The Nash equilibrium point in the LQ game on positive systems with two players

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Abstract: We consider the linear quadratic differential games for positive linear systems with the feedback information structure and two players. The Newton method to obtain the stabilizing solution of a corresponding set of Riccati equations is presented in the literature. Here, we modify the Newton method and propose a new faster iterative method. Moreover, the convergence properties of the modification are investigated and the sufficient condition to apply the modification is derived. The performances of the proposed algorithm are illustrated on some numerical examples.

Key–Words: feedback Nash equilibrium, generalized Riccati equation, stabilizing solution, nonnegative solution.

1 Introduction

The problem to compute the stabilizing nonnegative solution to the set of Riccati equation is a important problem with many practical applications. Our investigation is motivated from the paper of Azevedo-Perdicoulis and Jank [2], where the problem of finding a deterministic feedback Nash equilibrium for a two player infinite-horizon linear-quadratic differential game is studied. This equilibrium is defined as a pair of linear time-invariant state feedback strategies stabilizing the closed-loop system. However, the considered game is studied on positive systems and players's strategies are presented via the stabilizing solution of the associated coupled set of Riccati equations. Positive systems have attracted much attention as applications in economics [1, 9], and financial modelling [7].

Let us introduce some notations we are used in the paper. $\mathcal{R}^{n\times s}$ stands for $n \times s$ real matrices. The inequality $X \geq 0$ $(X > 0)$ means that all elements of the matrix (or vector) X are real nonnegative (positive) and we call the matrix X nonnegative (positive). For the matrices $A = (a_{ij}), B = (b_{ij})$, we write $A \geq B (A > B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ hold for all indexes i and j. The notation $\mathbf{X} \geq \mathbf{Y}$ with $\mathbf{X} = (X_1, \dots, X_N)$ means that $X_i \geq Y_i\, , i = 1, \dots, N$. A matrix A is called asymptotically stable (or Hurwitz) if the eigenvalues of A have a negative real part. A symmetric matrix A is called positive definite (semidefinite) matrix if all eigenvalues are positive (nonnegative). An $n \times n$ matrix A is called a Z-matrix if it has nonpositive off-diagonal entries. Any Z-matrix A can be presented as $A = \alpha I - N$ with N being a nonnegative matrix, and it is called a nonsingular M-matrix if $\alpha > \rho(N)$, where $\rho(N)$ is the spectral radius of N. In addition, a matrix is called nonnegative (nonpositive) if all of its entries are nonnegative (nonpositive).

We introduce the following set of Riccati equations:

$$\begin{aligned} 0 = \mathcal{R}_1(X_1, X_2) &:= -A^T X_1 - X_1 A - Q_1 \\ &+X_1 S_1 X_1 - X_2 S_{12} X_2 \\ &+X_1 S_2 X_2 + X_2 S_2 X_1 \, , \\ 0 = \mathcal{R}_2(X_1, X_2) &:= -A^T X_2 - X_2 A - Q_2 \\ &+X_2 S_2 X_2 - X_1 S_{21} X_1 \\ &+X_2 S_1 X_1 + X_1 S_1 X_2 \, , \end{aligned} \tag{1}$$

where $A, Q_1, Q_2 \in \mathcal{R}^{n\times n}$ and Q_1, Q_2 are symmetric nonnegative matrices, and $-A$ is a Z-matrix.

The concept of a Nash equilibrium in games with feedback information structure has been introduced [3, 4]. Following their findings we refer that the deterministic feedback Nash equilibria are characterized by the solutions of a set of coupled algebraic Riccati equations with a stability property. The Newton method to calculate the nonnegative stabilizing solution of a coupled system of generalized algebraic matrix Riccati equations in a two-player linear-quadratic differential game with infinite time horizon is proposed in [2]. In Theorem 8 proved in their paper the convergence properties of the Newton method for a two-player differential game, where the information structure of each player is of a feedback patten are derived. In fact, the Newton method is considered to find the stabilizing solution to set (1). In addition, some interesting applications are developed, among them being the papers of [5, 6, 8].

