Continuous time quantum walks and recurrences in the Hilbert space

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Abstract: An analytical consideration of quantum walks in the Hilbert space is suggested for dynamical systems. It is shown that in the semiclassical limit, statistics of the quantum recurrences relates to statistics of the Poincaré recurrences of the classical counterpart. It is shown that the statistics of the quantum recurrences is sensitive to the statistics of the corresponding quantum spectrum. The difference in the statistics of quantum recurrences in the Hilbert space for the chaotic and integrable systems follows from the essential difference between the level statistics of integrable and chaotic systems. In particular, when the integrable part of the phase space emerges due to bifurcation, and the exponential distribution of the Poincaré recurrences of chaotic trajectories is changed into the power law, the statistics of the quantum walks in the Hilbert space follows exactly its classical counterpart.

Key–Words: Chaos, Random Walk, Poincaré recurrences, Statistics of quantum spectrum, Almost periodic functions

1 Introduction

Statistics of Poincaré recurrences is a powerful method for studying anomalous transport in chaotic systems with generic phase space structures, where regular and chaotic regions coexist [1]. The reason is that the distribution of the Poincaré cycles (recurrence times) is sensitive to the topological structure of the phase space and to the probabilistic features of chaotic trajectories. Particularly, for the chaotic systems with a uniform mixing property, the distribution is exponential [2]

$$P(\tau) = \frac{1}{\tau_{rec}} \exp(-\tau/\tau_{rec}) \qquad (1)$$

with the mean recurrence time

$$\tau_{rec} = \int_0^\infty \tau P(\tau) d\tau \propto 1/h_0 \,, \qquad (2)$$

where h_0 is a metric entropy. It follows, due to the Kac lemma, that $\tau_{rec} < \infty$ for the area preserving and bounded dynamics [3]. In systems with non–uniform mixing and sticky island regions, as shown in Fig. 1, the distribution of recurrences can be algebraic in the asymptotics of the large recurrence times:

$$P(\tau) \sim 1/\tau^\gamma, \quad (\tau \to \infty) \,, \qquad (3)$$

where γ is called the recurrence exponent [1]. As in Eq. (2), it follows from the Kac lemma that $\gamma > 2$ [3] (see also [4, 5]). Distribution of the Poincaré recurrences is therefore proven to be a powerful method for verification of the space–time complexity in classical Hamiltonian dynamics with nonzero or zero Lyapunov exponents or systems that exhibit a strong intermittent behavior with flights, trappings, weak mixing, etc. See [6] and references therein.

In quantum systems, the classical methodology fails because of the absence of trajectories, and any possible generalization of the notion of the Poicaré recurrences is desirable, although it can be non–unique. It is worthwhile to mention that, for systems with chaotic dynamics, a sequence of recurrence times $\{t\}_{rec} = \{t_1, t_2, \ldots\}_{rec}$ is a stochastic process with properties that depend on the type of dynamics. One can expect a similar process in the quantum case[1]

In this paper, we exploit Zaslavsky's idea [1] to relate statistics of recurrences of quantum walks in the Hilbert space to the phase space topology of the classical counterpart. We mobilize the standard notion of recurrences for a finite length vector $\mathbf{C} = (C_1, \ldots, C_N)$ in the Hilbert space, where a distance

[1] One should not confuse this with the phenomenon of periodic revivals of the wave functions, see *e.g.*, Refs. [7]. In this case, it is possible a truncation of the energy expansion near some level n_0, namely, $E_n = E_{n_0} + E'_{n_0}(n-n_0) + E''_{n_0}(n-n_0)^2 + \mathrm{O}[(n-n_0)^3]$.

