Oscillatory and nonoscillatory criteria for solutions of second order linear differential functional equations

GEVORG GRIGORIAN
Institute of mathematics NAS of Armenia,
0019, c. Erevan, str. Bagramian, 24/5
ARMENIA
mathphys2@instmath.sci.am

Abstract: Riccati equation method is used to establish oscillatory and non oscillatory criteria for solutions of second order linear differential functional equations. On examples the obtained result is compared with some criteria of work of L. Berezansky and E. Braverman.

Key–Words: Riccati equations, oscillation, nonoscillation, suboscillation.

1 Introduction

Let $q(t),\ r(t),\ f(t),\ q_j(t),\ r_j(t), \alpha_j(t), \beta_j(t),\ j = \overline{1,n}$, be real valued continuous functions on $[t_0;+\infty)$. In the sequel we will assume, that the functions $\alpha_j(t),\ \beta_j(t),\ j=\overline{1,n}$ are bounded below. Denote: $T_0 \equiv \min\{t_0, \min\limits_{1\le j\le n}\{\inf\limits_{t\ge t_0}\alpha_j(t), \inf\limits_{t\ge t_0}\beta_j(t)\}\}$. Let $p(t)$ be a positive function on $[T_0;+\infty)$. Consider the equation

$$(p(t)\phi'(t))'+q(t)\phi'(t)+r(t)\phi(t)+f(t)+ \\ +\sum_{j=1}^{n}[q_j(t)\phi'(\alpha_j(t))+r_j(t)\phi(\beta_j(t))]=0, \quad (1.1)$$

$t \ge t_0$. Study the question of oscillation and non oscillation of solutions of the differential functional equations, in particular of eq. (1.1), is an important problem of qualitative theory of differential functional equations, and many works are devoted to him (see [1] and cited works in it, [2] - [11]).

In this work the Riccati equation method is used to establish oscillatory and nonoscillatory criteria for solutions of eq. (1,1) in terms of oscillation and nonoscillation of eq.

$$(p(t)\phi'(t))'+q(t)\phi'(t)+r(t)\phi(t)=0, \quad (1.2)$$

$t \ge t_0$. and (or) the functions $r(t),\ f(t),\ q_j(t),\ r_j(t),\ \alpha_j(t),\ \beta_j(t),\ j=\overline{1,n}$.

2 Auxiliary propositions

Let $a(t), b(t), c(t), a_1(t), b_1(t), c_1(t)$ be real valued continuous functions on $[t_0;+\infty)$.

Consider the Riccati equations

$$y'(t)+a(t)y^2(t)+b(t)y(t)+c(t)=0; \quad (2.1)$$

$$y'(t)+a_1(t)y^2(t)+b_1(t)y(t)+c_1(t)=0, \quad (2.2)$$

$t\ge t_0$. and the differential inequalities

$$\eta'(t)+a(t)\eta^2(t)+b(t)\eta(t)+c(t)\ge 0; \quad (2.3)$$

$$\eta'(t)+a_1(t)\eta^2(t)+b_1(t)\eta(t)+ \\ +c_1(t)\ge 0, \quad (2.4)$$

$t\ge t_0$. Note, that every solution of eq. (2.1) ((2.2)) is a solution of ineq. (2.3) ((2.4)). Note also, that for $a(t)\ge 0$ $(a_1(t)\ge 0)$, $t\ge t_0$, the real valued solutions of the equation $\eta'(t)+b(t)\eta(t)+c(t)=0$ $(\eta'(t)+b_1(t)\eta(t)+c_1(t)=0)$ are solutions of ineq. (2.3) ((2.4)). Therefore for $a(t)\ge 0$ $(a_1(t)\ge 0)$, $t\ge t_0$, ineq. (2.3) ((2.4)) has a solution, satisfying any initial real value condition. In the sequel we will assume, that the solutions of considered equations are real valued.

Theorem 2.1. *Let $y_0(t)$ be a solution of eq. (2.1) on $[t_1;t_2)$, and $\eta_0(t)$, $\eta_1(t)$ be solutions of ineq. (2.3) and (2.4) with $\eta_0(t_1)\ge y_0(t_1)$, $\eta_1(t_1)\ge y_0(t_1)$ respectively, and let $a_1(t)\ge 0$, $\lambda - y_0(t_1)+$*

$$+\int\limits_{t_1}^{t}\exp\left\{\int\limits_{t_1}^{\tau}[a_1(\xi)(\eta_0(\xi)+\eta_1(\xi))+b_1(\xi)]d\xi\right\}\times \\ \times[(a(\tau)-a_1(\tau))y_0^2(\tau)+(b(\tau)-b_1(\tau))y_0(\tau)+ \\ +c(\tau)-c_1(\tau)]d\tau\ge 0,\ \ t\in[t_1;t_2),$$

for some $\lambda\in[y_0(t_1);\eta_1(t_1)]$. Then eq. (2.2) has a solution $y_1(t)$ on $[t_1;t_2)$ with $y_1(t_1)\ge y_0(t_1)$, moreover $y_1(t)\ge y_0(t)$, $t\in[t_1;t_2)$.

