Generalization of Granger Causality in Continuous Time

LJILJANA PETROVIĆ
Faculty of Economics
University of Belgrade
Department of Mathematics and Statistics
Kamenička 6, Belgrade
SERBIA
petrovl@ekof.bg.ac.rs

Abstract: The paper considers a statistical concepts of causality in continuous time between flows of information and between stochastic processes which is based on Granger's definitions of causality. More precisely, we will see how conditional orthogonality and conditional independence can serve as a basis for a general probabilistic theory of causality for both stochastic processes and single events. These results are motivated by causality relationship between filtrations "$(\mathcal{G}_t)$ is a cause of $(\mathcal{E}_t)$ within $(\mathcal{F}_t)$" and which is based on Granger's definition of causality. Also, we consider causality relationships between σ-fields (filtrations) associated by stopping times, which are applicable to the stopped processes (see Petrović et al. 2016). Then we give some basic properties of causality up to some stopping time.

Key–Words: Hilbert space, filtration, causality, stopping time, stopped process

1 Introduction

Many scientific studies focus on finding causal relationships between observed processes. Often this cannot be done by experiments and researchers are restricted to observe the systems that they want to describe. This is the case in many fields, for example in economics, demography and etc. Granger-causality is one of the most popular measure to reveal causality influence of time series widely applied in economics, demography, neuroscience etc. The study of Granger–causality has been mainly preoccupied with time series. We shall instead concentrate on continuous time processes. Many of systems to which it is natural to apply tests of causality, take place in continuous time. For example, this is generally the case within economy. For those systems, it may be difficult to use a discrete time model.

The authors of first papers in which we can find definitions of causality in continuous time in term of Hilbert spaces, i.e. in L^2-framework, were Gill and Petrović (1987) and Petrović (1989, 1996). Also, definition of causality was given in continuous time, but given in terms of σ-algebras, i.e. natural filtrations of stochastic processes were Mykland (1986) and Florens and Fougères (1996). Recently, there have been several papers which deal with these themes, among other in Aalen and Frigressi (2007), Commenges and Gégout-Petit (2009), Gégout-Petit and Commenges (2010).

The paper is organized as follows. In Section 2 we present different concepts of causality between flows of information that are represented by families of Hilbert spaces. Also, we develop concept of causality for stochastic process with continuous time parameter, using conditional independence among observed filtrations, we work in σ-algebraic framework.

The main results are given in Section 3. In this section we give some basic properties of the stopping times, stopped filtrations and stopped processes. Then, we introduce the definition of statistical causality associated to some stopping time and give some properties of the concept of causality up to some stopping time.

2 Causality between families of Hilbert Spaces and between Filtrations

In the first part of this section we give various concepts of causality relationship between flow of informations (represented by families of Hilbert spaces).

Causality concepts expressed in terms of conditional orthogonality in Hilbert spaces of square integrable random variables were studied by Hosoya

(1977), Florens and Mouchart (1985). In the papers of Florens and Mouchart (1982), Mykland (1986), Gill and Petrović (1987), Petrović (1989, 1996) it is shown how conditional orthogonality can serve as a basis for a general probabilistic theory of causality for both processes and single events.

Let $\mathcal{F}$ be a Hilbert space whose inner product is defined by $(\cdot,\cdot)$. For arbitrary subspaces F_1 and F_2 of $\mathcal{F}$ (all subspaces are taken to be closed), $F_1 \perp F_2$ means that F_1 and F_2 are orthogonal. The orthogonal projection of $x \in F_1$ onto F_2 is denoted by $P(x|F_2)$ and $P(F_1|F_2)$ will denote the orthogonal projection of of F_1 onto F_2 and $F_1 \ominus F_2$ will denote a Hilbert space generated by all elements $x - P(x|F_2)$, where $x \in F_1$. If $F_2 \subseteq F_1$, then $F_1 \ominus F_2$ coincides with $F_1 \cap F_2^{\perp}$, where $F_2^{\perp}$ is the orthogonal complement of F_2 in $\mathcal{F}$; i.e. $H_2^{\perp} = \mathcal{H} \ominus H_2$.

Definition 1. *If F_1 and F_2 are arbitrary subspaces of Hilbert space $\mathcal{F}$, then it is said that X is* **splitting** *for F_1 and F_2 or that F_1 and F_2 are* **conditionally orthogonal** *given X (and written as $F_1 \perp F_2|X$) if*

$$(1) \qquad F_1 \ominus X \perp H_2 F \ominus X,$$

or, equivalently,

$$(x_1, x_2) = (P(x_1|X), P(x_2|X)) \textit{ for all } x_1 \in F_1, x_2 \in F_2.$$

When X is trivial, i.e. $X = \{0\}$, this reduces to the usual orthogonality $F_1 \perp F_2$.

