As a consequence of both Claim 1 and Claim 2 we conclude that there exists an edge $uv$ in $C$ such that $d(\{u,v\},X) \geq \lfloor (g-4)/2 \rfloor$.

(2) Suppose now that $\mu = (g-5)/2$ otherwise by item (1) we are done. And we denote $C_X = \{u \in V(C) : d(u,X) = (g-5)/2\}$. By item (1) we can take an edge $uv$ in $G[C_X]$.

Firstly, assume $(N(u)-v) \cap C_X \neq \varnothing$ or $(N(v)-u) \cap C_X \neq \varnothing$, say, $(N(v)-u) \cap C_X \neq \varnothing$. Notice that $X_v^+(u) = X_v^+(w) = X_{uw}^+(v) = \varnothing$ and that the sets $X_v^=(u), X_v^-(u), X_v^=(w), X_v^-(w), X_{uw}^=(v)$ and $X_{uw}^-(v)$ are pairwise disjoint. We will prove it by contradiction.

By contradiction, suppose that any vertex $u$ in $C_X$ satisfies $|N_{(g-5)/2}(u) \cap X| \geq 2$. Then we have $|N_{(g-5)/2}(X_v^=(u)) \cap X| \geq 2|X_v^=(u)|, |N_{(g-5)/2}(X_{uw}^=(v)) \cap X| \geq 2|X_{uw}^=(v)|$, and $|N_{(g-5)/2}(X_v^=(w)) \cap X| \geq 2|X_v^=(w)|$. Since the sets $N_{(g-5)/2}(X_v^=(u)) \cap X, N_{(g-7)/2}(X_v^-(u)) \cap X, N_{(g-5)/2}(X_{uw}^=(v)) \cap X, N_{(g-7)/2}(X_{uw}^-(v)) \cap X, N_{(g-5)/2}(X_v^=(w)) \cap X$ and $N_{(g-7)/2}(X_v^-(w)) \cap X$ are pairwise disjoint, it follows that

$$\begin{aligned}
\xi_3(G) &\geq |X| \\
&\geq |N_{(g-5)/2}(X_v^=(u)) \cap X| + \\
&\quad |N_{(g-7)/2}(X_v^-(u)) \cap X| + \\
&\quad |N_{(g-5)/2}(X_{uw}^=(v)) \cap X| + \\
&\quad |N_{(g-7)/2}(X_{uw}^-(v)) \cap X| + \\
&\quad |N_{(g-5)/2}(X_v^=(w)) \cap X| + \\
&\quad |N_{(g-7)/2}(X_v^-(w)) \cap X| \\
&\geq 2|X_v^=(u)| + |X_v^-(u)| + 2|X_{uw}^=(v)| + \\
&\quad |X_{uw}^-(v)| + 2|X_v^=(w)| + |X_v^-(w)| \\
&\geq \xi_3(G) + |X_v^=(u)| + |X_{uw}^=(v)| + |X_v^=(w)|.
\end{aligned}$$

Then $X_v^=(u) = X_{uw}^=(v) = X_v^=(w) = \varnothing$ and

$$\begin{aligned}
X &= (N_{(g-5)/2}(u) \cap X) \cup (N_{(g-5)/2}(v) \cap X) \cup \\
&\quad (N_{(g-5)/2}(w) \cap X). \qquad (5)
\end{aligned}$$

Furthermore, we can obtain $|N_{(g-5)/2}(u) \cap X| = |X_v^-(u)|, |N_{(g-5)/2}(v) \cap X| = |X_{uw}^-(v)|$ and $|N_{(g-5)/2}(w) \cap X| = |X_v^-(w)|$. This means that $\mu = (g-5)/2 \geq 2$. As $\delta \geq 3$, we have $|N(z) \cap (C_X - u)| \geq d(z) - 2 \geq 1$ for all $z \in X_v^-(u)$ (Otherwise a cycle of length at most $g-2$ would appear). Take a vertex $z \in X_v^-(u)$ and consider a vertex $z' \in N(z) \cap (C_X - u)$. Then from (5) a cycle of length at most $g-1$ would appear, a contradiction.