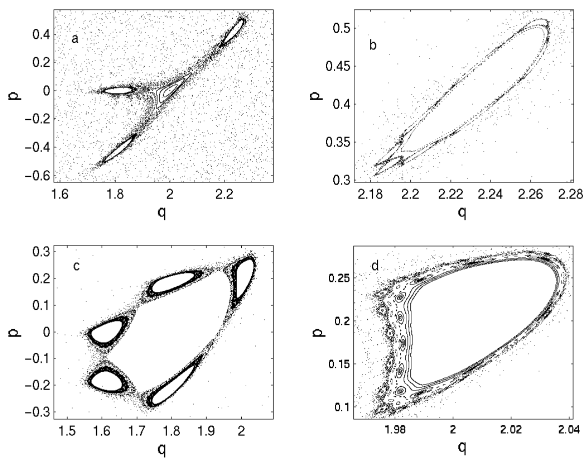

Figure 1: Phase portrait of the AMI, where (a) and (b) are sequences of the island chain of the first and the second generations correspondingly with periods 3 and 8 for $K^* = 6.908745$; plots (c) and (d) are the first and the second generations and correspond to $5 - 11 - \ldots$ island chain for $K^* = 6.476339$.

between any vectors $\mathbf{C}^a$ and $\mathbf{C}^b$ is defined as

$$d_{ab}^2 = |\mathbf{C}^a - \mathbf{C}^b|^2 = \sum_{j=1}^{N} |C_j^a - C_j^b|^2 \qquad (4)$$

and, particularly, the dynamics of an initial state $\mathbf{C}(0)$ reads

$$d^2(t) = |\mathbf{C}(t) - \mathbf{C}(0)|^2 . \qquad (5)$$

With definition (5), one can introduce a notion of the quantum recurrence (QR) as the condition

$$d_j^2(\tau) \le \epsilon, , \ (\forall j)$$
$$d_j = |C_j(\tau) - C_j(0)|, \ (\forall j) . \qquad (6)$$

Here, we use a complete analogy with the classical Poincaré recurrences: like a classical trajectory returns to some finite area of the chaotic region, the quantum state returns to some ϵ-cone in the Hilbert space, where τ is the QR time. We follow this heuristic definition to show how quantum walks in the Hilbert space reflect the topology of the classical phase space and how it can distinguish between different cases, particularly between the case with the classically almost uniformly mixing dynamics and the case of the strongly intermittent dynamics.

Nevertheless, the relation between the QR and the classical Poincaré recurrences is not so straightforward, since the quantum dynamics is a quasi-periodic process, and it is independent of the integrability property of the classical counterpart. Note that quantum localization time $\sim 1/\tilde{h}^2$, when the classical chaotic dynamics exhibits in quantum one, is incomparably small than QR times. Here, $\tilde{h}$ is a dimensionless effective Planck constant. Therefore, we follow the results on integrable and chaotic spectrum: in the semiclassical limit $\tilde{h} \to 0$, the quantum spectrum follows the classical dynamics. Namely, for the integrable systems with uncorrelated spectrum, the probability distribution of the spacing between successive levels is Poisson distribution [19]

$$P^{(i)}(\Delta) = \frac{1}{\Delta_0} \exp(-\Delta/\Delta_0) , \qquad (7)$$

where Δ_0 is the mean level spacing. In contrast, for chaotic systems, the corresponding quantum spectrum is strongly correlated (repelled), and the level spacing is described by the Wigner-Dyson statistics of random matrices [20, 21]. Following discussion in Ref. [22], it can be presented in the form

$$P^{(c)}(\Delta) = C_\beta(\Delta_0)\Delta^\beta \exp(-\Delta^2/\Delta_0^2) , \qquad (8)$$

where $C_\beta(\Delta_0)$ is the normalization constant and $\beta = 1, 2, 4$ for the orthogonal, unitary, and symplectic Gaussian ensembles, respectively. According to Berry-Robnik's assumption [22], there exist the both kind of the spectrum statistics simultaneously, Poisson and Wigner-Dyson, in the presence of any integrable/regular part of the phase space. Following their calculations [22], the authors expressed the idea that " each connected regular or irregular classical phase-space region in ΔE gives rise to its own sequence of regular or irregular levels" and the level spacing distribution follows Eq. (7) for the regular part of the phase space, and for the chaotic motion it follows Eq. (8).

