Factorial Designs as a Tool for Fuzzy Sensitivity Analysis Problems

ZDENĚK KALA
Department of Structural Mechanics
Brno University of Technology, Faculty of Civil Engineering
Veveří Str. 95, Brno
CZECH REPUBLIC
kala.z@fce.vutbr.cz; http://www.vutbr.cz

Abstract: - The paper is concentrated on the interval sensitivity analysis which can be useful for model-building, and can be a profitable tool of deeper understanding complex physical processes. The method presented has a known number of combinations of endpoints necessary for computation. Sensitivity indices enable the identification both of main effects and higher order interactions effects in such types of problems in which the uncertainty of input and output parameters is expressed by intervals. The presented strategy of calculation of sensitivity indices is an appropriate completion of such types of computation technologies which are tolerant of imprecision, uncertainty, partial truth, and approximation.

Key-Words: Fuzzy sets, sensitivity analysis, interval, interaction, model, stochastic, uncertainty, Zadeh

1 Introduction

In the hitherto literature, the global sensitivity analysis of model outputs has been solved within the framework of probabilistic methods of modelling of uncertainties [1, 2]. In the situations when uncertain parameters of model outputs have other than probabilistic descriptions of uncertainties (including interval analysis, convex function modeling and fuzzy variables), alternative frameworks for the uncertainty become relevant, see, e.g., [3-6]. For the evaluation of global sensitivity analysis, alternative solutions for the uncertainty analysis require the finding of alternative approaches.

One of basic alternative approaches to mathematical description, and analysis of limit states of building structures is based on fuzzy sets [7]. Fuzzy numbers represent the appropriate alternative of random quantities, in particular for computation models of serviceability limit states. The fuzzy analysis can be an alternative of statistical analysis, in particular as far as the parameters of the structure are concerned the limit values of which are determined by a human in a subjective manner. The fuzzy analysis methods are known [8, 9], but their adaptability to analysis of serviceability limit states of building structures [10, 11] is low.

The problems in which mathematical methods describing stochastic and epistemic uncertainties are combined are still more complicated [12]. Computer simulations of influence of both uncertainty types on model outputs are justified for reliability analyses. Numerous new problems are to be solved, when it is necessary to quantify the influences of inputs on model outputs. The sensitivity measurements of combined uncertainties require the introduction of various measures of model weights in the context of input and output random quantities which are combined with fuzzy numbers [3]. The research work and modelling of combined uncertainties lead to precision of more and more sophisticated reliability analyses, and of their methods of sensitivity analyses which form their part, as well [13-18].

The presented paper describes the global sensitivity interval analysis which can be applied to within the framework of the analysis of fuzzy uncertainties.

2 Fuzzy Sets

In mathematics, fuzzy sets are sets the elements of which have degrees of membership [7]. The definition of a fuzzy variable $\tilde{x}$ is an uncertain subset of the fundamental set **X** [7].

$$\tilde{x} = \{x, \mu_{\tilde{x}}(x) | x \in \mathbf{X}\} \tag{1}$$

A normalized membership function $\mu_{\tilde{x}}(x)$ is defined as

$$0 \le \mu_{\tilde{x}}(x) \le 1 \ \ \forall x \in \mathbb{R} \tag{2}$$

$$\exists\, x_l,\, x_r \text{ with } \mu_{\tilde{x}}(x) = 1 \ \ \forall x \in [x_l; x_r] \tag{3}$$

A fuzzy variable $\tilde{x}$ is said to be convex if its membership function $\mu_{\tilde{x}}(x)$ monotonously decreases on both sides of the maximum value, i.e.,

$$\mu_{\tilde{x}}(x_{II}) \geq \min[\mu_{\tilde{x}}(x_I); \mu_{\tilde{x}}(x_{III})] \;\; \forall x_I, x_{II}, x_{III} \in \mathbb{R} \quad \text{with } x_I \leq x_{II} \leq x_{III} \tag{4}$$

Membership functions may be either continuous or discrete. Continuous membership functions are common in engineering applications. From all $\alpha \in (0,1]$ closed finite intervals $[{}^{\alpha l}x; {}^{\alpha r}x]$ may be extracted from a convex fuzzy variable $\tilde{x}$, where

$${}^{\alpha l}x = \min[x \in \mathbb{R} \,|\, \mu_{\tilde{x}}(x) \geq \alpha] \tag{5}$$

and

$${}^{\alpha r}x = \max[x \in \mathbb{R} \,|\, \mu_{\tilde{x}}(x) \geq \alpha] \tag{6}$$

The intervals $[{}^{\alpha l}x; {}^{\alpha r}x]$ are referred to as α-cut sets X_α. In practical applications, a finite number of α-cut sets X_{α_j} are used, where j=0, 1, 2, ..., m and $0 \leq \alpha_j \leq \alpha_{j+1} \leq 1$. If j=0, then the interval defined by the cut α_0 =0 is referred to as the support.