Proof see in [12].

Let $t_0 \le t_1 < t_2 \le +\infty$. Denote: $T(t_1;t_2) \equiv$ $\equiv \min\{t_1, \min\limits_{1\le j\le n}\{\inf\limits_{t\in[t_1;t_2)} \alpha_j(t), \inf\limits_{t\in[t_1;t_2)} \beta_j(t)\}\}$, $U(t_1;t_2) \equiv$ $\equiv \max\{t_2, \max\limits_{1\le j\le n}\{\sup\limits_{t\in[t_1;t_2)} \alpha_j(t), \sup\limits_{t\in[t_1;t_2)} \beta_j(t)\}\}$.

We shall say, that $\phi(t)$ is a solution of eq. (1.1) on $[t_1;t_2)$, if: $\phi(t)$ is defined and continuously differentiable on $[T(t_1;t_2);U(t_1;t_2))$; $p(t)\phi'(t)$ is continuously differentiable on $[t_1;t_2)$; $\phi(t)$ satisfies (1.1) on $[t_1;t_2)$. By a solution of eq. (1.1) we shall mean its solution on $[t_0;+\infty)$.

Consider the equation

$$y'(t) + \frac{1}{p(t)}y^2(t) + \frac{q(t)}{p(t)}y(t) + r(t) + \frac{f(t)}{\mu}\exp\Big\{-\int\limits_{t_1}^{t}\frac{y(\tau)}{p(\tau)}d\tau\Big\} + \sum_{j=1}^{n}\Big[\frac{q_j(t)y(\alpha_j(t))}{p(\alpha_j(t))}\exp\Big\{\int\limits_{A_j(t)}\frac{y(\tau)}{p(\tau)}d\tau\Big\} + r_j(\tau)\exp\Big\{\int\limits_{B_j(t)}\frac{y(\tau)}{p(\tau)}d\tau\Big\}\Big] = 0, \qquad (2.5)$$

$t \ge t_1$ $(\ge t_0)$, $\mu = const \ne 0$, the symbol $\int\limits_{A_j(t)}(.)d\tau$ $\big(\int\limits_{B_j(t)}(.)d\tau\big)$ denotes integration by direction from t to $\alpha_j(t)$ $(\beta_j(t))$. We shall say, that $y(t)$ is a (nonnegative, nonpositive) solution of eq. (2.5) on $[t_1;t_2)$, if: $y(t)$ is defined and continuous on $[T(t_1;t_2);U(t_1;t_2))$; (is nonnegative, nonpositive on $[T(t_1;t_2);U(t_1;t_2))$) and satisfies (2.5) on $[t_1;t_2)$.

Let $\phi_0(t)$ be a solution of eq. (1.1) on $[t_1;t_2)$, and let $\phi_0(t) \ne 0$, $t \in [T(t_1;t_2);U(t_1;t_2))$. It is easy to show, that

$$y_0(t) \equiv \frac{p(t)\phi_0(t)}{\phi_0(t)}, \qquad (2.6)$$

$t \in [T(t_1;t_2);U(t_1;t_2))$, is a solution of eq. (2.5) on $[t_1;t_2)$, where $\mu = \phi_0(t_1)$. Consider the Riccati equation

$$y'(t) + \frac{1}{p(t)}y^2(t) + \frac{q(t)}{p(t)}y(t) + r(t) = 0, \qquad (2.7)$$

$t \ge t_0$.

Lemma 2.1. *Let eq. (2.5) has a (nonnegative, nonpositive) solution on* $[t_1;t_2)$*, and let* $q_j(t) \ge 0$, $(q_j(t) \le 0)$ $q_j(t) \equiv 0$, $r_j(t) \ge 0$, $j = \overline{1.n}$, $\frac{f(t)}{\mu} \ge 0$, $t \in [t_1;t_2)$. *Then eq. (2.7) has a solution on* $[t_1;t_2)$.