2 Quantum recurrences

Important part of our analysis is application of a theory of almost periodic functions. The theory of almost periodic functions is well developed and its main features has been created in the third decade of 20th century, see monograph by A.S. Besicovitch [23]. Some basic ideas of the theory, which are used here is presented in Appendix A.

Let us consider the dynamics of an initial wave function Ψ_0, which is controlled by the evolution operator $\hat{U}$ with the Hamiltonian H and with the energy/quasienrgy spectrum E_k such that $\hat{U}(t)\psi_k = e^{-iE_k t}\psi_k$ is due the expression

$$\Psi(t) = \hat{U}(t)\Psi_0 = \sum_k a_k \exp(-iE_k t)\psi_k . \qquad (9)$$

This yields the evolution of the distance, or the QRs (6) in the Hilbert space

$$d^2(t) = |\Psi(t) - \Psi_0|^2 = \sum_k |a_k|^2 |e^{-iE_k t} - 1|^2 . \qquad (10)$$

As seen, this expression is the squared translation function (A. 2). Therefore, the QRs exist with the recurrence, or translation time τ and integer N such that

$$d^2(\tau)=\sum_{k=1}^{N}|a_k|^2|e^{-iE_k\tau}-1|^2<\epsilon^2\,. \qquad (11)$$

Note that this result also follows from the quantum mechanical consideration [24, 25]. As shown in Ref. [24], the quasi-periodic quantum motion, reflected in Eq. (11), corresponds to the dynamics of a collection of N classical harmonic oscillators with frequencies $(E_1,\dots,E_N)$ taking place in the N-dimensional torus. The dynamics is described by the action-angle coordinates $(I_1,\phi_1,\dots,I_N,\phi_N)$. If the oscillators have the initial conditions $I_k=a_k$ and $\phi(0)=0$, therefore, a set of numbers $a_ke^{-iE_kt}$ corresponds to motion on the N-torus, and the results on the classical Poincaré recurrences of N oscillators are valid now.

Since the wave function is normalized $\sum_k|a_k|^2=1$, there exists an integer N such that

$$\sum_{k=N+1}^{\infty}|a_k|^2<\epsilon^2\ll 1\,.$$

In our analysis, we follow the theory of almost periodic functions [23], where all the translation numbers belong to set $\mathcal{E}=\mathcal{E}\left\{\epsilon^2,d^2(t)\right\}$. Therefore, due to the theorem (A. 5), all numbers τ of the set $\mathcal{E}$ satisfy the following N Diophantine inequalities

$$|e^{-iE_k\tau}-1|^2<\delta_1^2\,. \qquad (12)$$

Substituting Eq. (12) in Eq. (11), one obtains that

$$\sum_{k=1}^{N}|a_k|^2|e^{-iE_k\tau}-1|^2<\delta_1^2\sum_{k=1}^{N}|a_k|^2<\epsilon^2\,.$$

Therefore, $\delta_1\sim\epsilon\ll 1$. We also obtain from Eq. (12)

$$|e^{-iE_k\tau}-1|^2=4\sin^2\left(\frac{E_k\tau}{2}\right)<\delta_1^2\sim\epsilon^2\,. \qquad (13)$$

As seen, this condition is stronger than Theorem (A. 3), since now we have N Diophantine inequalities determined by $\delta_1\sim\epsilon\ll 1$ and not by $\delta<\pi$ as in (A. 3). Therefore, these conditions yield

$$|E_k\tau-2\pi n_k|<\epsilon(\delta_1)\,, \qquad (14)$$

where n_k are integer numbers, which corresponds to the energies E_k. Equation (13) can be rewritten in the form

$$E_k\tau=2\pi n_k+\eta_k\,,\quad |\eta_k|<\epsilon\,. \qquad (15)$$

From these expressions, one can also define n_k considering the level spacing of the ordered spectrum $E_1<E_2,\dots<E_N<E_{N+1}$

