The Dual Aspects of Accounting Transaction and the Assets-Claims on Assets Equality in Axiomatic Theory

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Abstract: - The purpose of this study is to analyze the structure of the financial statements’ balance sheet and the dual aspects of accounting transactions from the viewpoint of axiomatic theory showing the relationship between assets and claims on assets. The methodology is rationalistic and analytical; it uses a well-known axiomatic theory to analyzing the balance sheet. The procedure involves a definition of axioms, an application of axiomatic theory, and an analysis of the assets-claims on assets relationship. Results show that assets and claims on assets have a set structure and can be analyzed with the axiomatic theory, leading to the conclusion that they are not equal, under the analysis of set equality.

Key-Words: - Dual aspect, accounting transactions, axiomatic method, assets, claims on assets, financial statements.

1 Introduction
This paper addresses the issue of identifying a set structure to the balance sheet and determining the relationship between assets and claims on assets using the axiomatic method.

The axiomatic method has been mainly used to create theories about the entire accounting system. The use of this method in accounting is significant (see [1], [2], [3], [4], [5]) and the analysis of financial statements can also include different types of logics, such as belief, circumscription, paraconsistent logics and dialogic, providing a different perspective on topics such as the accounting equation ([6], [7], [8]). The axiomatic method is appropriate in any science to analyze structures [9].

However, the emphasis on creating entire accounting axiomatic systems led to difficulties in understanding the complex applications of this method. Another approach is to fit an existing axiomatic theory to the accounting system and use the axioms and rules of that theory to test the trustiness of the accounting assumptions. This approach has the advantage of avoiding creating new theories, based on the author preferences; this is the approach used in this paper.

Otherwise, the dual aspects of accounting transactions determine the structure of accounting system. According to this principle, every accounting transaction is recorded in two accounts with different signs in a double classification system [10]. When it is extended to the assets-claims on assets relationship, it becomes a type of accounting assumption and, along with the double-entry bookkeeping system, is crucial to the organization of financial information.

However, other approaches criticize the accounting principles (see [11], [12], [13], [14]); among them, the fair value approach ([15], see [16] for a critic) provides a different view of the dual aspects.

2 Problem Formulation
The purpose of this research is to analyze the structure of the balance sheet and the dual aspects of accounting transactions, in its assumption form, under the axioms and principles of axiomatic theory.

The dual aspect of accounting transactions is a convention to register the credits and debits. This convention is also the foundation of the double-entry bookkeeping system that fully supports the balance sheet.

Justification exists to use the axiomatic method to analyzing an accounting principle; this method is one of the most important components of classical science [9] and provides a logical structure to a
subject [17] and a scientific explanation of the basis of any field of knowledge.

The axiomatic method has been used in accounting on many occasions, usually to create a new axiomatic system for the accounting theory and practice. Nevertheless, this paper introduces a major difference to its use; instead of creating a new accounting-specific axiomatic system, as most of the authors do, it takes an existing, well-known, and not accounting-specific axiomatic method to analyzing the structure of the balance sheet and the dual aspects of accounting transaction. The purpose of doing so is to test the use of a solid axiomatic theory in analyzing accounting system structure.

Despite the accounting-specific axiomatic systems are well-defined, and they meet their goals of explaining the assumptions of accounting, they are quite diverse; no matter how good they are, no consensus exists about which one is the most appropriate to axiomatize the accounting principles ([10], [2]). Moreover, they are created to explain the appropriate to axiomatize the accounting principles ([10], [2]).

The axiomatic method is rationalistic and analytical; it uses axiomatic set theory along with predicate logic to develop rationales and conclusions. The method involves a set of axioms, and the logical rationale to apply them to any demonstration. The Zermelo-Fraenkel (ZF) axiomatic theory, used in this paper, comprises ten well-defined axioms that determine the possibility of applying logical operations to a predicate logic language. Initially, Zermelo created this system because advances in set theory did not involve a proper definition of sets; Fraenkel made some adjustments to the theory and added the replacement axiom [19]. This axiomatic theory remains as the most prevalent, and deals with infinite and finite sets.