3 General Extension Principle

Zadeh's general extension principle is one of the most important tools in the theory of fuzzy sets. The general extension principle enables the transformation of an arbitrary operation in the classical set into an operation in the fuzzy sets. The general extension principle makes it possible to apply operations and functions originally defined as functions of real variables, to fuzzy sets. The general extension principle is an alternative mathematical basis for the mapping $\tilde{y} = f(\tilde{x}_1, \tilde{x}_2, ...)$ of fuzzy variables $\tilde{x}_1, \tilde{x}_2$ onto $\tilde{y}$. The fuzzy resulting variable $\tilde{y} = f(\tilde{x}_1, \tilde{x}_2, ...)$ obtained according to the general extension principle is

$$\tilde{y} = \{y, \mu_{\tilde{y}}(y) \,|\, y = f(x_1, x_2, ...); \; y \in \mathbf{Y}; (x_1, x_2, ...) \in \mathbf{X_1} \times \mathbf{X_2} \times ...\} \tag{7}$$

for sets $\mathbf{X_1}$, $\mathbf{X_2}$, ..., $\mathbf{Y}$ and the membership function

$$\mu_{\tilde{y}}(y) = \begin{cases} \sup \min_{y = f(x_1, x_2, ...)} [\mu_{\tilde{x}_1}(x_1), \mu_{\tilde{x}_2}(x_2), ...] \\ \quad\quad \text{for } \exists \quad y = f(x_1, x_2, ...) \\ 0 \quad\quad \text{otherwise} \end{cases} \tag{8}$$

The fuzzy outputs variable $\tilde{y}$ can be evaluated using the general extension principle and the so-called α-cuts, which unequivocally characterize each fuzzy set.

4 Fuzzy Sensitivity Analysis

The sensitivity analysis studies the effect of input variables on the output variable. If the input and output variables are fuzzy numbers then the interval arithmetic performed on individual α-cuts may also be used to perform the sensitivity analysis. The α-cut of variable , for i=1, 2, ...n, is characterized by the interval $[{}^{\alpha l}x_i; {}^{\alpha r}x_i]$. Fuzzy sensitivity analysis can be performed with the aim to estimate the effect of the size of the endpoints of interval $[{}^{\alpha l}x_i; {}^{\alpha r}x_i]$ of input fuzzy variable $\tilde{x}_i$ on the response $\tilde{y}$. The effect of each interval $[{}^{\alpha l}x_i; {}^{\alpha r}x_i]$ on interval $[{}^{\alpha l}y; {}^{\alpha r}y]$ can be evaluated using so-called factorial designs [19]. Each input interval $[{}^{\alpha l}x_i; {}^{\alpha r}x_i]$ can be assumed to take two possible values, called 'levels' (lower and upper endpoints of the interval), which are denoted as '+' (${}^{\alpha r}x_i$) or '-' (${}^{\alpha l}x_i$). The sensitivity analysis simulates all possible combinations of lower '-' and upper '+' endpoints of intervals on the α-cut. The computational cost is 2^n runs. Let us consider three fuzzy variables $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ and α-cut, which defines three intervals $[{}^{\alpha l}x_1; {}^{\alpha r}x_1]$, $[{}^{\alpha l}x_2; {}^{\alpha r}x_2]$, $[{}^{\alpha l}x_3; {}^{\alpha r}x_3]$. There exist k=2^3=8 combinations of all endpoints of input intervals. We denote the outcome of the i-th combination as ${}^{j}y$, with j=1, ..., k. The design matrix determines the k=8 endpoint combinations, see Table 1.

Table 1: A 2^3 output combinations.

Run *j*	1	2	3	4	5	6	7	8
x_1	-	-	-	-	+	+	+	+
x_2	-	-	+	+	-	-	+	+
x_3	-	+	-	+	-	+	-	+
x_1-x_2	+	+	-	-	-	-	+	+
x_1-x_3	+	-	+	-	-	+	-	+
x_2-x_3	+	-	-	+	+	-	-	+
x_1-x_2-x_3	-	+	+	-	+	-	-	+
${}^{j}y$	${}^{1}y$	${}^{2}y$	${}^{3}y$	${}^{4}y$	${}^{5}y$	${}^{6}y$	${}^{7}y$	${}^{8}y$

