Vibration of Rectangular Plates on Elastic Foundations by Finite Grid Solution

ABDULHALİM KARASİN
Civil Engineering Department
Dicle University
Engineering Faculty, Diyarbakır
TURKEY
karasin@dicle.edu.tr

Abstract: - Assessment of stress conditions created by vertical or horizontal forces to the supporting medium is a frequent problem of design. In engineering practice, beside static case often dynamic effects must be taken into consideration for plate design problems. Plate vibration solutions have been available for regular geometries for a long time, but it will be necessary to describe the governing equation of motion in a general mathematical form. This is not easy. The intention of this study is to extend analytical solutions of the discrete one-dimensional beam elements resting on elastic foundation for solution of plate vibration problems. The solution can be stated as an extension of the so-called discrete parameter approach where the physical domain is broken down into discrete sub-domains, each endowed with a response suitable for the purpose of mimicking problem at hand.

Key-Words: Grillage of beams; vibration; elastic foundation; finite grid solution

1 Introduction

Plates on elastic foundations have received considerable attention due to their wide applicability in many engineering disciplines. Since the interaction between structural foundations and supporting soil has a great importance in many engineering applications, a considerable amount of research has been conducted. Many studies, such as [1-4] have been done to find a convenient representation of physical behavior of a real structural component supported on a foundation. There are several practical foundation models as well as their proper mathematical formulations. A broad range of the beam or plates as engineering problems has been solved numerically such as finite element and boundary element methods [5-9]. Owing to its convenience in solution of plate problems as a numerical method the finite strip method have attracted much attention from many authors as [10-12] suggested a procedure incorporating the finite strip method together with spring systems for treating plates on elastic supports. However series and closed form solutions for plates have been published for a limited number of cases as [13-19]. The orthogonalization of the series and other calculations are performed using Fourier expansion of Bernoulli polynomials under some realistic approximations for the limiting values of the boundary conditions. The studies can be summarized as series expansion consisting of some specially chosen trigonometric functions used for free vibrations of rectangular plates resting on elastic foundations with various boundaries and subjected to uniform and constant compressive, unidirectional forces and closed form solutions of free vibration problem of thin rectangular plates on Winkler and Pasternak elastic foundation model developed with some limitations such as mixed or fully-clamped boundary conditions etc.

This study is oriented toward the development of finite grid element. It is an application of the finite element method. The aim is to investigate an improved finite grid solution for vibration problems of plates on elastic foundation. This is possible for free as well as forced vibration cases. The solution can be stated as an extension of the so-called discrete parameter approach where the physical domain is broken down into discrete sub-domains, each endowed with a response suitable for the purpose of mimicking problem at hand. In another words this method the discretized plate element is reassembled by the matrix displacement method so that consistent mass matrix of the total structure is generated schme to compute all displacements for each nodal point in a convenient sequence. By this representation, it is possible to solve complicated plate problems such as non-uniform thickness and foundation properties, arbitrary boundary and loading conditions and discontinuous surfaces.

2 Theory of Problem Formulation

In engineering practice, dynamic effects need to be taken into consideration for a wide variety of plate problems. It will be necessary to describe the governing equation of motion of plates in a general mathematical form for such cases. This can be achieved by inserting the inertia forces due to the lateral translations, in an appropriate way, into the governing differential equation for static equilibrium. For dynamic problems of the plates on elastic foundations with arbitrary shapes and boundary conditions with most elements developed to date there exists no rigorous solution except in the form of infinite Fourier series for a Levy-type solution. The series solutions are valid for very limited cases such as when the second foundation parameter has been eliminated, and simple loading and boundary conditions exist. Grillages of beam elements that have no such limitations can represent the plates.

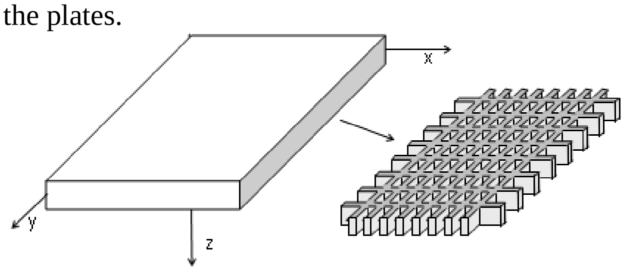

Fig.1 Representation of a rectangular plate by grids as parallel sets of one-dimensional beam elements replaces the continuous surface.