Secondly, if $(N(u)-v) \cap C_X = \varnothing$ and $(N(v)-u) \cap C_X = \varnothing$, then take a vertex $w$ in $N(v)$ with $d(w,X) = (g-7)/2$. Hence $uvw$ is a 2-path in $C$, it is analogous to the above case. □

Let $G = (V,E)$ be a $\lambda_3$-connected graph. An arbitrary $\lambda_3$-cut $F$ can be denoted by $[V(C), V(\overline{C})]$, where $C$ and $\overline{C}$ are the only two components of $G-F$. There are $X \subseteq V(C)$ and $Y \subseteq V(\overline{C})$ such that $X \cup Y$ is the set of the end vertices of $[V(C), V(\overline{C})]$, and so $[V(C), V(\overline{C})] = [X,Y]$.

A $\lambda_3$-connected graph $G$ is said to be *super*-$\lambda_3$, if $G$ is $\lambda_3$-optimal and every minimum 3-restricted edge cut isolates a component with exactly three vertices. A $\kappa_3$-connected graph $G$ is said to be *super*-$\kappa_3$, if $\kappa_3(G) = \xi_3(G)$ and the deletion of each minimum 3-restricted cut isolates a component with exactly three vertices.

**Lemma 2.2.** *Let $G$ be a connected graph with girth $g \geq 6$, and minimum degree $\delta \geq 3$. Let $[V(C), V(\overline{C})]$ = $[X,Y]$ be a $\lambda_3$-cut. Then the following assertions hold:*
*(1) If $V(C) = X$, then $G$ is super-$\lambda_3$.*
*(2) If $G$ is not super-$\lambda_3$, then $C - X$ has a component with at least three vertices.*

*Proof.* Since $g \geq 6$ and $\delta \geq 3$, by Theorem 1.1 $G$ is $\lambda_3$-connected.

(1) Suppose that $V(C) = X$, then each vertex of $C$ is incident with some edges of $[X,Y]$. If $|V(C)| = 3$, then we are done. So assume that $|V(C)| \geq 4$. Let $uvw$ be a 2-path of $C$. Because $\delta \geq 3$, we assume that $|X_v^=(u)| \geq 1$. Since girth $g \geq 6$, thus arguing as before, we have

$$\begin{aligned}
\xi_3(G) &\geq \lambda_3(G) = |[X,Y]| \\
&\geq |[u,Y]| + |[v,Y]| + |[w,Y]| \\
&+ |[X_v^=(u), Y]| + |[X_{uw}^=(v), Y]| \\
&+ |[X_v^=(w), Y]| \\
&\geq |[u,Y]| + |[v,Y]| + |[w,Y]| + \\
&\quad |X_v^=(u)| + |X_{uw}^=(v)| + |X_v^=(w)| \\
&\geq 3 + d(u) - 1 + d(v) - 2 + d(w) - 1 \\
&> \xi_3(G),
\end{aligned}$$

which is a contradiction.

(2) By item (1) we have $C - X \neq \varnothing$. Suppose that any component of $C - X$ has at most two vertices. Let $C_1, C_2, \cdots, C_k$ be the components of $C - X$.

**Case 1.** Each component $C_i$ satisfies $|C_i| = 1$.

Take $C_1$ from $C_1, C_2, \cdots, C_k$. Let $C_1 = \{v\}$. Then $N(v) \subseteq X$. And $\delta \geq 3$, we pick $u, w \in N(v)$, and thus $uvw$ is a 2-path in $C$. Arguing as item (1), we have

$$\begin{aligned}\xi_3(G) &\geq \lambda_3(G) = |[X,Y]| \\ &\geq |[N(u)-v,Y]| + |[N(w)-v,Y]| + \\ &\quad |[N(v)-u-w,Y]| \\ &\geq |N(u)-v| + |N(w)-v| + \\ &\quad |N(v)-u-w| \\ &= d(u)+d(v)+d(w)-4 \geq \xi_3(G).\end{aligned}$$