$$\Delta_k=E_k-E_{k+1}\,,\ \ k=1,2,\dots,N\,.$$

Note that the energy level E_{N+1} does exist. Using Eq. (15), one obtains the following chain of transformations for the argument of the sine-function in Eq. (13)

$$\sin^2\left(\tfrac{E_k\tau}{2}\right)=\sin^2\left[\tfrac{1}{2}(E_k\tau\pm E_{k+1}\tau)\right]=$$
$$\sin^2\left[\tfrac{1}{2}(\Delta_k\tau+E_{k+1}\tau)\right]=\sin^2\left[\tfrac{1}{2}(\Delta_k\tau\pm 2\epsilon)\right]$$
$$<\tfrac{\epsilon^2}{4}\,, \qquad (16)$$

where we used Eq. (15) $|E_k\tau-E_{k+1}\tau|=|2\pi(n_k-n_{k+1}+\eta_k-\eta_k|_{2\pi}=|\eta_k-\eta_k|<|\eta_k|+|\eta_k|<2\epsilon$. From Eq. (16), one obtains $|\sin\left[\frac{\Delta_k\tau}{2}\right]|-|\epsilon|<|\sin\left[\frac{1}{2}(\Delta_k\tau\pm 2\epsilon)\right]|<\epsilon/2$ This eventually yields N Diophantine inequalities

$$|\sin\left[\frac{\Delta_k\tau}{2}\right]|<\frac{3\epsilon^2}{2}\,. \qquad (17)$$

Substituting this result in Eq. (10), one obtains that QRs with the translation times τ are determined from the level spacings Δ_k

$$\begin{aligned} d^2(\tau) &= \sum_{k=1}^{N}|a_k|^2|e^{i\Delta_k\tau}-1|^2 \\ &= 4\sum_{k=1}^{N}|a_k|^2\sin^2\left[\frac{\Delta_k\tau}{2}\right]<9\epsilon^2\,. \qquad (18)\end{aligned}$$

Here we also use the change $-\Delta_k\to\Delta_k>0$. Therefore, the structure of the recurrent-translation times is determined by N Diophantine inequalities

$$|\Delta_k\tau-2\pi m_k|<\epsilon_1\,, \qquad (19)$$

where $\epsilon_1=3\epsilon$ and m_k are integers.

Now using the approach, developed for the QRs, which relates to a set of translations of u.a.p. functions, we consider the QRs as quantum returns to a finite area in the Hilbert space. Therefore, N Diophantine inequalities (19) determine QR, τ or translations forming a set $\mathcal{E}\{\epsilon_1,d(t)\}$, where the structure of the translations is

$$\tau=2\pi\frac{\tilde{m}(\{\Delta_k\})}{\Delta_k}\,. \qquad (20)$$

Here $\tilde{m}(\{\Delta_k\})$ is a functions of N random variables Δ_k such that

$$|\tilde{m}(\{\Delta_k\})-m_k|<\frac{\epsilon_1}{2\pi}\,,\quad k=1,\dots,N\,. \qquad (21)$$

These quantum walks correspond to independent random processes for every trial of returning/recurrence in the dynamics of the wave function in the Hilbert space with a set of translations - recurrences $\mathcal{E}\{\epsilon_1, d(t)\}$ constructed by a system of N Diophantine inequalities (14), (17), and (19). Obviously, this set can be easily enlarged by increasing ϵ_1 (see Eq. (A. 1) in Appendix).