3 Problem Solution

3.1 Primitives and Axioms of the Zermelo–Fraenkel theory

In the ZF theory (see [20]), the primitives are membership $\in$ and set $\{x\}$. ZF theory deals only with sets; thus, the elements of a set are, in turn, sets; it does not accept elements not linked to any set (urelements).

The ZF theory comprises ten axioms; nevertheless, in this paper, only three of them will be used. They are: a) the axiom of specification that allows creating sets based on a formula, b) the axiom of union that gives a proper definition to group some sets into another set, and c) the axiom of extensionality that defines the equality of sets. The axioms will be explained all along the analysis.

The ZF theory also accepts the definition of subset as a set that is a member of another set.

3.2 Accounting axioms

According to the ZF theory, some sets exist, so in the accounting system some sets exist too.

Otherwise, the axiomatic method in accounting requires additional accounting primitives and axioms. The primitive in this system is the monetary unit $u_i$, which is the value unit used to valuate every asset or claim on assets.

The accounting axioms are as follows:

Accounting axiom 1. The elements of any set of assets and claims on assets are sets that contain sets of monetary units. This axiom means that the lowest level sets are always sets of monetary units. Therefore

$$\forall A \exists! u_i (\forall A \forall C: (u_i \in A \land u_i \in C) \rightarrow \exists A_i \in C_i)$$

with $A =$ assets, $C =$ claims on assets, $A_i =$ element (subset) of assets, $C_i =$ element (subset) of claims on...
assets, and \( u_i \) = monetary units. A special type of set
is the single monetary unit \( \{u_i\} \).

The monetary unit can be in the legal tender or
any other unit; it does not make any difference to
the analysis and does not need additional definition;
once the monetary unit is chosen it is the same for
all sets. To the purpose of this paper, the accounts in
financial statements comprise a finite number of
monetary units.

Accounting axiom 2. Every monetary unit \( \{u_i\} \) is
different to another monetary unit \( \{u_j\} \).

\[ \forall u_i \forall u_j [u_i \neq u_j] \] (2)

This axiom is necessary, because if the monetary
units were equal, a set containing ten monetary units
would be equal to a set containing just one. Therefore,
by this axiom, to any pair of monetary units \( \{u_i\} \) and \( \{u_j\} \)

\[ \forall u_i \forall u_j \exists x [(u_i \in x \land u_j \in x) \rightarrow u_i \neq u_j] \] (3)

\[ \forall u_i \forall u_j \exists x \forall y [(u_i \in x \land u_j \in y) \rightarrow u_i \neq u_j] \] (4)

Accounting axiom 3. Every monetary unit has the
property of being an asset and a claim on asset set,
simultaneously. That is

\[ \forall u_i \exists C_1 \exists A \exists C \exists u_i \in A \land u_i \in C \rightarrow (u_i \in A_i \land u_i \in C_i) \] (5)

Therefore, a monetary unit \( \{u_i\} \) can belong to
two different sets \( A_i \) and \( C_i \), simultaneously. This
axiom represents the dual aspect of the accounting
transactions as an assumption, the duality assumption. However, it is not equal to the double-
entry bookkeeping, the practice of the dual aspects
of the accounting transactions.

### 3.3 The set structure of assets and claims on
assets under the axiomatic method

In financial statements, and specifically in the
balance sheet, assets \( (A) \) are equal to claims on
assets \( (C) \).

All of the financial resources of an organization
come from institutions, companies or individuals,
and they have the right to make a claim on these
resources. That is the rationale for this relationship.
However, both groups refer to the only capital that
exists.

From now on, the letters \( u, x, y, z, C, A, L, \) and \( E \)
are used to name sets, with no reference to elements
not included in a set. Let us characterize the terms of
the balance sheet, \( A \) and \( C \), in the form of sets.

By the axiom 1, every monetary unit is allocated
to some accounts (sets), and by the accounting
axiom 3, these accounts are in both assets and
claims on assets. Accordingly, every monetary unit
is in an asset and a claim on assets accounts. A
monetary unit is characterized as an asset, or claim
on asset as follows:

\( u_i \): monetary unit considered to be an asset under an
accepted definition.

\( u_c \): monetary unit considered to be a claim on assets
under an accepted definition.