The notion of splitting was first given in [19].

Let $\mathbf{F} = (F_t), t \in \mathbf{R}$ be a family of Hilbert spaces. We shall think about F_t as a basis for approximation an information available at time t, or as a basis for approximation current information. Total information $F_{<\infty}$ carried by $\mathbf{F}$ is defined by $F_{<\infty} = \vee_{t\in R} F_t$, while past and future information of $\mathbf{F}$ at t is defined as $F_{\leq t} = \vee_{s\leq t} F_s$ and $F_{\geq t} = \vee_{s\geq t} F_s$, respectively. It is to be understood that $F_{<t} = \vee_{s<t} F_s$ and $F_{>t} = \vee_{s>t} F_s$ do not have to coincide with $F_{\leq t}$ and $F_{\geq t}$ respectively; $F_{<t}$ and $F_{>t}$ are sometimes referred to as the real past and real future of $\mathbf{F}$ at t.

Analogous notation will be used for families $\mathbf{G} = (G_t)$ and $\mathbf{E} = (E_t)$.

Causality is, in any case, a prediction property and central question is: is it possible to reduce available information in order to predict a given stochastic process? Motivated by Granger nonlinear causality, we give the definition of causality via filtrations.

The intuitively plausible notion of causality between families of Hilbert spaces is given in Gill, Petrović (1987) and generalized in Petrović (1996).

Definition 2. *Let* $\mathbf{E}$, $\mathbf{G}$ *and* $\mathbf{F}$ *be arbitrary families of Hilbert spaces. It is said that* $\mathbf{G}$ **is a cause of** $\mathbf{E}$ *within* $\mathbf{F}$ *(and written as* $\mathbf{E} \,|\!< \mathbf{G}; \mathbf{F}$*) if* $E_{<\infty} \subseteq F_{<\infty}$, $\mathbf{G} \subseteq \mathbf{F}$ *and*

$$(2) \qquad E_{<\infty} \perp F_{\leq t} | G_{\leq t}$$

for each t.

The essence of (2) is that all information about $E_{<\infty}$ that gives $F_{\leq t}$ comes via $G_{\leq t}$ for arbitrary t; equivalently, $G_{\leq t}$ contains all the information from the $F_{\leq t}$ needed for predicting $E_{<\infty}$. It is s equivalent to $E_{<\infty} \perp F_{\leq t} \bigvee G_{\leq t} | G_{\leq t}$. The last relation means that condition $\mathbf{G} \subseteq \mathbf{F}$ does not represent essential restriction. Intuitively, $\mathbf{E} \,|\!< \mathbf{G}; \mathbf{F}$ means that, for arbitrary t, information about $E_{<\infty}$ provided by $F_{\leq t}$ is not "bigger" than that provided by $G_{\leq t}$.

If $\mathbf{G}$ and $\mathbf{F}$ are such that $\mathbf{G} \,|\!< \mathbf{G}; \mathbf{F}$, we shall say that $\mathbf{G}$ is its own cause within $\mathbf{F}$ (compare with [24]). It should be mentioned that the notion of subordination (as introduced in [36]) is equivalent to the notion of being one's own cause, as defined here.

If $\mathbf{G}$ and $\mathbf{F}$ are such that $\mathbf{G} \,| < \mathbf{G}; \mathbf{G} \bigvee \mathbf{F}$ (where $\mathbf{G} \bigvee \mathbf{F}$ is a family determined by $(G \bigvee F)_t = G_t \bigvee F_t$), we shall say that $\mathbf{F}$ does not cause $\mathbf{G}$. It is clear that the interpretation of Granger–causality is now that $\mathbf{F}$ does not cause $\mathbf{G}$ if $\mathbf{G} \,|\!< \mathbf{G}; \mathbf{G} \bigvee \mathbf{F}$ (see [24]). Without difficulty, it can be shown that this term and the term " $\mathbf{F}$ does not anticipate $\mathbf{G}$ " (as introduced in [37]) are identical.