The design matrix listed in Table 1 is used to calculate the main and interactions effects. The difference in output values y_1 and y_2, e.g., arises solely from variations on the level of variable x_1, whereas the other two variables x_2, x_3 are kept the same. Four individual measures exist for x_1. The main effect of a variable is defined as the average effect of that variable over all conditions of other factors. The main effect C_1 of x_1 , e.g., is calculated as the average of four individual measures. Main effects C_2 and C_3 are determined analogously.

$$C_1=\frac{\left({}^2y-{}^1y\right)+\left({}^4y-{}^3y\right)+\left({}^6y-{}^5y\right)+\left({}^8y-{}^7y\right)}{4} \tag{9}$$

$$C_2=\frac{\left({}^3y-{}^1y\right)+\left({}^4y-{}^2y\right)+\left({}^7y-{}^5y\right)+\left({}^8y-{}^6y\right)}{4} \tag{10}$$

$$C_3=\frac{\left({}^5y-{}^1y\right)+\left({}^6y-{}^2y\right)+\left({}^7y-{}^3y\right)+\left({}^8y-{}^4y\right)}{4} \tag{11}$$

The interaction between x_1-x_2 is evaluated by coefficient $C_{1\text{-}2}$, the interaction between x_1-x_3 is evaluated by coefficient $C_{1\text{-}3}$ etc. Coefficient $C_{1\text{-}2\text{-}3}$ is determined using the sign combination, which has not been previously used.

$$C_{1\text{-}2}=\frac{{}^1y+{}^2y-{}^3y-{}^4y-{}^5y-{}^6y+{}^7y+{}^8y}{4} \tag{12}$$

$$C_{1\text{-}3}=\frac{{}^1y-{}^2y+{}^3y-{}^4y-{}^5y+{}^6y-{}^7y+{}^8y}{4} \tag{13}$$

$$C_{1\text{-}4}=\frac{{}^1y-{}^2y-{}^3y+{}^4y+{}^5y-{}^6y-{}^7y+{}^8y}{4} \tag{14}$$

$$C_{1\text{-}2\text{-}3}=\frac{-{}^1y+{}^2y+{}^3y-{}^4y+{}^5y-{}^6y-{}^7y+{}^8y}{4} \tag{15}$$

5 Conclusion

The sensitivity analysis is a diagnostic tool that can guide the model calibration and verification, support the prioritization of efforts for uncertainty reduction, or help with model-based decision-making [20, 21]. The described sensitivity analysis has its fundamental application to the study of subjective information described by the fuzzy numbers which can be elaborated using the general extension principle. The working out of the sensitivity analysis methods which make possible the study of higher order interaction effects, is, as the fuzzy analysis is concerned, always insufficient in comparison with stochastic approaches [2]. The advantage of fuzzy sensitivity analysis is the fact that the necessary number of combinations of endpoints of intervals is known in advance. Stochastic methods of sensitivity analysis require the application of high number of simulation runs. If the number of simulations runs is too low to cover the input space, the analysis may not provide reliable results. On the other hand, very high numbers of simulation runs can improve the result of sensitivity analysis only very little. These problems can be very significant when studying the complicated nonlinear computation models, very demanding on the CPU time. The sequence of sensitivity coefficients is the output of the sensitivity analysis. If we have in mind the global sensitivity analysis, there are pairs and groups of three, not only independent factors in the sequence of dominant influences. The sequence describes the order of input factors according to their relative influence on the model outputs. Commonly, it is used to improve our knowledge on the model for identification of dominant controls of the model behaviour.

Recently, the sensitivity analysis finds more and more significant application to the such scientific branches where it was used only sporadically before. The sensitivity analysis is, e.g., one of interesting directions of research concerning the fuzzy linear programming. The linear programming is a branch of optimization. It solves the problem of finding the minimum (and/or maximum) of linear function of n variables on the set described by a system of nonlinear inequalities. The first work concentrated on the study and development of methods of sensitivity analysis of fuzzy problems of linear programming was [22]. The membership functions are basic fuzzy objective functions describing the uncertainty of membership of a numerical value into a set. Although there exist numerous types of membership functions, the most spread form of membership function is, in fuzzy linear programming, the linear form. The above described global fuzzy sensitivity analysis can be applied to problems which are solved using the linear programming. The intervals of values of linear fuzzy numbers are considered as parameters.

Acknowledgement
The article was elaborated within the framework of project GAČR 14-17997S and project FAST-S-16-3779.

References:

[1] E. Borgonovo, E. Plischke, Sensitivity analysis: A review of recent advances, *European Journal of Operational Research*, Vol.248, No.3, 2016, pp. 869–887.

[2] A. Saltelli, M. Ratto, T. Andress, F. Campolongo, J. Cariboni, D. Gatelli, M. Saisana, S. Tarantola, *Global Sensitivity Analysis Guiding the Worth of Scientific Models*, New York: John Wiley and Sons, 2007.