Then, the sets \( A \) and \( C \), in any financial
statements, need to be defined by formulae. The
specification axiom allows the identification of
subsets under certain conditions. This axiom states that

\[ \forall z \forall w_1 \forall w_2 \ldots \forall w_n \exists y \exists x \in y \leftrightarrow (x \in z \land \phi) \] (6)

It means that a formula \( \phi \) allows identification of
subset \( y \) such that it contains every element \( x \) of the
set \( z \) that has the property defined in the formula \( \phi \).

The sets \( A \) and \( C \) are subsets of the sets \( A_i \) and
\( C_i \), respectively. These sets \( A_i \) and \( C_i \) are also assets,
and claims on assets respectively, but they are more
comprehensive sets and comprise groups of
companies, the industry, the country, or any other
combination. In this sense, the sets \( A \) and \( C \) are
subsets of other sets.

Then, applying the specification axiom to \( A \) and
\( C \)

\[ \forall A_i \exists A \exists A_i \exists u_i [u_i \in A \leftrightarrow (u_i \in A_i \land \phi_i)] \] (7)

where \( \phi_i : u_i \) is a monetary unit of the company’s
assets. In the same form,

\[ \forall C_i \exists C \exists u_c [u_c \in C \leftrightarrow (u_c \in C_i \land \phi_c)] \] (8)

where \( \phi_c : u_c \) is a monetary unit of the company’s
claims on assets.

Claims on assets comprise the accounts (subsets)
liabilities and stockholder’s equity. Then, applying
this axiom to create the subsets \( L \) (liabilities) and \( E \)
(stockholder’s equity) of \( C \),

\[ \forall C \exists L \exists u_L [u_L \in L \leftrightarrow (u_L \in C \land \phi_L)] \] (9)

where \( \phi_L \): \( u_L \) is a monetary unit of the company’s
liability, and
\[ \forall C \exists E \exists u_E [u_E \in E \leftrightarrow (u_E \in C \wedge \phi_E)] \]  

(10)

where \( \phi_E \): \( u_E \) is a monetary unit of the company’s stockholder’s equity.

It is important to note that this is not a partition of sets because a partition has different properties to that of subsets, which are the ones being defined here.

For the sake of clarity, the analysis will address only a few items of the financial statements. Therefore, by the specification axiom, one can create subsets, in such a way that the set \( A \) contains the subsets current assets \( A_c \) and non-current assets \( A_nc \). Current assets \( A_c \), in turn, comprises cash \( A_{cc} \) and accounts receivable \( A_{car} \) whereas non-current assets \( A_nc \) contains long-term investments \( A_{nclt} \), property, plant, and equipment \( A_{ncppe} \), and intangible assets \( A_{nci} \).

As already mentioned, the formula \( \phi \) of the specification axiom allows the inclusion of monetary units in sets or subsets. This formula applies to any set or subset of financial statements.

### 3.3 The aggregated accounts as set and subsets

Financial statements allocate items to other items. Here, the ZF set theory assumes the definition of a subset as a set that is a member of another set. This definition is useful here; in predicate logic and set language, the definition of a subset is in the following form:

\[ (x \subseteq y) \leftrightarrow (\forall z (z \in x \rightarrow z \in y)) \]  

(11)

That means that if a set \( x \) contains a set \( z \) and \( y \) contains \( x \), then \( y \) contains \( z \), and \( x \) is a subset of \( y \). Regarding monetary units, and keeping in mind that ZF theory includes only sets,

\[ (u_i \subseteq u_j) \leftrightarrow (\forall u_a (u_a \in u_i \rightarrow u_a \in u_j)) \]  

(12)

In the equation, \( u_i \), \( u_j \), and \( u_a \) are sets, and it means that \( u_i \) is a subset of \( u_j \) because every element \( u_a \) of \( u_i \) is contained in \( u_j \).