It follows that $|[N(u)-v,Y]| = |N(u)-v|, |[N(v)-u-w,Y]| = |N(v)-u-w|, |[N(w)-v,Y]| = |N(w)-v|$ and $X = (N(u)-v) \cup (N(v)-u-w) \cup (N(w)-v)$. Hence $[\{u,w\},Y] = \varnothing$, which is a contradiction.

**Case 2.** There is a component $C_1$ with $|C_1| = 2$.

Assume that $V(C_1) = \{u,v\}$. Then $C_1 = K_2$, and $N(u)-v \subseteq X, N(v)-u \subseteq X$. Take $w \in X \cap (N(v)-u)$. Then $uvw$ is a 2-path in $C$. As $g \geq 6$, arguing as in (1), we have

$$\begin{aligned}\xi_3(G) &\geq \lambda_3(G) = |[X,Y]| \\ &\geq |[N(u)-v,Y]| + |[N(v)-u-w,Y]| + \\ &\quad |[(N(w)-v)\cap X,Y]| + |[w,Y]| \\ &= d(u)+d(v)+d(w)-4 \geq \xi_3(G).\end{aligned}$$

It follows that $|[N(u)-v,Y]| = |N(u)-v|, |[N(v)-u-w,Y]| = |N(v)-u-w|, |[(N(w)-v)\cap X,Y]| = |(N(w)-v)\cap X|$ and $X = (N(u)-v) \cup (N(v)-u-w) \cup ((N(w)-v)\cap X) \cup \{w\}$. Therefore, for any $x \in (N(u)-v) \cup (N(v)-u-w) \cup ((N(w)-v)\cap X)$, we have $|[x,Y]| = 1$. Since $g \geq 6$ and $\delta \geq 3$, it follows that $N(x) \cap (X-x) = \varnothing$. So $x$ is adjacent to some $C_i$'s ($2 \leq i \leq k$). If there is a $C_i = \{y\}$ such that $y \in N(x)$, then $N(y) \subseteq X$. As $g \geq 6$ and $\delta \geq 3$, we have $|N(y) \cap (N(u)-v)| \leq 1, |N(y) \cap (N(v)-u)| \leq 1$ and $|N(y) \cap (N(w) \cap X)| \leq 1$.

Without loss of generality, we assume that $|N(y) \cap (N(w) \cap X)| = 1$, then $N(y) \cap (N(v)-u) = \varnothing, \{u,v\} \nsubseteq N(y)$, and we have $|N(y) \cap (N(u)-v)| \geq 2$. There is a cycle with length smaller than $g$, a contradiction. If $|N(y) \cap (N(w) \cap X)| = 0$, then $|N(y) \cap (N(u)-v)| \geq 2$ or $|N(y) \cap (N(v)-u)| \geq 2$. There is also a cycle of length smaller than $g$, which is impossible.

If there is a $|C_j| = 2$ which $x$ is adjacent to, then it is analogous to the case of $|C_i| = 1$. We discuss the neighbors of each vertex in $C_j$, we can obtain the required result. □

Recall that in the line graph $L(G)$ of a graph $G$, each vertex represents an edge of $G$, and two vertices in a line graph are adjacent if and only if the corresponding edges of $G$ are adjacent. Let us consider the edges $x_1y_1, x_2y_2 \in E(G)$. The distance between the corresponding vertices of $L(G)$ satisfies

$$d_{L(G)}(x_1y_1, x_2y_2) = d_G(\{x_1,y_1\},\{x_2,y_2\})+1, \qquad (6)$$

which is useful to prove that $D(G)-1 \leq D(L(G)) \leq D(G)+1$.