The Newton method is given by the following set of recursive equations:

$$
\begin{array}{l}
-{A^{(k)}}^T X_1^{(k+1)} - X_1^{(k+1)} A^{(k)} \\
+W_{12}^{(k)} X_2^{(k+1)} + X_2^{(k+1)} {W_{12}^{(k)}}^T = Q_1^{(k)} , \\
-{A^{(k)}}^T X_2^{(k+1)} - X_2^{(k+1)} A^{(k)} \\
W_{21}^{(k)} X_1^{(k+1)} + X_1^{(k+1)} {W_{21}^{(k)}}^T = Q_2^{(k)} ,
\end{array}
\tag{2}
$$

where

$$
\begin{array}{l}
A^{(k)} = A - S_1 X_1^{(k)} - S_2 X_2^{(k)} , \\
W_{ij}^{(k)} = X_i^{(k)} S_j - X_j^{(k)} S_{ij} , \\
i, j = 1, 2; i \neq j , \\
Q_i^{(k)} = Q_i + X_i^{(k)} S_i X_i^{(k)} - \sum_{j\neq i} X_j^{(k)} S_{ij} X_j^{(k)} \\
+\sum_{j\neq i} [X_i^{(k)} S_j X_j^{(k)} + X_j^{(k)} S_j X_i^{(k)}] .
\end{array}
\tag{3}
$$

The execution of iteration (2) is required to solve a set of linear equations of the form in each step:

$$
\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}
\begin{pmatrix} vec(X_1^{(k)}) \\ vec(X_2^{(k)}) \end{pmatrix}
=
\begin{pmatrix} vec(Q_1^{(k)}) \\ vec(Q_2^{(k)}) \end{pmatrix}
$$

with

$$
\begin{cases}
L_{11} = L_{22} = -I_n \otimes {A^{(k)}}^T - {A^{(k)}}^T \otimes I_n \\
L_{12} = -I_n \otimes {W_{12}^{(k)}}^T - {W_{12}^{(k)}}^T \otimes I_n \\
L_{21} = -I_n \otimes {W_{21}^{(k)}}^T - {W_{21}^{(k)}}^T \otimes I_n
\end{cases}
$$

where $\otimes$ is the Kronecker product and the vec operator stacks the columns of a matrix into a column vector.

In this paper we will improve the Newton iteration. We introduce a new iterative method for computing the nonnegative stabilizing solution to (2), where two sequences of Lyapunov algebraic equations are constructed. Numerical examples have been introduced so as to demonstrate the effectiveness of the proposed algorithms. The second method is faster that the Newton method because it solves the Lyapunov matrix equations at each iterative step in comparison with the system of linear equations with high dimensional structure.

2 A new iterative method

We consider the matrix functions $\mathcal{R}_1(X_1, X_2)$ and $\mathcal{R}_2(X_1, X_2)$ introduced in (1).

Lemma 1 *For the matrix function $\mathcal{R}_i(X_1, X_2), i = 1, 2$ the following identities hold:*

$$
\begin{array}{lrcl}
\text{(i)} & \mathcal{R}_i(X_1, X_2) & = & {A_X}^T X_i - X_i A_X \\
 & & & -Q_i - X_i S_i X_i - \sum_{j\neq i} X_j S_{ij} X_j ,
\end{array}
\tag{4}
$$

with $A_X = A - S_1 X_1 - S_2 X_2$, and

$$
\begin{array}{l}
\text{(ii) } \mathcal{R}_i(X_1, X_2) = \mathcal{R}_i(Z_1, Z_2, X_1, X_2) := \\
-Q_i - Z_i S_i Z_i + (X_i - Z_i) S_i (X_i - Z_i) \\
+\sum_{j\neq i} [(X_j - Z_j) S_j X_i + X_i S_j (X_j - Z_j)] \\
-{A_Z}^T X_i - X_i A_Z - \sum_{j\neq i} X_j S_{ij} X_j ,
\end{array}
\tag{5}
$$

where $A_Z = A - S_1 Z_1 - S_2 Z_2$ and $Z_i = Z_i^T,\ i = 1, 2$.

Proof: The statements of Lemma 1 are verified by direct manipulations. □

We denote $\mathcal{R}_i(\mathbf{Z}, \mathbf{X})$ the presentation of $\mathcal{R}_i(\mathbf{X})$ through a symmetric matrix $\mathbf{Z}$.