[3] G. Greegar, C.S. Manohar, Global Response Sensitivity Analysis of Uncertain Structures, *Structural Safety*, Vol.58, 2016, pp. 94–104.

[4] H.U. Koyluoglu, A.S. Cakmak, S.R.K. Nielsen, Interval Algebra to Deal with Pattern Loading and Structural Uncertainties, *Journal of Engineering Mechanics*, Vol.121, 1995, pp. 1149–1157.

[5] S.S. Rao, L. Berke, Analysis of uncertain structural systems using interval analysis, *AIAA Journal*, Vol.35, No.4, 1997, pp. 727–735.

[6] R.S. Langley, Unified Approach to Probabilistic and Possibilistic Analysis of Uncertain Systems, *Journal of Engineering Mechanics*, Vol.126, No.11, 2000, pp. 1163–1172.

[7] B. Möller, U. Reuter, *Uncertainty Forecasting in Engineering*, Springer Press, 2007, p.202, ISBN 978–3–540–37173–1.

[8] B. Farhadinia, Sensitivity Analysis in Interval-valued Trapezoidal Fuzzy Number Linear Programming Problems, *Applied Mathematical Modelling*, Vol.38, No.1, 2014, pp. 50–62.

[9] P. Gupta, D. Bhatia, Sensitivity Analysis in Fuzzy Multiobjective Linear Fractional Programming Problem, *Fuzzy Sets and Systems*, Vol.122, No.2, 2001, pp. 229–236.

[10] Z. Kala, Global Sensitivity Analysis in Stability Problems of Steel Frame Structures, *Journal of Civil Engineering and Management*, Vol.22, No.3, 2016, pp. 417–424.

[11] Z. Kala, Reliability Analysis of the Lateral Torsional Buckling Resistance and the Ultimate Limit State of Steel Beams, *Journal of Civil Engineering and Management*, Vol.21, No.7, 2015, pp. 902–911.

[12] Z. Kala, A. Omishore, Fuzzy Stochastic Approaches for Analysing the Serviceability Limit State of Load Bearing Systems, *International Journal of Mathematics and Computers in Simulation*, Vol.10, 2016, pp. 294–301.

[13] Z. Kala, J. Kala, Sensitivity Analysis of Stability Problems of Steel Structures using Shell Finite Elements and Nonlinear Computation Methods, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.4, No.3, 2009, pp. 105-114.

[14] Z. Kala, Influence of Partial Safety Factors on Design Reliability of Steel Structures - Probability and Fuzzy Probability Assessments, *Journal of Civil Engineering and Management*, Vol.13, No.4, 2007, pp. 291-296.

[15] Z. Kala, Sensitivity and Reliability Analyses of Lateral-torsional Buckling Resistance of Steel Beams, *Archives of Civil and Mechanical Engineering*, Vol.15, No.4, 2015, pp. 1098-1107.

[16] F. Fedorik, J. Kala, A. Haapala, M. Malask, Use of Design Optimization Techniques in Solving Typical Structural Engineering Related Design Optimization Problems, *Structural Engineering and Mechanics*, Vol.55, No.6, 2015, pp. 1121-1137.

[17] P. Kral, J. Kala, P. Hradil, Verification of the Elasto-plastic Behavior of Nonlinear Concrete Material Models, *International Journal of Mechanics*, Vol.10, 2016, pp. 175-181.

[18] F. Hokes, J. Kala, O. Krnavek, Nonlinear Numerical Simulation of a Fracture Test with use of Optimization for Identification of Material Parameters, *International Journal of Mechanics*, Vol.10, 2016, pp. 159-166.

[19] G.E.P. Box, W.G. Hunter, J.S. Hunter, *Statistics for Experiment. An Introduction to Design*, Data Analysis and Model Building, Wiley, New York (1978).

[20] J. Antucheviciene, Z. Kala, M. Marzouk, E.R. Vaidogas, Solving Civil Engineering Problems by Means of Fuzzy and Stochastic MCDM Methods: Current State and Future Research, *Mathematical Problems in Engineering*, Vol.2015, 2015, Article number 362579.

[21] J. Antucheviciene, Z. Kala, M. Marzouk, E.R. Vaidogas, Decision Making Methods and Applications in Civil Engineering, *Mathematical Problems in Engineering*, Vol.2015, 2015, Article number 160569.

[22] H. Hamacher, H. Leberling, H.-J. Zimmermann, Sensitivity Analysis in Fuzzy Linear Programming, *Fuzzy Sets and Systems*, Vol.1, No.4, 1978, pp. 269-281.