# 3 Some sufficient conditions for graphs to be super-$\lambda_3$ (resp. super-$\kappa_3$)

Now, we will show Theorem 3.1 by contradiction.

**Theorem 3.1.** *Let $G$ be a connected graph with girth $g \geq 4$ and minimum degree $\delta \geq 3$. The following assertions hold:*

(1) *If $D(G) \leq g-4$, then $G$ is super-$\lambda_3$.*

(2) *If $D(G) \leq g-5$, then $G$ is super-$\kappa_3$.*

(3) *If the diameter of the line graph $D(L(G)) \leq g-4$, then $G$ is super-$\lambda_3$.*

(4) *If the diameter of the line graph $D(L(G)) \leq g-5$, then $G$ is super-$\kappa_3$.*

*Proof.* Since $g \geq 4$, clearly $G$ is different from the graphs in Fig.1. Thus, by Theorem 1.1, $G$ is $\lambda_3$-connected. Moreover, if $g \in \{4,5,6\}$, then theorem clearly holds. So we assume that $g \geq 7$. By part (2) of Theorem 1.2, $G$ is $\kappa_3$-connected.

(1) From Theorem 1.2 it follows that $\lambda_3 = \xi_3$. Assume that $G$ is not super-$\lambda_3$. Let $[V(C), V(\overline{C})] = [X,Y]$ be a $\lambda_3$-cut with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. By Lemma 2.2 we know that both $C-X$ and $\overline{C}-Y$ contain a connected component say $H$ and $K$, respectively, of cardinality at least three vertices. Hence both $X$ and $Y$ are cutsets with $|X|,|Y| \leq \xi_3(G)$. From Lemma 2.1 there exist two vertices $u \in V(H)$ and $\overline{u} \in V(K)$ such that $g-4 \geq D(G) \geq d(u,\overline{u}) \geq d(u,X)+1+d(\overline{u},Y) \geq 2\lfloor (g-4)/2 \rfloor + 1$, which is a contradiction if $g$ is even.

And for $g$ odd all the inequalities become equalities. This means that $\max\{d(u,X) : u \in V(H)\} = (g-5)/2$ and $\max\{d(\overline{u},Y) : \overline{u} \in V(K)\} = (g-5)/2$. Thus by Lemma 2.1, we can find $u \in V(H)$ with $d(u,X) = (g-5)/2$ such that $N_{(g-5)/2}(u) \cap X = \{x\}$ for some $x \in X$; and we can find $\overline{u} \in V(K)$ with $d(\overline{u},Y) = (g-5)/2$ such that $N_{(g-5)/2}(\overline{u}) \cap Y = \{\overline{x}\}$ for some $\overline{x} \in Y$. As $d(u,\overline{u}) = g-4$, it follows that $x\overline{x} \in [X,Y]$. Clearly we can find a vertex $v \in N(u)$ with $d(v,X) = (g-5)/2$, because otherwise $|N_{(g-5)/2}(u) \cap X| \geq |N(u)| \geq 2$. Since $d(v,\overline{u}) = g-4$ we must have $x \in N_{(g-5)/2}(v)$ or $\overline{x} \in N_{(g-3)/2}(v)$. As a consequence, the path from $u$ to $\overline{x}$ together with the path from $v$ to $\overline{x}$ and the edge $uv$ form a cycle of length at most $g-2$, which is a contradiction.

(2) From Theorem 1.2 it follows that $\kappa_3 = \xi_3$. Assume that $G$ is not super-$\kappa_3$. Let $X$ be an any $\kappa_3$-cut and consider two connected components $C, \overline{C}$ of $G-X$ with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. From Lemma 2.1 there exist two vertices $u \in V(C)$ and $\overline{u} \in V(\overline{C})$ such that $g-5 \geq D(G) \geq d(u, \overline{u}) \geq d(u, X) + d(\overline{u}, X) \geq 2\lfloor (g-4)/2 \rfloor$, which is a contradiction if $g$ is even.