Definition 2 can be applied to stochastic processes: it will be said that stochastic processes are in a certain relationship if and only if the Hilbert spaces they generate are in this relationship. So, from Definition 2 it follows that stochastic process $\mathbf{Y}$ is a cause of a process $\mathbf{X}$ within process $\mathbf{Z}$ relative to P if $F^X_{<\infty} \subseteq F^Z_{<\infty}$, $\mathbf{F}^{\mathbf{Y}} \subseteq \mathbf{F}^{\mathbf{Z}}$ and if $F^X_{<\infty}$ and $F^Z_{\leq t}$ are conditionally orthogonal of given F^Y_t for each t, i.e.

$$F^X_{<\infty} \perp F^Z_{\leq t} | F^Y_{\leq t} \text{ for each } t.$$

In the remaining part of this section we give a concept of causality for stochastic process with continuous time parameter, using conditional independence among observed filtrations, we work in σ-algebraic framework. The benefit of this approach is to obtain a theory invariant not only to linear transformation of the variables but also to any change of coordinates and theory which easily can deal with nonlinear transformations.

A probabilistic model for a time–dependent system is described by $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $(\Omega, \mathcal{F}, P)$ is a probability space and $\{\mathcal{F}_t,\ t \in I\}$ is a "framework" filtration, $\mathcal{F}_t$ is a set of all events in the model up to and including time t and $\mathcal{F}_t$ is a subset of $\mathcal{F}$. $\mathcal{F}_\infty$ is the smallest $\sigma-$algebra containing all the $\mathcal{F}_t$ (even if $\sup I < +\infty$), $\mathcal{F}_\infty = \bigvee_{t \in I} \mathcal{F}_t$. We suppose that the filtration $\{\mathcal{F}_t\}$ satisfies the "usual conditions", which means that $\{\mathcal{F}_t\}$ is right continuous and each $\mathcal{F}_t$ is complete. Analogous notation will be used for filtrations $\mathbf{E} = \{\mathcal{E}_t\}$ and $\mathbf{G} = \{\mathcal{G}_t\}$. It is said that filtration $\mathbf{G}$ is a subfiltration of $\mathbf{F}$ and written as $\mathbf{G} \subseteq \mathbf{F}$, if $\mathcal{G}_t \subseteq \mathcal{F}_t$ for each t.

Definition 3. (compare with (Rozanov, 1977). Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ arbitrary subσ-algebras from $\mathcal{F}$. It is said that $\mathcal{G}$ is **splitting** for $\mathcal{F}_1$ and $\mathcal{F}_2$ or that $\mathcal{F}_1$ and $\mathcal{F}_2$ are **conditionally independent** given $\mathcal{G}$ (and written as $\mathcal{F}_1 \perp \mathcal{F}_2 | \mathcal{G}$) if

$$(\forall A_1)\,(A_1 \in \mathcal{F}_1)(\forall A_2)\,(A_2 \in \mathcal{F}_2)\ P(A_1 A_2|\mathcal{G}) = P(A_1|\mathcal{G})P(A_2|\mathcal{G}).$$

We now give a definition of causality formulated in terms of σ-algebras (filtrations), which is analogous to the Definition 2 formulated in terms of Hilbert spaces.

Definition 4. *Let* $\mathbf{F} = \{\mathcal{F}_t\}$*,* $\mathbf{G} = \{\mathcal{G}_t\}$ *and* $\mathbf{E} = \{\mathcal{E}_t\}$*,* $t \in I$*, be filtrations on the same probability space. It is said that* **G is a cause of E** *within* $\mathbf{F}$ *relative to* P *(and written as* $\mathbf{E} \,|\!<\, \mathbf{G}; \mathbf{F}; P$*) if* $\mathcal{E}_\infty \subseteq \mathcal{F}_{<\infty}$*,* $\mathbf{G} \subseteq \mathbf{F}$ *and if* $\mathcal{E}_{<\infty}$ *is conditionally independent of* $\mathcal{F}_t$ *given* $\mathcal{G}_t$ *for each t,*

$$\mathcal{E}_{<\infty} \perp \mathcal{F}_t | \mathcal{G}_t$$

i.e.

$$(\forall t \in I)(\forall A \in \mathcal{E}_{<\infty})\ P(A|\mathcal{F}_t) = P(A|\mathcal{G}_t).$$

If there is no doubt about P, we omit "relative to P".

Intuitively, $\mathbf{E} \,|\!<\, \mathbf{G}; \mathbf{F}$ means that, for arbitrary t, information about $\mathcal{E}_{<\infty}$ provided by $\mathcal{F}_t$ is not "bigger" than that provided by $\mathcal{G}_t$.