Proof. Let $y_0(t)$ be a (nonnegative, nonpositive) solution of eq. (2.5) on $[t_1;t_2)$. Note, that $y_0(t)$ is a solution of the Riccati equation

$$y'(t) + \frac{1}{p(t)}y^2(t) + \frac{q(t)}{p(t)}y(t) + \widetilde{r}(t) = 0, \qquad (2.8)$$

$t \in [t_1;t_2)$, where $\widetilde{r}(t) \equiv r(t) +$ $+\frac{f(t)}{\mu}\exp\Big\{-\int\limits_{t_1}^{t}\frac{y_0(\tau)}{p(\tau)}d\tau\Big\} +$

$$+\sum_{j=1}^{n}\Big[\frac{q_j(t)y_0(\alpha_j(t))}{p(\alpha_j(t))}\exp\Big\{\int\limits_{A_j(t)}\frac{y_0(\tau)}{p(\tau)}d\tau\Big\} + r_j(\tau)\exp\Big\{\int\limits_{B_j(t)}\frac{y_0(\tau)}{p(\tau)}d\tau\Big\}\Big], \quad t \in [t_1;t_2).$$

It follows from conditions of the lemma, that

$$\widetilde{r}(t) \ge r(t), \qquad t \in [t_1;t_2), \qquad (2.9)$$

Let $y_1(t)$ be a solution of eq. (2.7) with $y_1(t_1) \ge$ $\ge y_0(t_1)$. Then by virtue of (2.8) and Theorem 2.1 from (2.9) it follows, that $y_1(t)$ exists on $[t_1;t_2)$. The lemma is proved.

Lemma 2.2. *Let* $y_0(t)$ *be a solution of eq. (2.7) on* $[t_1;t_2)$*, and let* $y_1(t)$ *be a (nonnegative) solution of eq. (2.5) on* $[t_1;t_2)$ *with* $y_1(t_1) \ge y_0(t_1)$. *Let* $(q_j(t) \le$ $\le 0)$, $q_j(t) \equiv 0$, $r_j(t) \le 0$, $j = \overline{1.n}$, $\frac{f(t)}{\mu} \le$ ≤ 0, $t \in [T(t_1;t_2);U(t_1;t_2))$. *Then*

$$y_1(t) \ge y_0(t), \qquad t \in [t_1;t_2), \qquad (2.10),$$

moreover, if $y_1(t_1) > y_0(t_1)$*, then*

$$y_1(t) > y_0(t), \qquad t \in [t_1;t_2), \qquad (2.11),$$

Proof. Note, that $y_1(t)$ is a solution of the Riccati equation

$$y'(t) + \frac{1}{p(t)}y^2(t) + \frac{q(t)}{p(t)}y(t) + \widetilde{\widetilde{r}}(t) = 0, \qquad (2.12)$$

$t \in [t_1;t_2)$, where $\widetilde{\widetilde{r}}(t) \equiv r(t) +$ $+\frac{f(t)}{\mu}\exp\Big\{-\int\limits_{t_1}^{t}\frac{y_1(\tau)}{p(\tau)}d\tau\Big\} +$

$$+\sum_{j=1}^{n}\Big[\frac{q_j(t)y_1(\alpha_j(t))}{p(\alpha_j(t))}\exp\Big\{\int\limits_{A_j(t)}\frac{y_1(\tau)}{p(\tau)}d\tau\Big\} + r_j(\tau)\exp\Big\{\int\limits_{B_j(t)}\frac{y_1(\tau)}{p(\tau)}d\tau\Big\}\Big], \quad t \in [t_1;t_2).$$

From conditions of the lemma it follows, that

$$\widetilde{\widetilde{r}}(t) \le r(t), \qquad t \in [t_1; t_2). \tag{2.13}$$

By virtue of Theorem 2.1 and (2.12) from here follows (2.10). Let $y_1(t_1) > y_0(t_1)$, and let $\widetilde{y}_0(t)$ be the solution of eq. (2.7) with $\widetilde{y}_0(t_1) = y_1(t_1) > y_0(t_1)$. Then (see [13]) $\widetilde{y}_0(t)$ exists on $[t_1; t_2)$ and

$$\widetilde{y}_0(t) > y_0(t), \qquad t \in [t_1; t_2). \tag{2.14}$$

By virtue of Theorem 2.1 and (2.12) from (2.13) it follows, that $y_1(t) \ge \widetilde{y}_0(t), \ t \in [t_1; t_2)$. From here and from (2.14) follows (2.11). The lemma is proved.

3 Oscillatory and nonoscillatory criteria

Definition 3.1. *A solution of eq. (1.1) is said to be oscillatory, if it has arbitrary large zeroes. Otherwise it is said to be nonoscillatory.*

Definition 3.2. *A solution of eq. (1.1) is said to be suboscillatory, if its derivative has arbitrary large zeroes.*

Definition 3.3. *Eq. (1.1) is said to be oscillatory, if its all solutions are oscillatory.*

Theorem 3.1. *Let eq. (1.2) is oscillatory, and let* $r_j(t) \ge 0, \ t \ge t_0, \ \lim\limits_{t\to+\infty} \alpha_j(t) = \lim\limits_{t\to+\infty} \beta_j(t) = +\infty, \ j = \overline{1,n}$. *Then the following assertions are valid:*