3 Statistics of quantum recurrences

The property of these random walks can be specified by their distribution function $\rho_{QR}(\tau)$. To find the distribution function $\rho_{QR}(\tau)$ of QRs, we determine the averaged value of the translation numbers and averaged value of the squared translation numbers. To this end we use the following properties of the QRs. First of all, recurrence times τ are huge values, while the second important property is the finitness of the mean value of the QRs, which follows from the Kac lemma [3]. Therefore, this reads for finite N

$$\langle\tau\rangle = \int \rho_{QR}(\tau)\tau d\tau < \infty \,, \tag{22}$$

where $\rho_{QR}(t)$ is the distribution function of the quantum recurrences defined above. Therefore, in our case, the Kac lemma states that for both integrable and chaotic spectral statistics the averaged recurrence times are the finite values:

$$\langle\tau\rangle = \int \tau(\{\Delta\})P(\{\Delta\})\prod_{k=1}^{N} d\Delta_k \,, \tag{23}$$

where $P(\{\Delta\})$ is a many dimensional joint level spacing distribution function.

Let us first calculate the averaged values of the QR/translations for the integrable case that corresponds to the Poisson statistics (7). In this case, the sequence of levels E_j is a random set without correlations, see for example [26], and the joint distribution $P(\{\Delta\})$ is a product of distributions (7)

$$\langle\tau\rangle^{(i)} = \frac{2\pi}{\Delta_0}\int_0^\infty \frac{\langle\tilde{m}(\Delta)\rangle}{\Delta} e^{-\frac{\Delta}{\Delta_0}} d\Delta < \infty \,. \tag{24}$$

Here $\langle\tilde{m}(\Delta)\rangle$ is the function of only one variable, obtained by integration of $\tilde{m}(\{\Delta_k\})$ over $N-1$ variables Δ_k. The main problem of the integration in Eq. (24) is the lower limit $\Delta \to 0$ due to the singular-pole behavior of the integrand. Due to the Kac lemma this integral is finite, therefore one obtains in the vicinity of the lower limit $\langle\tilde{m}(\Delta)\rangle \sim M\Delta^\gamma$ with $0 < \gamma \ll 1$ and $M \gg 1$. Note that it is an important condition since τ are very large numbers for any $\Delta_k \to 0$. Taking this into account, one obtains that the integral in eq. Eq. (24) is the Gamma function $\Gamma(\gamma)$

$$\langle\tau\rangle^{(i)} \sim 2\pi M\Gamma(\gamma) \,. \tag{25}$$

Therefore, the correct structure of QRs is

$$\tau = 2\pi\Delta_k^{\gamma-1}M(\{\Delta_k\}) \,, \tag{26}$$

where $M(\{\Delta_k\})$ can be singular in the vicinity of $\Delta \to 0$ not stronger than $\prod_{l\neq k}\Delta_l^{-\delta_l}$ with $0 < \delta_l < 1$. This also yields a crude estimation of the QRs length $\tau \propto \prod_{k=1}^{N}\Delta_k^{-\delta_k} \sim \Delta^{-N\delta}$, where $\Delta, \delta \ll 1$

Evidently, for this integrand, the second moment and the variance are divergent, $\langle\tau^2\rangle^{(i)} = \infty$. Therefore, the recurrent times are distributed according the power law

$$\rho_{QR}^{(i)}(\tau) \sim \left(\frac{\tau_0}{\tau_0+\tau}\right)^\alpha \,, \quad 2 < \alpha < 3 \,, \tag{27}$$

where τ_0 is a characteristic time scale that is taken in such a way that $\int \rho_{QR}^{(i)}(\tau)\tau d\tau = 2\pi M\Gamma(\gamma)$.

In contrast, for the chaotic dynamics, when the level spacing statistics is governed by the Wigner-Dyson distribution (8), the same arguments on the averaging procedure, as presented above, ensures the existence of the second moment/variance even for GOE with $\beta = 1$. The main picularity here is the correlations between the levels E_j. In this case the joint ditribution of levels is (see for example [26])

$$\begin{aligned} P(\{\Delta\}) &= C_\beta(A) \\ \times & \prod_{k<l}^{1\dots N}|E_k - E_l|^\beta \exp\left(-A\sum_{k=1}^{N}E_k^2\right) \,, \end{aligned} \tag{28}$$

where A fixes the unit of eneregе (for example it can be the mean squared level spacing, as Eq. (8)) and C_β is a normalization constant. The existence of the first and the second moments for the GUE and GSE follows immidietely from the joint distribution (28) and the structure of the QRs. In the rest we show that the second moment for the GOE is finite as well.