In Theorem 8 Azevedo-Perdicoulis and Jank [[2]] have proved the convergence properties of the Newton method for a two-player differential game, where the information structure of each player is of a feedback patten are derived. In order to improve the Newton method we introduce the Lyapunov iterative process, where the sequences of Lyapunov algebraic equations are constructed. We put $X_2^{(k)}$ instead of $X_2^{(k+1)}$ in the first equation of (2) and $X_1^{(k)}$ instead of $X_1^{(k+1)}$ in the second equation of (2). We obtain a new iterative method named the Lyapunov method:

$$
\begin{array}{l}
-{A^{(k)}}^T X_1^{(k+1)} - X_1^{(k+1)} A^{(k)} = \tilde{Q}_1^{(k)} \\
-{A^{(k)}}^T X_2^{(k+1)} - X_2^{(k+1)} A^{(k)} = \tilde{Q}_2^{(k)} ,
\end{array}
\tag{6}
$$

where

$$
\begin{array}{l}
\tilde{Q}_1^{(k)} = Q_1 + X_1^{(k)} S_1 X_1^{(k)} + X_2^{(k)} S_{12} X_2^{(k)} \\
\tilde{Q}_2^{(k)} = Q_2 + X_2^{(k)} S_2 X_2^{(k)} + X_1^{(k)} S_{21} X_1^{(k)} .
\end{array}
\tag{7}
$$

In our investigation we exploit the fact that the following statements are equivalent for a Z-matrix (-A):

(a) $-A$ is a nonsingular M-matrix;

(b) $I_n \otimes (-A^T) + (-A^T) \otimes I_n$ is a nonsingular M-matrix;

(c) A is asymptotically stable.

The convergence properties of the Lyapunov iteration (6) are established in the following theorem:

Theorem 2 *Assume there exist symmetric nonnegative matrices $\hat{X}_1, \hat{X}_2$ and $X_1^{(0)} = 0, X_2^{(0)} = 0$ such that $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \geq 0$, and $-A$ is a nonsingular M-matrix. Then, the matrix sequences $\{X_1^{(k)}, X_2^{(k)}\}_{k=0}^{\infty}$ defined by (6) satisfies:*

(i) $\hat{X}_i \geq X_i^{(k+1)} \geq X_i^{(k)}$ *and* $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \leq 0$ *for* $i = 1, 2\,,\ k = 0, 1, \ldots$*;*

(ii) *The matrix $-A^{(k)}$ is an M-matrix for $k = 0, 1, \ldots$;*

(iii) *The matrix sequences $\{X_1^{(k)}, X_2^{(k)}\}_{k=0}^{\infty}$ converge to the nonpositive solution $\tilde{X}_1, \tilde{X}_2$ to the set of Riccati equations (1) with $\tilde{X}_i \leq \hat{X}_i$ and the matrix $\tilde{A}$ is asymptotically stable.*

Proof: Using iteration (6) we construct the matrix sequences $X_1^{(1)}, X_2^{(1)}, X_1^{(2)}, X_2^{(2)}, \ldots, X_1^{(r)}, X_2^{(r)}$.

We will prove by induction the following statements for $r = 0, \ldots$:

(A) $\mathcal{R}_i(X_1^{(r)}, X_2^{(r)}) \leq 0, i = 1, 2$ and the matrix $-A^{(r)}$ is an M-matrix;

(B) $X_i^{(r+1)} \geq X_i^{(r)}\,, i = 1, 2;$

(C) $\hat{X}_i \geq X_i^{(r+1)}\,, i = 1, 2.$

Assume that $\mathcal{R}_i(X_1^{(k-1)}, X_2^{(k-1)}) \leq 0$ and the matrix $-A^{(k-1)}$ is an M-matrix and $\hat{X}_i \geq X_i^{(k)} \geq X_i^{(k-1)}\,, i = 1, 2$. We will prove the statements (A)-(B)-(C) for $r = k$.

First, we would prove $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \leq 0, i = 1, 2$ and $-A^{(k)}$ is an M-matrix. Secondly, we would compute $X_1^{(k+1)}, X_2^{(k+1)}$ as a unique solution of (6). Third, we would prove that $\hat{X}_i \geq X_i^{(k+1)} \geq X_i^{(k)}\,, i = 1, 2$.