The usual approach in formulating problems of beams, plates, and shells supported by elastic media is based on the inclusion of the foundation reaction in the corresponding differential equation of the beam, plate, or shell. In case of elastic foundation under the combined action of transverse load and vibration the governing differential equation of the plates can be obtained as;

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+k_1 w \\ -k_\theta\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-\overline{m}\frac{\partial^2 w}{\partial t^2}=q(x,y) \qquad (1)$$

where $w = w(x,y,t)$ is the transverse deflection of the plate, k1 is Winkler parameter with the unit of force per unit area/per unit length (force/length3), kθ is reaction moment per unit area per unit rotation, q(x,y) is the external loads, D is flexural rigidity of plate and $\overline{m}$ is the mass of the plate per unit area.

By representing the plate with assemblage of individual beam elements interconnected at their neighboring joints, the system cannot truly be equal to the continuous structure, however sufficient accuracy can be obtained similar to the static case. Therefore plates can be modeled as an assemblage of individual beam elements interconnected at their intersecting joints. There are many researches concerning analysis of beam element resting on elastic foundation as [20-24]. The properties of such beam elements on elastic foundations will be a very useful tool to solve such generalized problems.

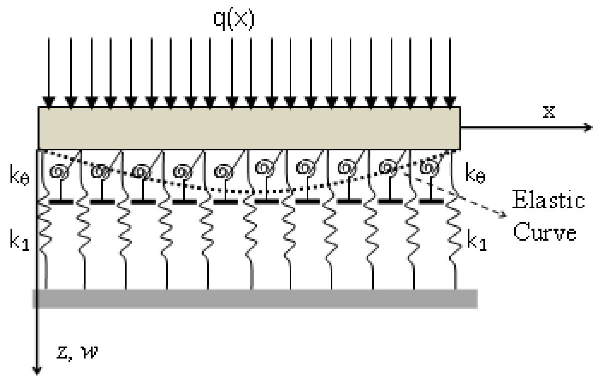

Fig.2 Representation of the beam element resting on a generalized foundation

By representing the plate shown in Fig.1 with individual beam elements the problem can be reduced to a one-dimensional one. On the other hand the similar elements can be formed in radial and tangential directions for circular plates as [25] . By representing the continuous plate with individual beam elements resting on continuous springs shown in Fig. 2 the problem will be reduced to one-dimensional one. Then Eq. (1) can be rewritten in reduced form of the governing equation for one-dimensional beam elements as;

$$D\frac{d^4w}{dx^4}+k_1w-k_\theta\frac{d^2w}{dx^2}+\overline{m}\frac{d^2w}{dt^2}=0 \qquad (2)$$

The main advantage of the reduction is that the exact geometric stiffness matrix can be determined for the beam elements and these matrices can be used as a basis of assembling the elements to apply to plate problems as [26]. Then dynamic problems of the plates resting on Winkler foundation with arbitrary loading and boundary conditions could be solved approximately. Assemblies of beam elements that have no limitations for loading and boundary conditions can represent plates adequately. The properties of beam elements resemble strips of plates resting on elastic foundations is a convenient tool to solve complicated plate problems.

The degrees of freedom of the element are the local torsion, rotation and translation at each end. Since the angular displacements are obtained from the pure torsion member, the torsional DOF's are independent of the foundation. Then it can be assumed that the displacements within the span are defined by the same interpolation functions those already derived for obtaining the element stiffness matrices. loading conditions and discontinuous surfaces.

3 Consistent Mass Matrices

Consider the beam element shown in Fig. 3 having a mass distribution m(x). If it is subjected to a unit angular acceleration at point a, the acceleration would be developed along its length as follow;

$$\ddot{w}(x)=\psi_2(x)\ddot{w}_2 \qquad (3a)$$

By d'Alembert's principle, the inertial force due to this acceleration is;

$$f_I(x)=m(x)\ddot{w}(x)=m(x)\psi_2(x)\ddot{w}_2 \qquad (3b)$$

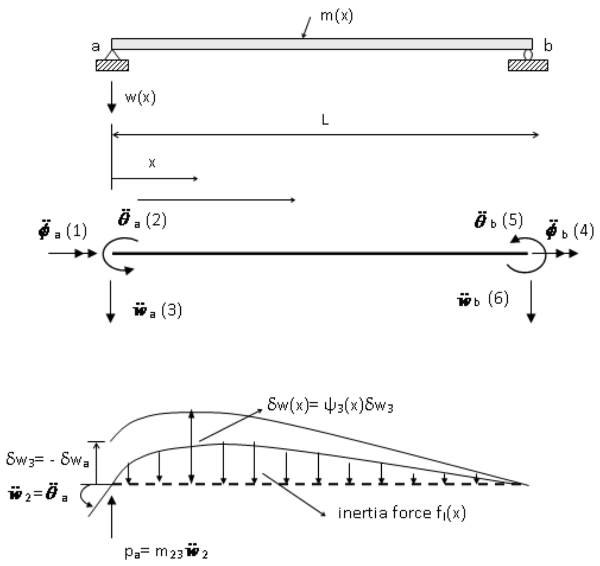

Fig.3 Representation of a beam element subjected to a unit real acceleration and virtual translation at the left side