Thus, total assets is a set \( A \) that consists of sets containing other sets:

\[ A = \{ \{ A_c \} , \{ A_{nc} \} \} \]  

(13)

\[ A_c = \{ \{ A_{cc} \} , \{ A_{car} \} \} \]  

(14)

\[ A_{nc} = \{ \{ A_{nclt} \} , \{ A_{ncppe} \} , \{ A_{nci} \} \} \]  

(15)

The definition of the subset allows the following structure to be built:

\[ (A_{cc} \subseteq A_c) \leftrightarrow (\forall u_i (u_i \in A_{cc} \rightarrow u_i \in A_c)) \]  

(16)

\[ (A_{car} \subseteq A_c) \leftrightarrow (\forall u_i (u_i \in A_{car} \rightarrow u_i \in A_c)) \]  

(17)

\[ (A_{nclt} \subseteq A_{nc}) \leftrightarrow (\forall u_i (u_i \in A_{nclt} \rightarrow u_i \in A_{nc})) \]  

(18)

\[ (A_{ncppe} \subseteq A_{nc}) \leftrightarrow (\forall u_i (u_i \in A_{ncppe} \rightarrow u_i \in A_{nc})) \]  

(19)

\[ (A_{nci} \subseteq A_{nc}) \leftrightarrow (\forall u_i (u_i \in A_{nci} \rightarrow u_i \in A_{nc})) \]  

(20)

\[ (A_{c} \subseteq A) \leftrightarrow (\forall A_i (A_i \in A_c \rightarrow A_i \in A)) \]  

(21)

\[ (A_{nc} \subseteq A) \leftrightarrow (\forall A_i (A_i \in A_{nc} \rightarrow A_i \in A)) \]  

(22)

In this structure, \( A \) is any subset of \( A_c \) or \( A_{nc} \). Likewise, the set \( L \) contains subsets, such as current liabilities \( L_c \) and non-current liabilities \( L_{nc} \). Current liabilities \( L_c \) include, in turn, subsets such as accounts payable \( L_{cap} \) and unearned revenues \( L_{cur} \), whereas non-current liabilities \( L_{nc} \) contains the set mortgage payable \( L_{ncm} \) and notes payable \( L_{ncnp} \). The set owners’ equity \( E \) includes issued capital \( E_{ic} \), common stocks \( E_{cs} \), and retained earnings \( E_{re} \). These sets, as in the total asset set, are in the form

\[ L = \{ \{ L_c \} , \{ L_{nc} \} \} \]  

(23)

\[ L_c = \{ \{ L_{cap} \} , \{ L_{cur} \} \} \]  

(24)

\[ L_{nc} = \{ \{ L_{ncm} \} , \{ L_{ncnp} \} \} \]  

(25)

\[ E = \{ \{ E_{ic} \} , \{ E_{cs} \} , \{ E_{re} \} \} \]  

(26)

According to the definition of subset, these sets are

\[ (L_{cap} \subseteq L_c) \leftrightarrow (\forall u_i (u_i \in L_{cap} \rightarrow u_i \in L_c)) \]  

(27)

\[ (L_{cur} \subseteq L_c) \leftrightarrow (\forall u_i (u_i \in L_{cur} \rightarrow u_i \in L_c)) \]  

(28)

\[ (L_{ncm} \subseteq L_{nc}) \leftrightarrow (\forall u_i (u_i \in L_{ncm} \rightarrow u_i \in L_{nc})) \]  

(29)

\[ (L_{ncnp} \subseteq L_{nc}) \leftrightarrow (\forall u_i (u_i \in L_{ncnp} \rightarrow u_i \in L_{nc})) \]  

(30)

\[ (L_c \subseteq L) \leftrightarrow (\forall L_{ci} (L_{ci} \in L_c \rightarrow L_{ci} \in L)) \]  

(31)

\[ (L_{nc} \subseteq L) \leftrightarrow (\forall L_{nci} (L_{nci} \in L_{nc} \rightarrow L_{nci} \in L)) \]  

(32)

Furthermore,

\[ (E_i \subseteq E) \leftrightarrow (\forall u_i (u_i \in E_i \rightarrow u_i \in E)) \]  

(33)

The sets and subsets \( \{ A \} , \{ A_c \} , \{ A_{nc} \} , \{ A_{cc} \} , \{ A_{car} \} , \{ A_{nclt} \} , \{ A_{ncppe} \} , \{ A_{nci} \} , \{ L \} , \{ L_c \} , \{ L_{nc} \} , \{ L_{cap} \} , \{ L_{cur} \} , \{ L_{ncm} \} , \{ L_{ncnp} \} , \{ E \} , \{ E_{ic} \} , \{ E_{cs} \} , \) and \( \{ E_{re} \} \) are created by formulae; this grouping has three levels for assets and liabilities and two for equity.