Gégout-Petit and Commenges (2010) use conditional independence of filtrations to establish some causality relations, too. In their terminology causality relationship $\mathbf{F}^X \,|\!<\, \mathbf{F}^X; \mathbf{F}^{X,Z}$ would be interpreted as $(\mathcal{F}_t^X)$ is filtration-based strong local independent of filtration $(\mathcal{F}_t^Z)$.

A family of σ-algebras (filtrations) induced by a stochastic process $X = \{X_t, t \in I\}$ is given by $\mathbf{F}^X = \{\mathcal{F}_t^X, t \in I\}$, where $\mathcal{F}_t^X = \sigma\{X_u, u \in I, u \le t\} \vee \mathcal{N}$ being the smallest σ-algebra with respect to which all random variables X_u, $u \le t$, are measurable. The process $X = \{X_t\}$ is $(\mathcal{F}_t)$-adapted if $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for each t.

Definition 4 can be applied to stochastic processes. It will be said that stochastic processes are in a certain relationship if we were talking about the corresponding filtrations. Specially, $(\mathcal{F}_t)-$adapted stochastic process $X = \{X_t\}$ is its own cause if $\mathbf{F}^\mathbf{X} = (\mathcal{F}_\mathbf{t}^\mathbf{X})$ is its own cause within $\mathbf{F} = (\mathcal{F}_\mathbf{t})$ if $\mathbf{F}^\mathbf{X} \,|\!<\, \mathbf{F}^\mathbf{X}; \mathbf{F}; \mathbf{P}$.

Remark. The condition of Granger causality is actually a condition of transitivity largely used in sequential analysis (in statistics), see (Bahadur, 1954) and (W.J. Hall, R.A. Wijsman, J.K. Gosh, 1965).

3 Causality between Stopped Processes

We now extend Definition 4 from fixed times to stopping times, i.e. we give characterization of causality using σ-field associated to stopping times. This generalization involves stopping times – a class of random variables that plays the essential role in the Theory of Martingales (for details see Elliot, 1982).

Let us briefly recall some basics about stopping times and σ-algebras.

- We say that $T : \Omega \to R \cup \{\infty\}$ is stopping time with respect to filtration $\mathbf{F}$, provided that $\{\omega \mid T(\omega) \le t\} \in \mathcal{F}_t$, for all t.
- $\mathcal{F}_T = \{A \in \mathcal{F} \mid A \cap \{T \le t\} \in \mathcal{F}_t, \text{for all } t\}$ is σ-field and intuitively $\mathcal{F}_T$ is the information available at time T.
- If S and T are stopping times with respect to the filtration $\mathbf{F}$, then $S \wedge T$ is a stopping time with respect to the filtration $\mathbf{F}$, too. Specially, if T is a stopping time and t some real number, then $t \wedge T$

defined by

$$t \wedge T(\omega) = \min(t, T) = \begin{cases} T(\omega), & T(\omega) < t \\ t, & T(\omega) \geq t \end{cases}$$

is a stopping time.

- If S and T are stopping times such that $S \leq T$ then $\mathcal{F}_S \subseteq \mathcal{F}_T$, and as a consequence we get that $\mathcal{F}_{s\wedge T} \subseteq \mathcal{F}_{t\wedge T}$ for all $s < t$.

In many situations we observe some systems up to some random time, for example till the time when something happens for the first time. For a process X, we set $X_T(\omega) = X_{T(\omega)}(\omega)$, whenever $T(\omega) < +\infty$. We define the stopped process $X^T = \{X_{t\wedge T}, t \in I\}$ with

$$X_t^T(\omega) = X_{t\wedge T(\omega)}(\omega) = X_t 1_{\{t<T\}} + X_T 1_{\{t\geq T\}}.$$

Theorem 1. (see Knill, 2009) If process X is progressively measurable with respect to the filtration $\mathbf{F} = \{\mathcal{F}_t\}$ and T is an $(\mathcal{F}_t)$-stopping time, then the stopped process X^T is progressively measurable with respect to the filtration $\mathbf{F}^T = \{\mathcal{F}_{t\wedge T}\}$.

The aim of this paper is to give some properties of the concept of causality for the stopped processes as a generalization of the concept given by the Definition 4. More precisely, we define the concept of causality for the stopped (progressively measurable) process X^T using the stopped filtration $\mathbf{F}^T = \{\mathcal{F}_{T\wedge t}\}$, i.e. using the σ-algebras associated to stopping times. Since T is a $(\mathcal{F}_t)$-stopping time and filtration $\mathbf{F}$ is right continuous, we have that $\mathcal{F}_{(t\wedge T)+} = \mathcal{F}_{t\wedge T}$, i.e. the filtration $\mathcal{F}_{t\wedge T}$ is right continuous, too.