By the principle of virtual displacements the mass influence coefficients associated with this acceleration as the nodal inertial forces can be evaluated. As an example, it is possible to evaluate the vertical force pa, equating work done by the external force due to virtual displacement, to the work done on the distributed inertial forces fI(x). That is,

$$p_a\delta w_3=\int_0^L f_I(x)\delta w(x)dx \qquad (4)$$

Substituting the vertical virtual displacement in terms of the shape functions into the equation then,

$$m_{23}=\int_0^L m(x)\psi_2(x)\psi_3(x)dx \qquad (5)$$

By this analogy, this equation can be extended to evaluate for the other degrees of freedoms such as;

$$m_{ij}=\int_0^L m(x)\psi_i(x)\psi_j(x)dx \qquad (6)$$

By using the proper shape functions, the corresponding shape functions derived for conventional beam or beam element resting one or two-parameter elastic foundations, this equation lets to evaluate all of the mass matrix terms. Computing the mass coefficients by the same shape functions with same procedures as done for determining the stiffness matrices is called consistent-mass matrices.

3.1 Consistent mass matrices for Two-Parameter foundation

For one-parameter foundation case it is possible to evaluate mass influence coefficients of a structural element with the procedures similar to that obtaining the element stiffness matrix by making the use of finite element concept. The consistent mass matrix of beam elements resting on two-parameter elastic foundations can also be evaluated by the same procedures as Winkler parameter case[27]. Substituting the proper shape functions of the beam elements resting on two-parameter derived by [26] for both $A<2\sqrt{B}$ and $A>2\sqrt{B}$ cases respectively, into Equation (6) leads to evaluate the consistent mass matrices. The terms of the mass matrix, $m_{ij} = f(p,t,\mu,L)$, obtained as functions of foundation parameters, length of the elements and mass per unit length. Since the terms for two-parameter cases are too complex and extremely long functions, they are not presented in this study. However, by letting both of the foundation parameters tend to zero, the correctness of the terms is checked. When foundation parameter k_1 and k_θ tend to zero (or p→0 and t→0), the terms in the equations reduce to the conventional beam consistent mass terms obtained by Hermitian functions as for Winkler case.

$$\lim_{p\to 0}[\underline{M}] = \frac{\mu L}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 4L^2 & -22L & 0 & -3L^2 & -13L \\ 0 & -22L & 156 & 0 & 13L & 54 \\ 70 & 0 & 0 & 70 & 0 & 0 \\ 0 & -3L^2 & 13L & 0 & 4L^2 & 22L \\ 0 & -13L & 54 & 0 & 22L & 156 \end{bmatrix}$$

In two parameter case same new parameters introduced as

$$\alpha = \sqrt{\lambda^2+\delta} = \lambda\sqrt{1+t} \qquad \text{For } A<2\sqrt{B}$$
$$\beta = \sqrt{\lambda^2-\delta} = \lambda\sqrt{1-t}$$

and

$$\alpha = \sqrt{\lambda^2+\delta} = \lambda\sqrt{1+t} \qquad \text{For } A>2\sqrt{B}$$
$$\beta = \sqrt{\delta-\lambda^2} = \lambda\sqrt{t-1}$$

where t is dimensionless as; $t = \dfrac{\delta}{\lambda^2} = \dfrac{\dfrac{k_\theta}{4EI}}{\sqrt{\dfrac{k_1}{4EI}}}$

The influence of the foundation parameters k_1 and k_θ on the consistent mass terms for $A<2\sqrt{B}$ with corresponding terms of Eq. (6) can be normalized as shown in Fig. 4. Note that, as the second parameter tends to zero (i.e. t →0) the same two-dimensional curves of one-parameter case given in Fig. 4 are obtained.