Another application of the subset definition leads to define the set \( C \) as comprising the subsets \( L \) and \( E \), in the form

\[ (E \subseteq C) \leftrightarrow (\forall E_i (E_i \in E \rightarrow E_i \in C)) \]  

(34)

\[ (L \subseteq C) \leftrightarrow (\forall L_i (L_i \in L \rightarrow L_i \in C)) \]  

(35)

Therefore,
\[ C = \{\{L\}, \{E\}\} \]

### 3.4 The relationship between assets and claims on assets

In all of the previous analyzes, the lowest level sets contain the monetary unit sets \(\{u_i\}\). However, these sets \(\{u_i\}\) have no financial meaning because they lack proper identification in financial statements. They acquire financial meaning by their inclusion in the next higher category, such as \(A_{cc} \ldots E_{re}\). The accounting axiom 1 states that sets in accounting system are sets that contain sets of monetary units. That allows aggregating sets of monetary units into higher order sets, which can be done by the union axiom of the ZF theory.

The axiom of union says that the union of sets is a set that contains the elements of the elements of another set. According to the axiom of union, the union of sets is

\[ \forall X \exists Y \forall z \forall w \ [(w \in z \land z \in X) \rightarrow w \in Y] \]  \hspace{1cm} (37)

It means that if a set \(X\) contains subsets \(z\) and these elements contain subsets \(w\), the union of the elements \(w\) of the subsets \(z\) of the set \(X\) is another set \(Y\). In the case of \(L\) (liabilities), current liabilities \(L_c\) and non-current liabilities \(L_{nc}\), it is

\[ \forall L \exists L_c \forall L_j \forall L_i \ [(L_i \in L_j \land L_j \in L) \rightarrow L_i \in L_u] \]  \hspace{1cm} (38)

The set \(L\) contains the subsets \(L_j\) (\(L_c\) and \(L_{nc}\)); \(L_i\) is every element of the sets \(L_c\) and \(L_{nc}\), and \(L_u\) is the union of the elements of the elements of all of \(L_j\). That is, the set \(L_u\) includes all the subsets \(L_i\) of \(L_c\) and \(L_{nc}\). With the definition of subset, \(L_u\) is included in set \(C\):

\[ (L_u \subseteq C) \leftrightarrow (\forall L_i \,(L_i \in L_u \rightarrow L_i \in C)) \]  \hspace{1cm} (39)

where \(L_i\) is any subset of \(L_u\).

Likewise, there are two sets on the claims on assets side: one is \(L_u\) and the other is \(E\); \(E\) contains all its subsets defined above. The set \(C\) contains both sets. The union \(C_u\) of these sets is

\[ \forall C \exists C_c \forall C_j \forall C_i \ [(C_i \in C_j \land C_j \in C) \rightarrow C_i \in C_u] \]  \hspace{1cm} (40)

where \(C\) is the set that contains the sets \(C_j\) (\(L_u\) and \(E\)) and \(C_i\) any subset of \(L_u\) and \(E\). Then, the set \(C_u\) comprises all \(C_i\) elements of \(L_u\) and \(E\).

The union of the subsets of \(A\) is

\[ \forall A \exists A_u \forall A_j \forall A_i \ [(A_i \in A_j \land A_j \in A) \rightarrow A_i \in A_u] \]  \hspace{1cm} (41)

where \(A\) contains the subsets \(A_j\) (\(A_c\) and \(A_{nc}\)); \(A_i\) is any element of the sets \(A\) and \(A_{nc}\); \(A_u\) is the union of the elements of the \(A_j\) subsets. That is, the set \(A_u\) includes all the subsets of \(A_c\) and \(A_{nc}\).

As a result, there are two sets, \(A_u\) and \(C_u\), which contain all the subsets of assets and all the subsets of claims on assets, respectively. These subsets are the lowest level sets with financial meaning because they have relevant item labels, such as cash, accounts receivable, accounts payable, mortgage payable, and so on. They contain all the subsets of monetary units \(\{u_i\}\).