The following definition is a generalization od Definition 4 from fixed time to stopping time, i.e. it gives causality between filtrations $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{E}$ up to stopping time T.

Definition 5. (see Petrović et al. 2016) Let $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$ and $\mathbf{E} = \{\mathcal{E}_t\}, t \in I$, be given filtrations on the probability space $(\Omega, \mathcal{F}, P)$ and let T be a stopping time relative to filtration $\mathbf{E}$. It is said that filtration $\mathbf{G}$ entirely causes $\mathbf{E}$ within $\mathbf{F}$ relative to P up to stopping time T or that filtration $\mathbf{G}^T$ entirely causes $\mathbf{E}^T$ within $\mathbf{F}^T$ relative to P (and written as $\mathbf{E}^T \mid\!< \mathbf{G}^T; \mathbf{F}^T; P$) if $\mathbf{E}^T \subseteq \mathbf{F}^T$, $\mathbf{G}^T \subseteq \mathbf{F}^T$ and if $\mathcal{E}_T$ is conditionally independent of $\mathcal{F}_{t\wedge T}$ given $\mathcal{G}_{t\wedge T}$ for each t, i.e.

$$\mathcal{E}_T \perp \mathcal{F}_{t\wedge T} | \mathcal{G}_{t\wedge T},$$

or, equivalently,

$$(\forall t \in I)(\forall A \in \mathcal{E}_T) \quad P(A|\mathcal{F}_{t\wedge T}) = P(A|\mathcal{G}_{t\wedge T}).$$

From the following result it follows that the relationship "being its own cause" for filtrations associated to stopping times is transitive relationship.

Theorem 1 *Let $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$, $\mathbf{E} = \{\mathcal{E}_t\}$ be filtrations on the probability space $(\Omega, \mathcal{F}, P)$. If T is a stopping time relative to $\mathbf{E}$, then from*

$$\mathbf{E}^T \mid\!< \mathbf{E}^T; \mathbf{G}^T; P \quad \text{and} \quad \mathbf{G}^T \mid\!< \mathbf{G}^T; \mathbf{F}^T; P,$$

it follows that

$$\mathbf{E}^T \mid\!< \mathbf{E}^T; \mathbf{F}^T; P.$$

If relationship "being one's own cause" holds up to stopping time T and if S is another stopping time such that $S \leq T$, it is natural to expect that the same relationship will hold up to stopping time S, as is shown in the next theorem.

Theorem 2 *Let $\mathbf{F} = \{\mathcal{F}_t\}$ and $\mathbf{G} = \{\mathcal{G}_t\}$ be filtrations on the probability space $(\Omega, \mathcal{F}, P)$ such that $\mathbf{G} \subseteq \mathbf{F}$ and let T and S be a two stopping times relative to $\mathbf{G}$, such that $S \leq T$. Then, from*

$$\mathbf{G}^T \mid\!< \mathbf{G}^T; \mathbf{F}^T; P \quad \text{it follows} \quad \mathbf{G}^S \mid\!< \mathbf{G}^S; \mathbf{F}^S; P.$$

4 Conclusion - Some Applications

The study of Granger–causality has been mainly preoccupied with time series. In this paper we considered continuous time processes. Many of systems to which it is natural to apply tests of causality, take place in continuous time. For example, this is generally the case within economy, for example, in labor economics (see Heckman and Singer, 1984, Geweke, Marshall and Zarkin, 1986), in modern finance theory (see Merton, 1990 and Melino, 1994). In this case, it may be difficult to use a discrete time model. Also, the observed "causality" in a discrete time model may depend on the length of interval between each two successive samplings, as in the case with Granger–causality.

The given causality concept can be applied to regular solutions of stochastic differential equations. The equivalence between some models of causality and weak uniqueness (for weak solutions of stochastic differential equations) is shown in Petrović, Stanojević, 2010 and Petrović, Valjarević, 2014.

The given concept of causality is related to the *orthogonality* of martingales and local martingales in Valjarević, Petrović, 2012. This connection is considered for the stopped local martingales, too.

Petrović, D.Valjarević, 2012, considered a stable subspaces of H^p, which contains the right continuous uniformly integrable $(\mathcal{F}_t; P)$-martingales and the necessary and sufficient conditions, in terms of statistical causality, for these spaces to coincide with H^p are given.

Some special cases of given causality concept links Granger–causality with **adapted distribution**. Some results are given in paper Petrović, S. Dimitrijević, 2011.

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