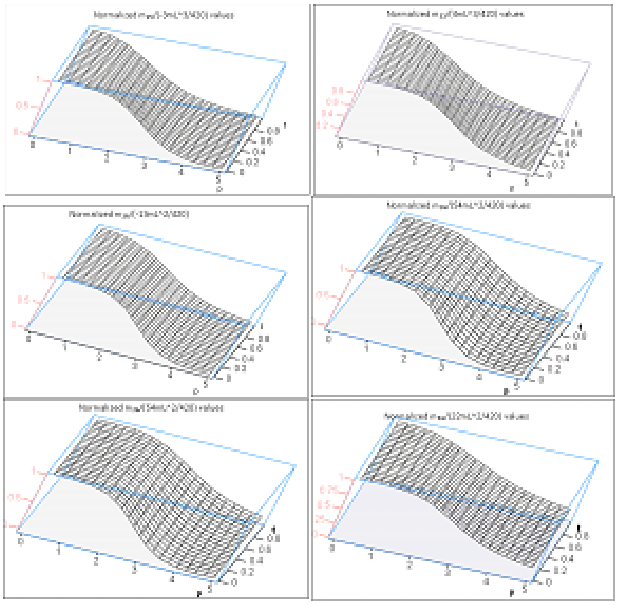

Fig.4 Influence of two-parameter foundation on the m_{22}, m_{25}, m_{26}, m_{33}, m_{36} and m_{56} normalized consistent mass terms

From the figure it is inferred that presence of second foundation parameter k_θ in the analysis is remarkably dominant. This might have been anticipated because strain energy density functional includes one more term in the case of two parameter foundation than that of the Winkler foundation.

3.2 Assembling the consistent mass matrix of the total structure

After obtaining the consistent mass matrices of each one dimensional elements the discretized plate element reassembled by the matrix displacement method to obtain free vibration frequencies of a total structure. That is, the stiffness and consistent mass matrices of the total structure is generated by using a proper numbering shame to collect all displacements for each nodal point in a convenient sequence of the system for rectangular grids can be generated as follow;

$$\underline{\boldsymbol{M}}_{sys} = \sum_{i=1}^{NE} \underline{\boldsymbol{a}}_i^{\boldsymbol{T}}\, \underline{\boldsymbol{M}}_i\, \underline{\boldsymbol{a}}_i$$

where i is the individual element number, NE is the number of elements depending on boundary conditions, a_i is the individual rotation element matrix, M_i is the proper element consistent mass matrix for a beam conventional resting on one-parameter elastic foundation and M_{sys} is the system consistent mass matrix. Then the equations of motion for a system in a free vibration as an eigenvalue problem may be written as;

$$(\underline{\boldsymbol{k}}_{sys} - \omega^2 \underline{\boldsymbol{M}}_{sys})\underline{w} = 0$$

where the quantities ω^2 are the eigenvalues indicting the square of free vibration frequencies that satisfy the above equation, while the corresponding displacement vector *w* express the fitting shapes of the vibrating system as the eigenvectors of mode shapes and k_{sys} is the stiffness matrix of the total structure defined by the same interpolation functions those already derived for obtaining the element stiffness matrices.

4 Conclusion

The solution of free vibration problems for rectangular plates resting on elastic foundations is considered to be too complex. In many cases there is apparently no analytical solution other than simple cases. A grid work analogy called the Finite Grid Solution involving discretized plate properties mapped onto equivalent beams with adjusted parameters and matrix displacement analysis are used to develop a more general simplified numerical approach for such complicated problems. It is shown that after obtaining solutions of the governing differential equations of beam elements, the derived exact shape functions (interpolation functions) have extended to determine consistent mass matrices by finite element method.

It is noted that the consistent mass terms related to one dimensional beam elements on elastic foundations are very sensitive to variation of foundation parameters. It can be concluded that the finite grid solution as a combination of finite element method, lattice analogy and matrix displacement analysis of grid works is a useful tool to improve the solution for various vibration of plate problems.

References:

[1] A.P.S. Selvadurai, Elastic analysis of soil – foundation interaction, Elsevier Scientific Publishing Co. Amsterdam, Netherlands, 1979.

[2] M. Hete´nyi, Beams on elastic foundation, Ann Arbor, MI: University of Michigan Press, 1964.

[3] A.D. Kerr, “A study of a new foundation model,” Acta Mech, vol.1, no.2, pp.135–47,1965.

[4] F.R. Scott, Foundation analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1981.

[5] M.H. Omurtag and F. Kadıoğlu, “Free vibration analysis of orthotropic plates resting on Pasternak foundation by mixed finite element formulation,” Computers and Structures, vol.67, no.4, pp.253-265, 1988.

[6] W.Q. Chen, C.F. Lü and Z.G. Bian, “A mixed method for bending and free vibration of beams resting on a Pasternak elastic foundation,” Applied Mathematical Modeling, vol.28, no.10, pp. 977-890, 2004.

[7] P.Ye. Tovstik, “The vibrations and stability of a prestressed plate on an elastic foundation,” Journal of Applied Mathematics and Mechanics, vol.73, no.1, pp. 77-87, 2009.