Let us consider the integration

$$\begin{aligned} \langle\tau^2\rangle^{(c)} &= \int_0^\infty \tau^2\rho_{QR}(\tau)d\tau \\ &= \tilde{C}\int_{-\infty}^{\infty}\prod_{k<l}^{1\dots N}|E_k - E_l|\exp\left(-A\sum_{k=1}^{N}E_k^2\right) \\ &\times \prod_{r\neq s}|E_r - E_{r+1}|^{-2\delta_l}|E_s - E_{s+1}|^{2\gamma-2}d^N E \\ &\equiv \tilde{C}\int_{-\infty}^{\infty}|E_s - E_{s+1}|^{2\gamma-2}\mathcal{F}(\{E_j\})d^N E \,, \end{aligned} \tag{29}$$

Here, for brevity sake, we define the rest of the integrand in Eq. (29) by $\mathcal{F}(\{E_j\})$ and $d^N E \equiv \prod_{j=1}^N dE_j$, and $\tilde{C} = (2\pi)^2 C_1(A)$. Let us rewrite this integration in the form of an additional integration with the δ function. Using the definition $\Delta = E_{s+1} - E_s$, one obtains

$$\begin{aligned}\langle\tau^2\rangle^{(c)} &= \int_0^\infty d\Delta|\Delta|^{2\gamma-2} \\ &\times \int_{-\infty}^\infty \delta(\Delta + E_s - E_{s+1})\mathcal{F}(\{E_j\})d^N E \\ &\equiv \int_0^\infty d\Delta|\Delta|^{2\gamma-2}P^{(c)}(\Delta)\,. \end{aligned} \tag{30}$$

Illuminating discussions presented *e.g.*, in [26, 27] explain impossibility to obtain analytical expressions for the nearest neighbor spacing destributions for $N \times N$ matrices. However, the destribution (8) obtained for 2×2 random matrices is a very good approximation for $N \times N$ matrices [20, 26, 27]. Therefore, integration over the N dimensional energy space can be replaced by the distribytion (8) for $P^{(c)}(\Delta$. Eventually, one arives at the following integration

$$\begin{aligned}\langle\tau^2\rangle^{(c)} &\sim \pi M C_1(\Delta_0)\int_0^\infty \Delta^{2\gamma-1}e^{-\frac{\Delta^2}{\Delta_0^2}}d\Delta \\ &= \pi M C_1(\Delta_0)\Delta_0^{2\gamma-2}\Gamma(\gamma)\,. \end{aligned} \tag{31}$$

The existence of the first and the second moments for the Gaussian recurrent process means that the recurrent times (as some trapping times outside the ϵ_1-cone) are distributed exponentially, see *e.g.*, [28]

$$\rho_{QR}^{(c)}(\tau) \sim \frac{1}{\tau_0}\exp\Big(-\tau/\tau_0\Big)\,, \tag{32}$$

where τ_0 now is the averaged recurrence time:

$$\begin{aligned}\tau_0 &= 2\pi M C_1(\Delta_0)\int_0^\infty \Delta^{\beta-1+\gamma}e^{-\frac{\Delta^2}{\Delta_0^2}}d\Delta \\ &= 2\pi M C_1(\Delta_0)\Delta_0^{\beta+\gamma}\Gamma\left(\frac{\beta+\gamma}{2}\right)\,. \end{aligned} \tag{33}$$

4 Conclusion

The present analysis is an analytical estimation of statistics of quantum recurrences in the Hibert space. As shown, the statistics of the quantum recurrences is sensitive to the statistics of the corresponding quantum spectrum. This leads to the essential difference in the statistics of the quantum recurrences in the Hilbert space for the chaotic and integrable systems, which results from the essential difference between the level statistics of integrable and chaotic systems. This also results from the fact that the quantum walks in the Hilbert space are random and the returning times are functions of the level spacings Δ, which are random variables with different distributions. Although, the analytical form of $\tau = \tau(\Delta)$ is the same for both integrable and chaotic cases, the averaged behavior is completely different that reflects the different natures of the quantum recurrences for chaotic and integrable (or generic) quantum systems. Obviously, for the generic case, the Poisson level spacing distribution is dominant, since the latter leads to the divergent second moment. Another important point is the semiclassical limit for $\tilde{h} \to 0$, when there are so many levels inside the integrable islands that the averaged procedures in Eqs. (23) have meaning.

