_{n_i}} = k_i$.

□

Corollary 9 *A necessary condition for controllability of the system 3 is that the matrix $\mathcal{L}$ has full rank.*

Example We consider 3 identical agents with the following dynamics of each agent

$$\begin{array}{ll} \dot{x}^1 & = Ax^1 + Bu^1 \\ \dot{x}^2 & = Ax^2 + Bu^2 \\ \dot{x}^3 & = Ax^3 + Bu^3 \end{array} \tag{4}$$

with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B \in M_{2\times \ell}(\mathbb{C})$, $1 \leq 2$.

The communication topology is defined by the undirected graph $(\mathcal{V}, \mathcal{E})$:

$\mathcal{V} = \{1, 2, 3\}$

$\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} = \{(1, 2), (1, 3)\} \subset \mathcal{V}\times\mathcal{V}$

and the adjacency matrix:

$$G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The neighbors of the parent nodes are $\mathcal{N}_1 = \{2, 3\}, \mathcal{N}_2 = \{1\}, \mathcal{N}_3 = \{1\}$.

The Laplacian matrix of the graph is

$$\mathcal{L} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$.

$$\text{rank}\begin{pmatrix} sI_6 - (I \otimes A) & \mathcal{L} \otimes B \end{pmatrix} =$$

$$\text{rank}\begin{pmatrix} s & -1 & 0 & 0 & 0 & 0 & 2a & 2c & -a & -c & -a & -c \\ 0 & s & 0 & 0 & 0 & 0 & 2b & 2d & -b & -d & -b & -d \\ 0 & 0 & s & -1 & 0 & 0 & -a & -c & a & c & 0 & 0 \\ 0 & 0 & 0 & s & 0 & 0 & -b & -d & b & d & 0 & 0 \\ 0 & 0 & 0 & 0 & s & -1 & -a & -c & 0 & 0 & a & c \\ 0 & 0 & 0 & 0 & 0 & s & -b & -d & 0 & 0 & b & d \end{pmatrix}$$

$$= \begin{cases} 6 \text{ for all } s \neq 0 \\ 5 \text{ for } s = 0 \end{cases}$$

In fact, for all matrix $B \in M_{2\times \ell}(\mathbb{C})$ for all $\ell \geq 0$

$$\begin{array}{l} \text{rank}\begin{pmatrix} sI_6 - (I \otimes A) & \mathcal{L} \otimes B \end{pmatrix} = \\ \begin{cases} 6 \text{ for all } s \neq 0 \\ 5 \text{ for } s = 0 \end{cases} \end{array}$$

If the matrix $\mathcal{L}$ has full rank, then the number of columns for exact controllability of matrix $I_k \otimes A$ depends on the multiplicity of the eigenvalues of the matrix A and we have the following result.

Proposition 10 *Let $\mathcal{L}$ be the Laplacian matrix of a graph having full rank. Then, the exact controllability $n_D(I_k \otimes A)$ for the system $\dot{\mathcal{X}} = (I_k \otimes A)\mathrm{X} + (\mathcal{L} \otimes B)\bar{\mathcal{U}}$ coincides with the exact controllability $n_D(A)$ for the system $\dot{x} = Ax + Bu$.*

Example We consider 3 identical agents with the following dynamics of each agent

$$\begin{aligned} \dot{x}^1 &= Ax^1 + Bu^1 \\ \dot{x}^2 &= Ax^2 + Bu^2 \\ \dot{x}^3 &= Ax^3 + Bu^3 \end{aligned} \qquad (5)$$

with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B \in M_{2\times\ell}(\mathbb{C})$, $1 \leq 2$.

The communication topology is defined by the undirected graph $(\mathcal{V}, \mathcal{E})$:
$\mathcal{V} = \{1, 2, 3\}$
$\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1)\} \subset \mathcal{V} \times \mathcal{V}$
and the adjacency matrix:

$$G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The neighbors of the parent nodes are $\mathcal{N}_1 = \{1, 2\}, \mathcal{N}_2 = \{1, 3\}, \mathcal{N}_3 = \{1\}$.