[8] M.K.Singha, T. Prakash, and M. Ganapathi, “Finite element analysis of functionally graded plates under transverse load,” Finite Elements in Analysis and Design, vol. 47, no. 4, pp. 453–460, 2011.

[9] K. Frydrýšek, R. Jančo, and H. Gondek, “Solutions of Beams, Frames and 3D Structures on Elastic Foundation Using FEM,” International Journal of Mechanics, , vol. 7, no. 4, pp., 362-369, 2013.

[10] P. Gagnon, C. Gosselin and L. Cloutier, “A finite strip element for the analysis of variable thickness rectangular thick plates,” Computers and Structures, vol.63 no.2, pp.349-362, 1997.

[11] M.A El -Sayad, A.M.Farag, “Semi-Analytical solution based on strip method for buckling and vibration of isotropic plate,” Journal of Applied Mathematics, vol.2013, p.p. 1-10, 2013.

[12] M.H. Huang and D.P. Thambiratnam, “Analysis of plate resting on elastic supports and elastic foundation by finite strip method,” Computers and Structures, vol.79, no. 29-30, 2547-2557, 2001.

[13] R. Szilard, Theory and analysis of plates: classical and numerical methods, Prentice Hall, Englewood Cliffs, New Jersey, 1974.

[14] S.S. Tameroğlu, “Vibrations of clamped rectangular plates on elastic foundations subjected to uniform compressive forces,” Journal of Engineering Mechanics, vol.122, no.8, pp.714-718, 1996.

[15] S. Hurlebaus, L. Gaul, J.T. Wang, ”An exact series solution for calculating the natural frequencies of orthotropic plates with completely free boundary,” Journal of Sound and Vibration, Vol.244, no.5, pp.747–759, 2001.

[16] I.E. Avramidis and K. Morfidis, “Bending of beams on three-parameter elastic foundation,” International Journal of Solids and Structures, vol.43 no. 2, 357–375, 2006.

[17] S. H. Hashemi, K. Khorshidi, and H. Rokni Damavandi Taher, “Exact acoustical analysis of vibrating rectangular plates with two

opposite edges simply supported via Mindlin plate theory," Journal of Sound and Vibration, vol. 322, no. 4-5, pp. 883–900, 2009.

[18] X. Kai, W. Jiufa, L. Qiuhong, W. Weiyuan, and W. Ping, "An exact series solution for the vibration of Mindlin rectangular plates with elastically restrained edges," Shock and Vibration, Vol. 2014, pp.1-11,2014.

[19] S. Motaghian, M. Mofid, and J.E. Akin, "On the free vibration response of rectangular plates, partially supported on elastic foundation," Applied Mathematical Modelling, vol.36 no.9, pp.4473– 4482, 2012.

[20] A. Karasin, "Extension the matrices of one dimensional beam elements for solution of rectangular plates resting on elastic foundation problems," International Journal of Mechanics, vol.9, pp.189-197, 2015.

[21] M. Eisenberger and J. Clastornik, "Beams on variable two parameter elastic foundations," ASCE Journal of Engineering Mechanics, vol.113, no.10, pp. 1454-1466,1987.

[22] J. Ying, C.F. Lü and W.Q. Chen, "Two-dimensional elasticity solutions for functionally graded beams resting on elastic foundations," Composite Structures, vol.84, no.3, pp. 209–219, 2008.

[23] M. Simsek, "Vibration analysis of a functionally graded beam under a moving mass by using different beam theories," Composite Structure, vol.92 no.4, pp. 904–917, 2010.

[24] B. Öztürk and S.B. Coşkun, "Analytical solution for free vibration analysis of beam on elastic foundation with different support conditions," Journal of Applied Mathematics, vol.2013, pp.1-7, 2013.

[25] A. Karasin, P. Gülkan and G. Aktas, "A finite grid solution for circular plates on elastic foundations," KSCE Journal of Civil Engineering, vol.19, no.4), 1157-1163, 2015.

[26] A. Karasin and P. Gulkan, "Finite grid analogy for Levy plates on generalized foundations," Turkish Journal of Engineering and Environment Science, vol. 31 no.3, pp.157-170, 2007.

[27] A. Karasin, M.E. Oncu, M. Suer, "Extension the Consistent Mass Matrices of Beam Elements for Vibration Problems of Rectangular Plates on Winkler Foundation," 17th International Conference on Mathematical and Computational Methods in Science and Engineering, 23-25 April 2015, Kuala Lumpur, Malaysia, p.101-106, 2015.