Using identity (5) for $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)})$ we present $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) = \mathcal{R}_i(X_1^{(k-1)}, X_2^{(k-1)}, X_1^{(k)}, X_2^{(k)})$. However:

$$-A^{(k-1)^T} X_i^{(k)} - X_i^{(k)} A^{(k-1)} = Q_i$$
$$+X_i^{(k-1)} S_i X_i^{(k-1)} + \textstyle\sum_{j \neq i} X_j^{(k-1)} S_{ij} X_j^{(k-1)}\,.$$

We obtain:

$$\begin{aligned} &\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \\ &= +(X_i^{(k)} - X_i^{(k-1)}) S_i (X_i^{(k)} - X_i^{(k-1)}) \\ &\quad + \textstyle\sum_{j \neq i} [(X_j^{(k)} - X_j^{(k-1)})\, S_j\, X_i^{(k)} \\ &\quad + X_i^{(k)}\, S_j (X_j^{(k)} - X_j^{(k-1)})]\,. \end{aligned}$$

Since $S_i \leq 0, i = 1, 2$ and hence, together with $X_i^{(k)} \geq X_i^{(k-1)} \geq 0, i = 1, 2$ we infer that $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \leq 0\,, i = 1, 2$.

Next, we will prove that $-A^{(k)}$ is an M-matrix. We consider the difference

$$\begin{aligned} &\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) - \mathcal{R}_i(\hat{X}_1, \hat{X}_2) \\ &= \mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) - \mathcal{R}_i(X_1^{(k)}, X_2^{(k)}, \hat{X}_1, \hat{X}_2) \\ &= -A^{(k)^T} (X_i^{(k)} - \hat{X}_i) - (X_i^{(k)} - \hat{X}_i)\, A^{(k)} \\ &- \textstyle\sum_{j \neq i} [(\hat{X}_j - X_j^{(k)})\, S_j\, \hat{X}_i + \hat{X}_i\, S_j (X_j - X_j^{(k)})]\,, \end{aligned}$$

and therefore

$$\begin{aligned} &-A^{(k)^T} (X_i^{(k)} - \hat{X}_i) - (X_i^{(k)} - \hat{X}_i)\, A^{(k)} \\ &= \mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) - \mathcal{R}_i(\hat{X}_1, \hat{X}_2) \\ &+ \textstyle\sum_{j \neq i} [(\hat{X}_j - X_j^{(k)})\, S_j\, \hat{X}_i + \hat{X}_i\, S_j (\hat{X}_j - X_j^{(k)})]\,. \end{aligned}$$

Since $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \geq 0, \mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \leq 0$ and $S_i \leq 0, i = 1, 2$ and hence, together with $\hat{\mathbf{X}} \geq \mathbf{X}^{(k)} \geq 0$ we infer that the right hand of the above identity is nonpositive. Therefore, the matrix $-A^{(k)}$ is an M-matrix.

Thus, we can apply the recursive equation (6) to find the matrix $X_1^{(k+1)}, X_2^{(k+1)}$. We will prove $\hat{X}_i \geq X_i^{(k+1)}, i = 1, 2$. After some matrix manipulations we obtain

$$\begin{aligned} &-A^{(k)^T} (X_i^{(k+1)} - \hat{X}_i) - (X_i^{(k+1)} - \hat{X}_i)\, A^{(k)} \\ &= -\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \\ &+ (\hat{X}_i - X_i^{(k)}) S_i (\hat{X}_i - X_i^{(k)}) \\ &+ \textstyle\sum_{j \neq i} [(\hat{X}_j - X_j^{(k)})\, S_j\, \hat{X}_i + \hat{X}_i\, S_j (X_j - X_j^{(k)})]\,. \end{aligned}$$

Now let us analyze the last set of matrix equations. The matrix $-A^{(k)}$ is an M-matrix. The right-hand side of each equation is nonpositive. Thus $X_i^{(k+1)} - \hat{X}_i \leq 0\,, i = 1, 2$ and $\hat{\mathbf{X}} \geq \mathbf{X}^{(k+1)}$.

For proving $\mathbf{X}^{(k+1)} \geq \mathbf{X}^{(k)}$ we consider the identity:

$$\begin{aligned} &-A^{(k)^T} (X_i^{(k)} - X_i^{(k+1)}) - (X_i^{(k)} - X_i^{(k+1)})\, A^{(k)} \\ &= \mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \end{aligned}$$

Since $\mathcal{R}_i(\mathbf{X}^{(k)})$ is a nonpositive matrix and $-A^{(k)}$ is an M-matrix we obtain $X_i^{(k)} - X_i^{(k+1)} \leq 0, i = 1, 2$. Thus $\mathbf{X}^{(k+1)} \geq \mathbf{X}^{(k)}$. Hence, the induction process has been completed.