The Laplacian matrix of the graph is

$$\mathcal{L} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 0.3820$, $\lambda_2 = 2$, $\lambda_3 = 2.6180$.

$$\operatorname{rank} \begin{pmatrix} s & -1 & 0 & 0 & 0 & 0 & 2a & -a & 0 \\ 0 & s & 0 & 0 & 0 & 0 & 2b & -b & 0 \\ 0 & 0 & s & -1 & 0 & 0 & -a & 2a & -a \\ 0 & 0 & 0 & s & 0 & 0 & -b & 2b & -b \\ 0 & 0 & 0 & 0 & s & -1 & -a & 0 & a \\ 0 & 0 & 0 & 0 & 0 & s & -b & 0 & b \end{pmatrix}$$

6 for all s and $b \neq 0$.

Obviously the system $\dot{x} = Ax + Bu$ with $B = \begin{pmatrix} a \\ b \end{pmatrix}$ and $b \neq 0$.

4 Conclusions

In this paper, the exact controllability for multi-agent systems where all agents have an identical linear dynamic mode are analyzed.

References:

[1] C.T. Chen, "Introduction to Linear System Theory". Holt, Rinehart and Winston Inc, New York, (1970).

[2] A. Fax, R. Murray, *Information flow and cooperative control of vehicle formations*, IEEE Trans. Automat. Control. **49**, (9), pp. 1453-1464, (2004).

[3] M.I. García-Planas, Sensivity and stability of singular systems under proportional and derivative feedback, Wseas Transactions on Mathematics, **8**, (11), pp 635-644, (2009).

[4] M.I. García-Planas, *Obtaining Consensus of Multi-agent Linear Dynamic Systems*, Advances in Applied and Pure Mathematics, pp. 91-95, (2014)

[5] M.I. García-Planas, J.L. Domínguez, "Introducción a la teoría de matrices positivas. Aplicaciones". Ed. Iniciativa Digital Politècnica, Barcelona, 2013.

[6] M.I. García-Planas, M.D. Magret, *Miniversal deformations of linear systems under the full group action*. Systems and control letters, **35**, pp. 279–286, (1998).

[7] M.I. García-Planas, S. Tarragona, A. Diaz, *Controllability of time-invariant singular linear systems*. From physics to control through an emergent view. pp. 112 -117. World Scientific, 2010.

[8] A. Jadbabaie, J. Lin, A.S. Morse, *Coordination of groups of mobile autonomous agents using nearest neighbor rules*, IEEE Transaction on Automatic Control. **48** (6), pp. 943-948, (2007).

[9] P. Lancaster, M. Tismenetsky, "The Thoery of Matrices". Academic Press. San Diego (1985).

[10] R.O. Saber, R.M. Murray, *Consensus Protocols for Networks of Dynamic Agents*, Report.

[11] R.O. Saber, R.M. Murray, *Consensus problems in networks of agents with switching topology and time-delays*, IEEE Trans. Automat. Control. **49**, (9), pp. 1520-1533, (2004).

[12] J. Wang, D. Cheng, X. Hu, *Consensus of multi-agent linear dynamics systems*, Asian Journal of Control **10**, (2), pp. 144-155, (2008).

[13] D. West "Introduction to Graph Theory" Prentice Hall (3rd Edition), (2007).

[14] G. Xie, L. Wang, *Consensus control for a class of networks of dynamic agents: switching topology*, Proc. 2006 Amer. Contro. Conf., pp. 1382-1387, (2007).

[15] Z.Z. Yuan, C. Zhao, W.X.Wang, Z.R. Di, Y.C. Lai, *Exact controllability of multiplex networks*, New Journal of Physics, **16**, pp. 1-24, (2014).

[16] Z.Z. Yuan, C. Zhao, W.X.Wang, Z.R. Di, Y.C. Lai, *Exact controllability of complex networks* Nature Communications. **4**, pp. 1-12, (2013).