And for $g$ odd all the inequalities become equalities. This means that $\max\{d(u, X) : u \in V(C)\} = (g-5)/2$ and $\max\{d(\overline{u}, Y) : \overline{u} \in V(\overline{C})\} = (g-5)/2$. Thus by Lemma 2.1, we can find $u \in V(C)$ with $d(u, X) = (g-5)/2$ such that $N_{(g-5)/2}(u) \cap X = \{x\}$ for some $x \in X$; and we can find $\overline{u} \in V(\overline{C})$ with $d(\overline{u}, Y) = (g-5)/2$ such that $N_{(g-5)/2}(\overline{u}) \cap Y = \{\overline{x}\}$ for some $\overline{x} \in Y$. As $d(u, \overline{u}) = g-5$, it follows that $x = \overline{x}$. Clearly we can find a vertex $v \in N(u)$ with $d(v, X) = (g-5)/2$. Since $d(v, \overline{u}) = g-5$ we must have $x \in N_{(g-5)/2}(v)$. As a consequence, the path from $u$ to $x$ together with the path from $v$ to $x$ and the edge $uv$ form a cycle of length at most $g-4$, which is a contradiction.

(3) Since $D(L(G)) \leq g-4$, then the diameter $D(G) \leq g-3$, which means that $\lambda_3 = \xi_3$ by Theorem 1.2. Assume that $G$ is not super-$\lambda_3$. Let $[V(C), V(\overline{C})]$ = $[X, Y]$ be a $\lambda_3$-cut with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. By Lemma 2.2 we know that both $C-X$ and $\overline{C}-Y$ contain a connected component say $H$ and $K$, respectively, of cardinality at least three. Hence both $X$ and $Y$ are cutsets with $|X|, |Y| \leq \xi_3(G)$. From Lemma 2.1 there exists an edge $uv$ in $C-X$ and there exist an edge $\overline{u}\,\overline{v}$ in $\overline{C}-Y$ satisfying $d(\{u, v\}, X) \geq \lfloor (g-4)/2 \rfloor$ and $d(\{\overline{u}, \overline{v}\}, Y) \geq \lfloor (g-4)/2 \rfloor$. Then by using (6) we have

$$\begin{aligned} g-4 \geq D(L(G)) &\geq d_{L(G)}(uv, \overline{u}\,\overline{v}) \\ &= d_G(\{u, v\}, \{\overline{u}, \overline{v}\}) + 1 \\ &\geq d_G(\{u, v\}, X) + 1 + \\ &\quad\ d_G(Y, \{\overline{u}, \overline{v}\}) + 1 \\ &\geq 2\lfloor (g-4)/2 \rfloor + 2, \end{aligned}$$

which is impossible.

(4) Now $D(L(G)) \leq g-5$. Thus the diameter $D(G) \leq g-4$, which means that $\kappa_3 = \xi_3$ by Theorem 1.2. Assume that $G$ is not super-$\kappa_3$. Let $X$ be an any $\kappa_3$-cut and consider two connected components $C, \overline{C}$ of $G-X$ with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. From Lemma 2.1 there exists an edge $uv$ in $C-X$ and there exists an edge $\overline{u}\,\overline{v}$ in $\overline{C}-X$ satisfying $d(\{u, v\}, X) \geq \lfloor (g-4)/2 \rfloor$ and $d(\{\overline{u}, \overline{v}\}, X) \geq \lfloor (g-4)/2 \rfloor$. Then by using (6) we have

$$\begin{aligned} g-5 \geq D(L(G)) &\geq d_{L(G)}(uv, \overline{u}\,\overline{v}) \\ &= d_G(\{u, v\}, \{\overline{u}, \overline{v}\}) + 1 \\ &\geq d_G(\{u, v\}, X) + d_G(X, \{\overline{u}, \overline{v}\}) \\ &\quad +1 \\ &\geq 2\lfloor (g-4)/2 \rfloor + 1, \end{aligned}$$

which is impossible. □

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