However, the nature of recurrences in the classical and quantum dynamics are completely different. As seen from the analysis, this phenomenon of quantum random walks in the Hilbert space is more general than Zaslavsky's conjecture on quantum recurrences, which is strictly related to the bifurcative emerging of regular islands. One should understand that the nature of the classical recurrences of a chaotic trajectory differs essentially from the quantum recurrences in the Hilbert space. First of all in the classical case, the dynamics is performed inside some invariant volume (measure) of the $2D$ phase space related to a chaotic trajectory [1]. The quantum dynamics is integrable and is described by the almost periodic wave functions [23, 24, 25]. It should be stressed, that the relation between the Poincaré recurrences in the classical and quantum recurrences is due to the reconstruction of the spectrum from $E^{(c)}$ to $E^{(i)}$ and changes the level spacing distribution from the Wigner-Dayson (8) to the Poisson (7).

Another important difference is that the statistics of the classical recurrences is numerically achievable. Contrary to that, the quantum recurrences take place on the N dimensional torus that corresponds to the dynamics of N harmonic oscillators, where $N \gg 1$, since Zaslavsky's conjecture is valid for the semiclassical limit $\tilde{h} \sim 1/N \ll 1$. In this case, numerical observation of quantum recurrences is impossible, since the probability that $N \gg 1$ phases $E_k\tau$ are simultaneously equitable, for example, in the interval $(0, \theta < \pi/2)$ modulo 2π is extremely small and τ of QRs are prohibitive times, and their numerical observation is irrelevant.

Acknowledgements: This work on the statistics of the quantum recurrences in the Hilbert space was initiated by Prof. G.M. Zaslavsky to whose memory this result is dedicated with my deep gratitude and respect. It is my great pleasure to thank Prof. S. Fishman for very informative and instructive discussions. This research was supported by the Israel Science Foundation (ISF).

Appendix A: Few properties of almost periodic functions

Important part of our analysis is application of a theory of almost periodic functions. The theory of almost periodic functions is well developed and its main features has been created in the third decade of 20th century, see monograph by A.S. Besicovitch [23]. In this section, we present some basic ideas of the theory, which are used in the present analysis. We follow Ref. [23].

- *Definition 1*: A set $\mathcal{E}$ of real numbers is said to be *relatively dense* (*r.d.*) if there exists a number $l > 0$ such that any interval of length l contains at least one number of $\mathcal{E}$. Any such number is called an inclusion interval of the set $\mathcal{E}$.

  Thus, the sets of numbers $\pm n$, or $\pm\sqrt{n}$, where n takes all the positive integer values, are both *r.d.* On the other hand, neither of the sets of all positive numbers, of all prime numbers $\pm p_n$ is *r.d.*

- *Definition 2*:

  Let $f(t)$ be a real or complex function defined for all real values of t. A number τ is called a translation number of $f(t)$ belonging to $\epsilon \geq 0$ if

  $$\sup_{-\infty<t<\infty}|f(t+\tau) - f(t)| \leq \epsilon\,.$$

We admit here the complete correspondence of this definition with the definition of the distance in the Hilbert space in Eq. (4) and the QR in Eq. (5). The following properties of translation numbers are followed from the above definition

- *Property (i)*: A translation number belonging to ϵ, belongs also to any $\epsilon' > \epsilon$.
- *Property (ii)*: If τ is a translation number belonging to ϵ, then so is $-\tau$.
- *Property (iii)*: If $\tau_1,\ \tau_2$ are translation numbers belonging respectively to $\epsilon_1,\ \epsilon_2$, then $\tau_1 \pm \tau_2$ is a translation number belonging to $\epsilon_1 + \epsilon_2$.