Thus the matrix sequences $\{X_1^{(k)}, X_2^{(k)}\}_{k=0}^{\infty}$ are monotonically increasing and bounded above by $(\hat{X}_1, \hat{X}_2)$ (in the elementwise ordering). We denote $\lim_{k\to\infty}(X_1^{(k)}, X_2^{(k)}) = (\tilde{X}_1, \tilde{X}_2)$. By taking the limits in (6) it follows that $(\tilde{X}_1, \tilde{X}_2)$ is a solution of $\mathcal{R}_i(\mathbf{X}) = 0, i = 1, 2$ with the property $(\tilde{X}_1, \tilde{X}_2) \leq (\hat{X}_1, \hat{X}_2)$ and $-\tilde{A}$ is an M-matrix and therefore $\tilde{A}$ is asymptotically stable.

□

3 Numerical examples

We carry out some numerical experiments for computing the stabilizing solution to the set of generalized Riccati equations (1). The Newton method (2) and the Lyapunov method (6) are applied and compared on some examples.

We consider a two-player game where the matrix coefficients: A, B_i, Q_i and R_{ij} for $i, j = 1, 2$ are the following. We define them using the Matlab description.

B_1=full(abs(sprandn(n,4,0.7))/10);

B_2=full(abs(sprandn(n,3,0.7))/10);

R_{11} = [-400 0 0 -40; 0 -150 0 0; 0 0 -300 0; -40 0 0 -300];

R_{22} = [-90 0 0; 0 -120 -5; 0 -5 -120];

R_{12} = [220 190 190; 190 180 22; 190 22 190];

R_{21} = [100 88 0 99; 88 250 190 0; 0 190 240 130; 99 0 130 300];

Q_1=0.375*eye(n,n); Q_1(1,n)=0.45; Q_1(n,1)=0.45;

Q_2=0.285*eye(n,n); Q_2(1,n)=1.5; Q_2(n,1)=1.5;

Test 1: A=(abs(rand(n,n))/1-7*eye(n,n))/10;

Test 2: A=(abs(rand(n,n))/1-15*eye(n,n))/10;

The latter example is executed for different values of n, also 100 runs are completed for each values of n.

The latter tests are executed Test 1 for n=12 and 100 runs; Test 2 for n=27 for 30 runs. We take $X_1^{(0)} = X_2^{(0)} = 0$ and thus $\mathcal{R}_i(\mathbf{X}^{(0)}) = -Q_i \leq 0$ (i.e. the matrix is nonpositive). Regrading the outlined choice, we might note that the conditions of theorem 2 are fulfilled, i.e. $\mathbf{X}^{(0)} \leq \hat{\mathbf{X}}$, $\mathcal{R}_i(\mathbf{X}^{(0)}) \leq 0$ and $\mathcal{R}_i(\hat{\mathbf{X}}) \geq 0$, $i = 1, 2$.

On the basis of the experiments, performed for $n = 12$, the following summary of results might be outlined. The Newton iteration (2)-(3) requires 3 iteration steps while finding the stabilizing nonnegative and positive definite solution $\tilde{\mathbf{X}}_N$ for each run. Yet, the Lyapunov iteration (6)-(7) requires 8 iteration steps so as to find the stabilizing nonnegative and positive definite solution $\tilde{\mathbf{X}}_L$. The CPU time is 0.88s and and 0.5s respectively for executing the Newton iteration with 100 runs and the Lyapunov iteration with 100 runs.

Considering our results obtained for Test 2 with $n = 27$, we could summarize that the Newton iteration (2)-(3) requires 3 iteration steps while finding the stabilizing nonnegative and positive definite solution $\tilde{\mathbf{X}}_N$. On the other hand, the Lyapunov iteration (6)-(7) requires also once again 8 iteration steps so as to find the stabilizing nonnegative and positive definite solution $\tilde{\mathbf{X}}_L$. Neverthelss, the CPU time for executing the Newton iteration with 30 runs is 22.4s and it is found to be 0.6s for the Lyapunov iteration .

4 Conclusion

We study the Lyapunov iterative process for finding the nonnegative stabilizing solution to a set of Riccati equations (1). The convergence properties of the Lyapunov method is derived in Theorem 2. Numerical experiments are carried out and the obtained results are used for comparison purposes. Thus, the following conclusions might be outlined. On one hand, the effectiveness of the proposed new iterative method (6)-(7) is confirmed. The Lyapunov iterative process is found to be faster than the Newton iteration, thus the new Lyapunov method represents an acceptable alternative to compute the nonnegative stabilizing solution to (1).

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