I. if $f(t) \ge 0 \ (\le 0), \ q_j(t) \ge 0 \ (\le 0), \ j = \overline{1,n}, \ t \ge t_0$, *then every solution* $\phi(t)$ *of eq. (1.1) is or else suboscillatory or else there exists* $t_\phi \ge t_0$ *such, that* $sign\,\phi(t) = -sign\,\phi'(t) \ne 0 \ (sign\,\phi(t) = sign\,\phi'(t) \ne 0), \ t \ge t_\phi$;

II. if $f(t) \equiv 0, \ q_j(t) \equiv 0, \ j = \overline{1,n}$, *then eq. (1.1) is oscillatory.*

Proof. Let us prove I. Let the solution $\phi(t)$ of eq. (1.1) is not suboscillatory. Then $\phi(t) \ne 0, \ \phi'(t) \ne 0, \ t \ge t_1$, for some $t_1 \ge t_0$. We must show, that

$$\frac{\phi'(t)}{\phi(t)} < 0 \ \ (> 0), \qquad t \ge t_1. \tag{3.1}$$

Suppose, that it is not so. Then

$$\frac{\phi'(t)}{\phi(t)} > 0 \ \ (< 0), \qquad t \ge t_1. \tag{3.2}$$

Since $\lim\limits_{t\to+\infty} \alpha_j(t) = \lim\limits_{t\to+\infty} \alpha_j(t) = +\infty$, then $T(t_2; +\infty) \ge t_1$ for some $t_2 \ge t_1$. Then by virtue of (2.6) $y_1(t) \equiv \frac{p(t)\phi'(t)}{\phi(t)}$ is a solution of eq. (2.5) on $[t_2; +\infty)$. By virtue of Lemma 2.1 from here, from (3.2) and from conditions of the theorem it follows, that eq. (2.7) has a solution $y_0(t)$ on $[t_2; +\infty)$. Then $\phi_0(t) \equiv \exp\Bigl\{\int\limits_{t_2}^{t} \frac{y_0(\tau)}{p(\tau)} d\tau\Bigr\}$ is a solution of eq. (1.2) on $[t_2; +\infty)$, which is continuable (as a solution of eq. (1.2)) on $[t_0; +\infty)$ and which does not vanish on $[t_2; +\infty)$. Therefore, (1.2) is not oscillatory, which contradicts condition of the theorem. The obtained contradiction proves (3.1). The assertion I is proved. Let us prove II. Suppose (1.1) is not oscillatory. Then there exists a solution $\phi(t)$ of eq. (1.1) such, that $\phi(t) \ne 0, \ t \ge t_1$ for some $t_1 \ge t_0$. Since $\lim\limits_{t\to+\infty} \alpha_j(t) = \lim\limits_{t\to+\infty} \alpha_j(t) = +\infty$, then $T(t_2; +\infty) \ge t_1$ for some $t_2 \ge t_1$. Therefore by virtue of (2.6) $y(t) \equiv \frac{p(t)\phi'(t)}{\phi(t)}$ is a solution of eq. (2.5) on $[t_2; +\infty)$. To complete the proof of II should be repeat the arguments of the last part of the proof of I. The theorem is proved.

Example 3.1. Consider the equation

$$\phi''(t) + \sum_{k=1}^{m} a_k(t)\phi(g_k(t)) = 0, \tag{3.3}$$

$t \ge t_0$, where $a_k(t) \ (k = \overline{1,m})$ are continuous functions on $[0; +\infty)$, $\int\limits_0^{+\infty} a_1(\tau)d\tau = +\infty$ ($a_1(t)$ is real valued), $a_k(t) \ge 0, \ k = \overline{2,m}, \ g_1(t) = t$, $g_k(t) = \ln^{s_k}(1 + t) + \cos(\lambda_k t) + \sin^2(\nu_k t)e^{\mu_k t}, \ \lambda_k, \ \nu_k, \ \mu_k$ are some real constants, $s_k > 0, \ k = \overline{2,m}$. For this equation the conditions of the theorems 8 and 9 of work [1] (see [1], pp. 733, 734), imposed on $g_k(t), \ k = \overline{2,m}$, are not fulfilled, and the condition of nonnegativity, imposed on $a_1(t)$, may not be satisfied. Therefore the last ones are not applicable to eq. (3.3). Applying Theorem 3.1 to (3.3) we see, that eq. (3.3) is oscillatory.

Denote: $I_{p,q,r}(\xi; t) \equiv \int\limits_{\xi}^{t} \exp\Bigl\{\int\limits_{\tau}^{t} \frac{q(s)}{p(s)} ds\Bigr\} r(\tau)d\tau, \ \xi, t \ge t_0$. Let $t_0 < t_1 < ... < t_n < ...$ be a infinite large sequence, and let

$$I_k(t) \equiv \int\limits_{t_k}^{t} \exp\Bigl\{\int\limits_{t_k}^{\tau} \Bigl[\frac{q(\zeta)}{p(\zeta)} - \frac{1}{p(\zeta)} I_{p,q,r}(t_k; \zeta)\Bigr] d\zeta\Bigr\} \times$$

$$\times r(\tau)d\tau, \ \ t \in [t_k; t_{k+1}), \ \ k = 0, 1, 2, ...$$

Theorem 3.2. *Let the following conditions are satisfied:*

1) $I_k(t) \le 0, \ t \in [t_k; t_{k+1}), \ k = 0, 1, 2, ...$;

2) $\alpha_j(t) \le t, \ \beta_j(t) \le t, \ j = \overline{1,n}, \ t \ge t_0$;

3) $f(t) \le 0 \ (\ge 0), \ r_j(t) \le 0, \ j = \overline{1,n}, \ t \ge t_0$.

Then the following assertions are valid:
I if*
$3_1)$ $q_j(t) \equiv 0, \ j = \overline{1,n}$,
then every solution $\phi(t)$ *of eq. (1.1) with* $\phi(t) > 0 \ (< 0), \ t \in [T_0; t_0], \ \phi'(t_0) \geq 0 \ (\leq 0)$ *is a nondecreasing (nonincreasing) function on* $[t_0; +\infty)$, *moreover if* $\phi'(t_0) > 0 \ (< 0)$, *then* $\phi'(t) > 0 \ (< 0), \ t > t_0$*;*
II if*
$3_2)$ $q_j(t) \leq 0, \ j = \overline{1,n}, \ t \geq t_0$,
then for every solution $\phi(t)$ *of eq. (1.1) with* $\phi(t) > 0 \ (< 0), \ \phi'(t) \geq 0 \ (\leq 0), \ t \in [T_0; t_0], \ \phi'(t_0) > 0 \ (< 0)$ *the inequality* $\phi'(t) > 0 \ (< 0), \ t \geq t_0$, *takes place.*

Proof. From the conditions 1) it follows, that eq. (2.7) has nonnegative solution $y_0(t)$ on $[t_0; +\infty)$, satisfying the initial condition $y_0(t_0) = 0$ (see [14], p. 26, Theorem 4.1). Let us prove I*. Let $\phi(t)$ be a solution of eq. (1.1) with $\phi(t) > 0 \ (< 0), \ t \in [T_0; t_0], \ \phi'(t_0) \geq 0 \ (\leq 0)$. Let us show, that

$$\phi(t) > 0 \ (< 0), \qquad t \geq t_0. \qquad (3.4)$$

Suppose, that it is not so. Then there exists $t_1 > t_0$ such, that

$$\phi(t) > 0 (< 0), t \in [t_0; t_1), \phi(t_1) = 0. \qquad (3.5)$$

By virtue of (2.6) from the conditions 2) it follows, that $y_1(t) \equiv \frac{p(t)\phi'(t)}{\phi(t)}$ ia a solution of eq. (2.5) on $[t_0; t_1)$ with $\mu = \phi(t_0)$, moreover $y_1(t_0) \geq y_0(t_0)$. By virtue of Lemma 2.2 from here and from the conditions 3), $3_1)$ it follows, that $y_1(t) \geq y_0(t), \ t \in [t_0; t_1)$. Taking into account (3.5) from here we conclude: $\phi'(t) \geq 0 \ (\leq 0), \ t \in [t_0; t_1)$. Therefore, $\phi(t_1) \geq \phi(t_0) > 0 \ (\phi(t_1) \leq \phi(t_0) < 0)$, which contradicts (3.5). The obtained contradiction proves (3.4). By virtue of (2.6) from (3.4) it follows, that $y_1(t)$ is a solution of eq. (2.5) on $[t_0; +\infty)$ with $\mu = \phi(t_0)$. By virtue of Lemma 2.2 from here and from the conditions 3), $3_1)$ it follows, that

$$y_1(t) \geq y_0(t) \geq 0, \qquad t \geq t_0, \qquad (3.6)$$

for $y_1(t_0) \geq y_0(t_0)$, and

$$y_1(t) > y_0(t) \geq 0, \qquad t \geq t_0, \qquad (3.7)$$

for $y_1(t_0) > y_0(t_0)$. From (3.4) and (3.6) it follows, that $\phi(t)$ is a nondecreasing (nonincreasing) function on $[t_0; +\infty)$, and from (3.4) and (3.7) it follows inequality $\phi'(t) > 0 \ (< 0), \ t > t_0$. The assertion I* is proved. Let us prove II*. Let $\phi(t)$ be a solution of eq. (1.1) with $\phi(t) > 0 \ (< 0), \ \phi'(t) \geq 0 \ (\leq 0), \ t \in [T_0; t_0], \ \phi'(t_0) > 0 \ (< 0)$. Then by virtue of (2.6) from the conditions 2) it follows, that $y_1(t) \equiv \frac{p(t)\phi'(t)}{\phi(t)}$ is a solution of eq (2.5) on $[t_1; t_2)$ with $\mu = \phi(t_0)$ for some $t_1 \in (t_0; +\infty]$. Let us show, that

$$y_1(t) \geq 0, \qquad t \in [T_0; t_1). \qquad (3.8)$$

Suppose, that it is not so. Then by virtue of initial value conditions, imposed on $\phi(t)$, we have

$$y_1(t) \geq 0, \qquad t \in [T_0; t_2), \qquad (3.9)$$

for some $t_2 \in (t_0; t_1)$ and

$$y_1(t) < 0, \qquad t \in [t_2; t_3), \qquad (3.10)$$

for some $t_3 \in (t_2; t_1)$. Let $\widetilde{y}(t)$ be the solution of eq. (2.1) with $\widetilde{y}(t_0) = y_1(t_0) > y_0(t_0) = 0$. Then (see above) $\widetilde{y}(t)$ exists on $[t_0; +\infty)$ and $\widetilde{y}(t) > 0, \ t \geq t_0$. By virtue of Lemma 2.2 from the conditions 3), $3_2)$ and from (3.9) it follows, that $y_1(t) \geq \widetilde{y}(t) > 0, \ t \in [t_0; t_2]$. Therefore, $y_1(t) > 0, \ t \in [t_0; t_2 + \varepsilon)$, for some $\varepsilon > 0$, which contradicts (3.10). The obtained contradiction proves (3.8). To complete the proof of II* (repeating the arguments of the proof of I*) on the basis of Lemma 2.2 and conditions 3) and $3_2)$ one should show, that $y_1(t)$ is a solution of eq. (2.5) on $[t_0; +\infty)$ and $y_1(t) \geq \widetilde{y}_0(t) > 0, \ t \geq t_0$. The proof of the theorem is complete.

Example 3.2. Let in eq. (3.3) $a_1(t) =$

$$= \begin{cases} -\sin t, \ t \in [2n\pi; (2n+1)\pi], \\ \qquad n = 0, 1, , 2, ...; \\ \\ -\lambda \sin t, \ t \in [(2n+1)\pi; (2n+2)\pi], \\ \qquad n = 0, 1, 2, ..., \end{cases}$$

$$I \equiv \int\limits_0^{2\pi} \exp\biggl\{-\int\limits_0^{\tau} I_{1,0,a_1}(0;\zeta)d\zeta\biggr\} a_1(\tau)d\tau \leq 0, \lambda > 0$$

(it is evident, that for $\lambda = 0$ we have: $I < 0$ and I continuously depends on λ, therefore there exists $\lambda > 0$ such, that $I \leq 0$); $g_1(t) = t, \ a_k(t) \leq 0, \ g_k(t) = t - \omega_k, \ k = \overline{2.m}, \ t \geq 0, \ 0 < \omega_2 < ... < \omega_m$. It is not difficult to see, that for such $a_k(t)$ and $g_k(t)$ the condition of Theorem 7 of the work [1] (see [1], p. 732) is not fulfilled. Therefore, for such conditions Theorem 7 is not applicable to eq. (3.3). Applying Theorem 3.2 to (3.3) one can readily verify (putting $t_k = 2\pi k, \ k = 0, 1, 2, ...$ and taking into account, that $I_k(t) \leq I_k(2\pi) = I \leq 0, \ t \in [t_k; t_{k+1}), \ k = 0, 1, 2, ...$), that for mentioned restrictions every solution $\phi(t)$ of eq. (3.3) with $\phi(t) > 0 \ (< 0), \ t \in [-\omega_m; 0], \ \phi'(0) \geq 0 \ (\leq 0)$ is nondecreasing (nonincreasing) function on $[0; +\infty)$ (therefore $\phi(t)$ is nonoscillatory), moreover if $\phi'(0) > 0 \ (< 0)$, then $\phi'(t) > 0 \ (< 0), \ t > 0$.

Theorem 3.3. *Let the conditions 2), 3),* $3_1)$ *of Theorem 3.2 are satisfied, and let the solution* $\phi(t)$ *of eq. (1.1) satisfies the initial conditions* *a)* $\phi(t) > 0 \ \ (< 0), \ \ t \in [T_0; t_0], \ \ \phi'(t_0) > 0 \ \ (< 0)$, *and the condition*

b) $p(t)r(t)\left[\frac{\phi(t_0)\exp\left\{\int\limits_{t_0}^{t}\frac{q(s)}{p(s)}ds\right\}}{2p(t_0)\phi(t_0)} + \right.$
$\left. + \int\limits_{t_0}^{t}\exp\left\{\int\limits_{\tau}^{t}\frac{q(s)}{p(s)}ds\right\}\frac{d\tau}{p(\tau)}\right]^2 \le \frac{1}{4}, \ \ t \ge t_0.$

Then

$$|\phi(t)| \ge \Big\{\phi^2(t_0) + 2\phi(t_0)\phi'(t_0)\times$$
$$\times\int\limits_{t_0}^{t}\exp\Big\{-\int\limits_{t_0}^{t}\frac{q(s)}{p(s)}ds\Big\}\frac{d\tau}{p(\tau)}\Big\}^{1/2}, \qquad (3.11)$$

$t \ge t_0$,

$$|\phi'(t)| \ge \phi(t_0)\phi'(t_0)\exp\Big\{-\int\limits_{t_0}^{t}\frac{q(s)}{p(s)}ds\Big\}\Big/$$
$$\Big/\Big(\phi^2(t_0) + 2\phi(t_0)\phi'(t_0)\times$$
$$\times\int\limits_{t_0}^{t}\exp\Big\{-\int\limits_{t_0}^{t}\frac{q(s)}{p(s)}ds\Big\}\frac{d\tau}{p(\tau)}\Big)^{1/2}, \ \ t \ge t_0. \quad (3.12)$$

Proof. In eq. (2.7) we make a change $y(t) = \exp\Big\{-\int\limits_{t_0}^{t}\frac{q(s)}{p(s)}ds\Big\}z(\alpha(t)), \ \ t \ge t_0$, where $\alpha(t) \equiv \int\limits_{t_0}^{t}\exp\Big\{-\int\limits_{t_0}^{t}\frac{q(s)}{p(s)}ds\Big\}\frac{d\tau}{p(\tau)}$. We come to the equation

$$z'(\alpha(t)) + z^2(\alpha(t)) +$$
$$+ p(t)r(t)\exp\Big\{2\int\limits_{t_0}^{t}\frac{q(s)}{p(s)}ds\Big\} = 0, \qquad t \ge t_0.$$

It is evident, that this equation is equivalent to the following Riccati equation

$$z'(t) + z^2(t) +$$
$$+ p(\beta(t))r(\beta(t))\exp\Big\{2\int\limits_{t_0}^{\beta(t)}\frac{q(s)}{p(s)}ds\Big\} = 0, \quad (3.13)$$

$t \in [0; \alpha(+\infty))$, where $\beta(t)$ is the inverse function of $\alpha(t)$ (since $\alpha'(t) > 0, \ \ t \ge t_0$, then $\beta(t)$ exists). Denote: $N \equiv \frac{\phi(t_0)}{2p(t_0)\phi'(t_0)}$. Consider the Riccati equation

$$z'(t) + z^2(t) + \frac{1}{4(t+N)^2} = 0, \ \ t \ge 0.$$

One can readily check, that $z_0(t) \equiv \frac{1}{2(t+N)}$ is a solution of this equation on $[0; +\infty)$. Let $z_1(t)$ be the solution of eq. (3.13) with $z_1(0) = z_0(0) = \frac{1}{N}$. By virtue of Theorem 2.1 it follows from here and from conditions a), b), that $z_1(t)$ exists on $[0; \alpha(+\infty))$, moreover

$$z_1(t) \ge z_0(t), \qquad t \in [0; \alpha(+\infty)). \qquad (3.14)$$

Then $y_1(t) \equiv \exp\Big\{-\int\limits_{t_0}^{t}\frac{q(s)}{p(s)}ds\Big\}z_1(\alpha(t))$, is a solution of eq. (2.7) on $[t_0; +\infty)$. It follows from (3.14), that

$$y_1(t) \ge \frac{p(t)\alpha'(t)}{2(\alpha(t)+N)}, \qquad t \ge t_0. \qquad (3.15)$$

Let us show, that

$$\phi(t) \ne 0, \qquad t \ge t_0. \qquad (3.16)$$

Suppose, that it is not so. Then from a) it follows, that

$$\phi(t) \ne 0, \qquad t \in [t_0; t_1), \qquad \phi(t_1) = 0, \quad (3.17)$$

for some $t_1 > t_0$. By virtue of (2.6) from here and from 2) and a) it follows, that $y_2(t) \equiv \frac{p(t)\phi'(t)}{\phi(t)}$ is a solution of eq. (2.5) on $[t_0; t_1)$. Since $y_2(t_0) = y_1(t_0)$, then by virtue of Lemma 2.2 from (3.15) and conditions 3), 3_1) it follows, that $y_2(t) \ge y_1(t) \ge \frac{p(t)\alpha'(t)}{2(\alpha(t)+N)} > 0, \ \ t \in [t_0; t_1]$. So, $sign\,\phi(t) = sign\,\phi'(t) \ne 0, \ \ t \in [t_0; t_1]$. Therefore $|\phi(t_1)| \ge |\phi(t_0)| \ne 0$, which contradicts (3.17). The obtained contradiction proves (3.16). By virtue of (2.6) from a) and (3.16) it follows, that $y_2(t)$ is a solution of eq. (2.5) on $[t_0; +\infty)$. By virtue of Lemma 2.2 from here, from conditions 2), 3) and from (3.12) it follows, that

$$y_2(t) \ge \frac{p(t)\alpha'(t)}{2(\alpha(t)+N)} > 0, \qquad t \ge t_0. \qquad (3.18)$$

Therefore, $|\phi(t)| \ge |\phi(t_0)|\exp\Big\{\int\limits_{t_0}^{t}\frac{y_2(s)}{p(s)}ds\Big\} \ge |\phi(t_0)|\exp\Big\{\frac{1}{2}\ln(1+\frac{1}{N}\alpha(t))\Big\}, \ \ t \ge t_0$. From here follows (3.11), and by virtue of (2.6) from (3.11), (3.16) and (3.18) follows (3.12). The theorem is proved.

By analogy can be proved

Theorem 3.4. *Let the conditions 2), 3),* 3_2) *of Theorem 3.2 are satisfied, and let the solution* $\phi(t)$ *of eq. (1.1) satisfies the initial value conditions*

$$\phi(t) > 0 \ \ (< 0), \ \ \phi'(t) \ge 0 \ \ (\le 0), \ \ t \in [T_0; t_0],$$
$$\phi'(t_0) > 0 \ \ (< 0)$$

and the condition b) *of Theorem 3.3. Then for* $\phi(t)$ *the inequalities (3.11) and (3.12) hold.*

4 Conclusion

The use of comparison and global solvability criteria for scalar Riccati equations ([12], [14]) allowed us to obtain new oscillatory and non oscillatory criteria for second order linear differential - functional equations. The approach used in this work allowed us to much weaken the restrictions on the deviations of the argument of solution of the equations, presented in formulations of propositions of work [1]. A new result of this work is estimations of nonoscillatory solutions and their derivative of differential - functional equations.

References:

[1] L. Berezansky, E. Braverman. Some oscillation problems for a second order linear delay differential equations. Mathematical analysis and applications, 220, pp. 719 - 740, 1998,

[2] J. Durina. Oscillation theorems for second order advanced neutral differential equations. Tatra Mt. Math. Pub. 48, pp. 61 - 71. 2011.

[3] 3. Ch. Zeng, R. P. Agraval, M. Bohner and T. Li. Properties of higher - order half - linear functional - differential equations with noncanonical operators. Advances in Difference equations, 2013, 54, pp. 1 - 10, 2013.

[4] J. Ohrishka. Oscillation of second - order linear delay differential equations. Cent. Eur. J. Math. N° 6 (3), pp. 439 - 452, 2008.

[5] J. Durina, D Hudakova. Oscillation of second order neutral delay differential equations. Mathematica Bohemica, vol. 134, N° 1, pp. 31 - 38, 2009.

[6] Z. Oplustil, J. Sremir. Some oscillation criteria for the second - order linear delay differential equation. Mathematica Bohemica, vol. 136, N° 2, pp. 195 - 204, 20011.

[7] Y. Tian, P. Weng, J. Yang. Nonoscillation for a second order linear differential equation with impulses. Acta Mathematica Applicata Sinica. English series, vol. 20, N° 1, pp. 101 - 114, 2004.

[8] T. Li, Yu. V. Rogovchenko and Ch. Zhang. Oscillation of second - order neutral differential equations. Funkcialaj Ekvacioj, N° 56, pp. 111 - 120, 2013.

[9] G. P. Peliuqh, D. V. Belskij, On asymptotic properties of the solutions of the differential- - functional equations. Nelinejni kolyvanija, vol 10, N° 1, 2007, pp. 144 - 160,

[10] J. Ohririska, Oscillation of second - order linear delay differential equations. Cent. Eur. J. Math., 6(3) 2008, pp. 439 - 452.

[11] L. Berezansky, E. Braverman and H. Acka, Oscillation of linear delay integro - differential equations. Dinamic systems and Applications, v. 8 1999, pp. 219 - 235.

[12] G. A. Grigorian, On two comparison tests for second-order linear ordinary differential equations (Russian) Differ. Uravn. 47 (2011), no. 9, 1225 - 1240; translation in Differ. Equ. 47 (2011), no. 9 1237 - 1252, 34C10.

[13] G. A. Grigorian, On some properties of solutions of the Riccati equation. Izvestiya NAS of Armenia, vol. 42, N° 4, 2007, pp. 11 - 26

[14] G. A. Grigorian, Two comparison criteria for the scalar Riccai equations and some their applications. Izv. vuzov, Matematica, N° 11, 2012, pp. 20 - 35.