The set of all translation numbers of a function $f(t)$ belonging to ϵ is denoted by $\mathcal{E}\{\epsilon, f(t)\}$. From *Property (i)* follows that

$$\mathcal{E}\{\epsilon, f(t)\} \subset \mathcal{E}\{\epsilon', f(t)\} \qquad \text{(A. 1)}$$

for any $\epsilon' > \epsilon$.

- *Definition 3*: A continuous function $f(t)$ is called *uniformly almost periodic* (u.a.p.) if for any $\epsilon > 0$ the set $\mathcal{E}\{\epsilon, f(t)\}$ is r.d.
- *Corollary*: A uniformly convergent series $\sum_{n=1}^{\infty} a_n e^{i\lambda_n t}$, where $\lambda_1,\ \lambda_2\,,\ldots$ are real, is a u.a.p. function.

This Corollary expresses an important relation between a class u.a.p. functions and the evolution of a wave functions in quantum mechanics $\Psi(t) = \sum a_n e^{iE_n t}\psi_n$, where E_n is the spectrum with corresponding eigenfunctions ψ_n.

The following properties of a translation function establishes the complete relation between the dynamics of wave functions in the Hilbert space and u.a.p. functions. This willbe relates to the distance (4) and (5).

- *The translation function* $v_f(\tau)$ of a u.a.p. function $f(t)$ is defined by equation

  $$v_f(\tau) = \sup_{-\infty<t<\infty} |f(t+\tau) - f(t)|\,. \qquad \text{(A. 2)}$$

Evidently, the set $\mathcal{E}\{\epsilon, f(t)\}$ is identical with the set $\mathcal{E}\{v_f(\tau) \leq \epsilon\}$ of values of τ for which $v_f(\tau) \leq \epsilon$. The function $v(\tau) \equiv v_f(\tau)$ satisfies the following conditions:

(a) $v(\tau) \geq 0\,, \quad v(0) = 0,$

(b) $v(-\tau) = v(\tau),$

(c) $v(\tau_1 + \tau_2) \leq v(\tau_1) + v(\tau_2),$

(d) $v(\tau$ is u.a.p.

Any function $v(\tau)$ satisfying the conditions (a),(b),(c),(d) is a translation function of a u.a.p. function.

The following theorems will be helpful to provide the main results of the present research.

- *Theorem 1*: Given a u.a.p. function $f(t) \sim \sum_{n=1}^{\infty} A_n e^{i\Lambda_n t}$, to any positive integer N and a positive number $\delta < \pi$ corresponds a positive ϵ such that all numbers τ of the set $\mathcal{E}\{\epsilon, f(t)\}$ satisfy the following Diophantine inequalities

  $$|\Lambda_n\tau - 2\pi k| < \delta\,, \qquad (n = 1, 2\ldots, N) \quad \text{(A. 3)}$$

  Evidently the inequality (A. 3), or $|\Lambda_n\tau| < \delta(\mathrm{mod}2\pi)$ is equivalent to the ordinary inequality

  $$|e^{i\Lambda_n\tau} - 1| < |e^{i\delta} - 1| = \delta_1\,. \qquad \text{(A. 4)}$$

- *Theorem 2*: Given a u.a.p. function $f(t) \sim \sum_{n=1}^{\infty} A_n e^{i\Lambda_n t}$, to any $\epsilon > 0$ corresponds a positive integer N and a positive $\delta < \pi$ such that any number τ satisfying the N Diophantine inequalities

$$|\Lambda_n \tau| < \delta(\text{mod} 2\pi)\,, \quad (n = 1, 2 \ldots, N) \tag{A. 5}$$

belong to $\mathcal{E}\{\epsilon, f(t)\}$.

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