The accounting axiom 3 states that every monetary unit is simultaneously located in both assets and claims on assets. Therefore, one can look for the type of relationship between assets and claims on assets, taking into account the set structure they have. The test to be conducted is

\[ A_u = C_u \]  \hspace{1cm} (42)

According to the axiom of extensionality, the equality of sets is

\[ \forall x \forall y \,(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)] \]  \hspace{1cm} (43)

This formula means that set \(x\) is equal to set \(y\) if for every \(z\), whenever \(z\) is a subset of \(x\), \(z\) is a subset of \(y\), and, conversely, whenever \(z\) is a subset of \(y\), \(z\) is a subset of \(x\). Then, for \(A_u\) and \(C_u\),

\[ \exists A_u \forall C_u[\forall x_i(x_i \in A_u \leftrightarrow x_i \in C_u) \rightarrow A_u = C_u] \]  \hspace{1cm} (44)

Accordingly, for \(A_u\) and \(C_u\) to be equal, they need to have the same subsets \(x_i\). It means that the subsets \(A_i\) must be equal to the subsets \(C_i\).

Also, for the subsets \(C_i\) and \(A_i\) to be equal, all of the monetary units \(\{u_i\}\) in a set \(C_i\) should only be in a set \(A_i\). Therefore, there must be a subset \(A_i\) for each \(C_i\), such that both of them have the same elements \(\{u_i\}\). Consequently, using the sets \(C_i\) and \(A_i\) of \(C_u\) and \(A_u\) respectively, for every \(C_i\) to be equal to an \(A_i\),

\[ \forall A_i \exists C_i[\forall u_i(u_i \in A_i \leftrightarrow u_i \in C_i) \rightarrow A_i = C_i] \]  \hspace{1cm} (45)

To assume that the \(C_i\) subsets are equal to the \(A_i\) subsets the \(\{u_i\}\) elements should be the same in each set. That is, the monetary unit sets \(\{u_i\}\) in a set \(C_i\)
are also in a single set $A_i$, and both sets have to have the same elements.

However, it is not a requirement of accounting axiom 3 to have the subsets of monetary units $\{u_i\}$ of each set $C_i$ located in a unique set $A_i$. It can happen that some of the monetary units of $C_i$ are also members of a given set $A_i$; therefore, the requirement that the $C_i$ subsets are equal to the $A_i$ would be in contradiction with accounting axiom 3.

Another mean to explain this, it is creating a new set by the axiom of specification. This axiom would be

$$\forall C_i \forall A_i \forall A_j [\forall u_i \forall u_j (u_i \in C_i \land u_j \in A_i) \rightarrow \exists A_i (u_i \in A_i)]$$

$$\rightarrow \exists u_j (u_j \in C_i \land u_j \in A_j)]$$

Therefore, the requirement that the $C_i$ subsets are equal to the $A_i$ would be in contradiction with accounting axiom 3.

Again, the result is a set $A_{uu}$ consisting of subsets of the type $\{u_i\}$. The sets $A_{uu}$ and $C_{uu}$ have all the monetary units $\{u_i\}$ because they are not included in any other item and, according to the axiom of extension,

$$\forall C_{uu} \forall A_{uu} [\forall u_i (u_i \in A_{uu} \leftrightarrow u_i \in C_{uu}) \rightarrow A_{uu} = C_{uu}]$$

All subsets $\{u_i\}$ are members of the sets $A_{uu}$ and $C_{uu}$, and thus

$$A_{uu} = C_{uu}$$

Although these sets are equal, this is meaningless in financial accounting. An amount of monetary units is equal to the same amount of monetary units, removing their financial classification. Yet, this classification is the essence of financial accounting.

### 4 Conclusion

The application of axiomatic theory to the balance sheet leads to the conclusions that the assets and claims on assets can be analyzed with an existing axiomatic theory combined with a small number of accounting axioms, avoiding creating new theories. Also, it yielded the conclusion that assets and claims on assets are not equal considering their different set structures.

The results obtained needs to be understood within the framework of the axiomatic theory. Also, to achieve these results the analysis took only a few items on the balance sheet. However, the same results would have been reached with any number of items or levels.
References: